A Constructive Introduction to Quantum Field Theory

Stochastic quantisation and stochastic observables

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SISSA lectures, Lecture 3 — notes: Lecture 3, §§4–6

Where we are

So far. A Euclidean field is a measure on $\mathscr{S}'(\mathscr{M})$; the GFF μ_m is its Gaussian part; local operators are functions of φ. Wick powers $:\varphi^n:$ are random distributions iff $n(d-2)<d$, and **not even tight** otherwise.

Today. The GFF as an equilibrium; then index everything by a **resolution** $t \in [0,\infty)$.

1. **Stochastic quantisation**: the GFF as the stationary marginal of a Langevin dynamics, and as the one-point marginal of an elliptic equation.
2. The GFF as the terminal value of a martingale of smooth fields; **observables** = martingales.
3. Observables of the GFF: Wick powers in every dimension; **products and OPE**.

Parabolic stochastic quantisation

Lecture 3

Langevin dynamics (Parisi–Wu 1981)

Finite dimensions: $\mathrm{d}X_t = -\nabla S(X_t)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t$ has generator $\mathscr{L} = \Delta - \nabla S \cdot \nabla$ and

$$\mathscr{L}^* \mathrm{e}^{-S} = \nabla \cdot (\nabla \mathrm{e}^{-S} + \mathrm{e}^{-S} \nabla S) = 0:$$

the Gibbs measure $\mathrm{e}^{-S}\mathrm{d}x\,/\,Z$ is invariant.

Apply this to the Euclidean action, with a fictitious **stochastic time** t:

$$\partial_t \Phi = -(m^2 - \Delta)\,\Phi - V'(\Phi) + \sqrt{2}\,\xi \qquad \text{on } \mathbb{R} \times \mathscr{M},$$

ξ space-time white noise: $\mathbb{E}\,[\xi(u)\,\xi(v)] = \langle u, v \rangle_{L^2(\mathbb{R}\times\mathscr{M})}$. (stationary solutions)

For $V = 0$: the stochastic heat equation (infinite-dimensional OU process).

History: Parisi–Wu 1981 (gauge theories without gauge fixing); physics review Damgaard–Hüffel 1987–88; first mathematical construction for $P(\varphi)_2$ on $\mathbb{T}^2$: Jona-Lasinio–Mitter 1985. Review: Gubinelli, *Encyclopedia of Mathematical Physics* 2025.

DS ⇒ invariance

Itô's formula: for $F = g(\varphi(f_1), \dots, \varphi(f_n))$ the generator is

$$\mathscr{L} F(\varphi) = \sum_{i,j} \langle f_i, f_j \rangle \, \partial_{ij} g(\dots) - \sum_i \varphi(A f_i) \, \partial_i g(\dots).$$

Proposition. $\mathbb{E}_{\mu_m}[G \, \mathscr{L} F] = -\mathbb{E}_{\mu_m}[\langle DG, DF \rangle_{L^2(\mathscr{M})}]$ *for* $F, G \in \mathscr{F} C_b^\infty$*; in particular* $\mathbb{E}_{\mu_m}[\mathscr{L} F] = 0$.

Proof. Apply DS to $k \, \partial_i g$ with $h = f_i$ and sum over i:

$$\sum_i \mathbb{E}[k \, \partial_i g \, \varphi(A f_i)] = \sum_{i,j} \langle f_i, f_j \rangle \, \mathbb{E}[\partial_j k \, \partial_i g + k \, \partial_{ij} g].$$

Static (DS) and dynamic (Langevin) characterisations are two faces of the same integration by parts.

The stationary solution

Theorem. *For $t \in \mathbb{R}$, $f \in \mathscr{S}(\mathscr{M})$ let $\Phi_t(f) = \sqrt{2}\ \xi(K_{t,f})$, $K_{t,f}(r,x) = 1_{r<t}(\mathrm{e}^{-(t-r)A}f)(x)$.*

a) $\mathbb{E}[\Phi_t(f)\,\Phi_s(g)] = \langle f, \mathrm{e}^{-|t-s|A}Cg\rangle$*: stationary, and* $\mathsf{Law}(\Phi_t) = \mu_m$ *for every* t.

b) Φ *is a mild and weak solution: for* $s<t$, $\Phi_t(h) = \Phi_s(h) - \int_s^t \Phi_r(Ah)\,\mathrm{d}r + \sqrt{2}\ \xi(1_{(s,t)} \otimes h)$.

c) From any initial datum $\Psi_0 \in \mathscr{S}'(\mathscr{M})$ *independent of* ξ: $\mathbb{E}\,\mathrm{e}^{\mathrm{i}\Psi_t(f)} \to \mathrm{e}^{-\frac{1}{2}\langle f, Cf\rangle}$. *Hence* μ_m *is the* ***unique invariant measure***.

Proof of (a). $\langle K_{t,f}, K_{s,g}\rangle = \int_{-\infty}^{t\wedge s} \langle f, \mathrm{e}^{-(t+s-2r)A}g\rangle\,\mathrm{d}r = \langle f, \mathrm{e}^{-|t-s|A}(2A)^{-1}g\rangle$; an $\mathscr{S}'$-valued version exists by Itô–Nawata/Mitoma regularisation.

Proof of (b), (c)

(a), on $\mathbb{T}^d$. The modes $\Phi_t(e_n)$ are independent stationary OU processes $\mathrm{d}X = -\lambda_n X \mathrm{d}t + \sqrt{2}\mathrm{d}B$ with variance λ_n^{-1}.

(b) Mild form from the semigroup identity $K_{t,h} = K_{s,\mathrm{e}^{-(t-s)A}h} + 1_{(s,t)}\mathrm{e}^{-(t-\cdot)A}h$; weak form by stochastic Fubini (Exercise 4.3).

(c) By independence,

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}\Psi_t(f)} = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\Psi_0(\mathrm{e}^{-tA}f)}\big]\,\mathrm{e}^{-\frac{1}{2}\langle f,(1-\mathrm{e}^{-2tA})Cf\rangle}.$$

For $m>0$, $\mathrm{e}^{-tA}f \to 0$ in $\mathscr{S}(\mathscr{M})$, so $\Psi_0(\mathrm{e}^{-tA}f) \to 0$ a.s. If $\Psi_0 \sim \mu_m$ the product is exactly $\mathrm{e}^{-\frac{1}{2}\langle f,Cf\rangle}$ (invariance); an invariant ν must have the same characteristic functional.

Comments

- As a field on $\mathbb{R} \times \mathscr{M}$, the stationary solution has covariance $2\,(\partial_t + A)^{-1}(-\partial_t + A)^{-1} = 2\,(-\partial_t^2 + A^2)^{-1}$; its equal-time slice is $2\int_{\mathbb{R}} (\theta^2 + A^2)^{-1} \frac{\mathrm{d}\theta}{2\pi} = A^{-1}$.
- The stochastic time t has nothing to do with the Euclidean time τ.
- Interacting case: a **singular SPDE**.
  - Φ^4_2: Da Prato–Debussche 2003 (also Jona-Lasinio–Mitter, Albeverio–Röckner).
  - Φ^4_3: Hairer 2014; Gubinelli–Imkeller–Perkowski 2015; Catellier–Chouk 2018.
  - Global bounds: Mourrat–Weber 2017; Gubinelli–Hofmanová 2019; construction of Euclidean Φ^4_3 via the dynamics: Gubinelli–Hofmanová 2021.

Elliptic stochastic quantisation

Lecture 3

Dimensional reduction (Parisi–Sourlas 1979)

The elliptic SPDE in $d+2$ dimensions

$$(m^2 - \Delta_{\mathbb{R}^2 \times \mathscr{M}})\,\Psi + V'(\Psi) = \zeta, \qquad \zeta \text{ white noise on } \mathbb{R}^2 \times \mathscr{M},$$

is related to the d-dimensional Gibbs measure: formally

$$\mathrm{Law}(\Psi(z,\cdot)) \propto \exp\left(-4\pi\left[\tfrac{1}{2}\langle \varphi, A\,\varphi\rangle + \int V(\varphi)\right]\right) \mathscr{D}\,\varphi.$$

- Mechanism: a hidden **supersymmetry** (Parisi–Sourlas; Klein–Landau–Perez 1984).
- Rigorous nonlinear results: $d=0$ (Albeverio–De Vecchi–Gubinelli 2020); polynomial on $\mathbb{T}^2$ and exponential on $\mathbb{R}^2$ (ADG 2021); elliptic Φ^4 in 4 and 5 dimensions (Gubinelli–Hofmanová 2019).
- For $V=0$ it is elementary, and it explains the 4π.

The linear case

Theorem. *Let* $\mathsf{L} = m^2 - \Delta_{\mathbb{R}^2 \times \mathscr{M}} = -\Delta_{\mathbb{R}^2} + A$ *and* ζ *white noise on* $\mathbb{R}^2 \times \mathscr{M}$.

a) $\mathsf{L}\Psi = \sqrt{4\pi}\,\zeta$ *has a unique tempered solution* $\Psi(u) = \sqrt{4\pi}\,\zeta(\mathsf{L}^{-1}u)$.

b) $K_{z,f} = \mathsf{L}^{-1}(\delta_z \otimes f) \in L^2$, *continuous in* z; $\Psi(z,f) = \sqrt{4\pi}\,\zeta(K_{z,f})$ *is the section of* Ψ *at* z.

c) $\mathbb{E}[\Psi(z,f)\,\Psi(z,g)] = \langle f, Cg\rangle$: *every section has law* μ_m, *and* $(\Psi(z,\cdot))_{z\in\mathbb{R}^2}$ *is stationary and isotropic.*

Uniqueness: L maps $\mathscr{S}$ onto $\mathscr{S}$, so $D(\mathsf{L}u) = 0\ \forall u$ forces $D = 0$.

Proof: where 4π comes from

Fourier transform in z only: $\widehat{K_{z,f}}(q) = e^{-iq\cdot z}(|q|^2 + A)^{-1} f$. By Plancherel and the spectral theorem,

$$\langle K_{z,f}, K_{z,g}\rangle = \left\langle f, \left[\int_{\mathbb{R}^2} \frac{dq}{(2\pi)^2} (|q|^2 + A)^{-2}\right] g\right\rangle = \frac{1}{4\pi}\langle f, A^{-1} g\rangle,$$

since

$$\int_{\mathbb{R}^2} \frac{dq}{(2\pi)^2} \frac{1}{(|q|^2 + a)^2} = \frac{1}{2\pi}\int_0^\infty \frac{r\,dr}{(r^2 + a)^2} = \frac{1}{4\pi a}.$$

Multiply by 4π. Stationarity and isotropy: the covariance $4\pi \int e^{iq\cdot(z-z')}\langle f, (|q|^2 + A)^{-2} g\rangle \frac{dq}{(2\pi)^2}$ depends only on $|z - z'|$.

Slices of Gaussian fields

Field on $\mathbb{R}^k \times \mathscr{M}$ with covariance $(-\Delta_{\mathbb{R}^k} + A)^{-p}$; its slice $\{0\} \times \mathscr{M}$ has covariance

$$c_{k,p}\, A^{k/2-p}, \qquad c_{k,p} = \frac{\Gamma\left(p - \frac{k}{2}\right)}{(4\pi)^{k/2}\Gamma(p)}.$$

(k,p)	slice covariance	interpretation
$(1,1)$	$\frac{1}{2}A^{-1/2}$	time-zero field, $(2\omega)^{-1}$
$(2,2)$	$(4\pi)^{-1}A^{-1}$	elliptic stochastic quantisation
$(1,1), A \to A^2$	$2(-\partial_t^2 + A^2)^{-1} \to A^{-1}$	Langevin dynamics
$(1,2)$	$\propto A^{-3/2}$	not a GFF

The GFF arises iff $p = 1 + \frac{k}{2}$. For $\mathscr{M} = \mathbb{R}^d$, every codimension-two slice of the elliptic field is a GFF.

The idea

- Wilson–Kadanoff: physics is organised **scale by scale**. The measure-on-$\mathscr{S}'$ picture hides this, and forces singular objects from the start.
- Observe the field at resolution t: a **smooth** function φ_t. Information at resolution t: $\mathscr{F}_t = \sigma(\varphi_s : s \leqslant t)$.
- A physical quantity is not a random variable but a **consistent family of finite-resolution estimates**, i.e. a martingale.

Slogan

Observables are martingales in the filtration of finite-resolution measurements. They need not have a terminal value.

Setting: covariance profiles and observables

Lecture 3

Covariance profiles

Definition. *A **covariance profile** is a family $(Q_t)_{t\geqslant 0}$ of symmetric, translation invariant operators with $Q_0=0$, $\dot{Q}_t:=\partial_t Q_t\geqslant 0$, $Q_t\to C=(m^2-\Delta)^{-1}=A^{-1}$, and smooth kernels for $t<\infty$.*

Heat profile:

$$Q_t=C\,\mathrm{e}^{-A/t^2},\qquad \dot{Q}_t=\frac{2}{t^3}\mathrm{e}^{-A/t^2},\qquad \dot{Q}_t(x-y)=\frac{2\,t^{d-3}\mathrm{e}^{-m^2/t^2}}{(4\pi)^{d/2}}\mathrm{e}^{-t^2|x-y|^2/4}\text{on }\mathbb{R}^d.$$

$\dot{Q}_t$ lives on distances $\lesssim 1/t$; Q_t keeps the modes $|k|\lesssim t$: t is a **momentum scale**.

Others: sharp Fourier cutoff $1_{|k|\leqslant t}(|k|^2+m^2)^{-1}$; **finite range** decompositions, $\dot{Q}_t(x,y)=0$ for $|x-y|\geqslant 1/t$.

Notation: $c_t=Q_t(x,x)$, $Q_{T,t}=Q_T-Q_t$. For $d\geqslant 2$, $c_t\to\infty$ ($\frac{1}{2\pi}\log t$ in $d=2$, $\asymp t^{d-2}$ in $d\geqslant 3$).

The field at resolution t

W cylindrical Brownian motion on $L^2(\mathscr{M})$,

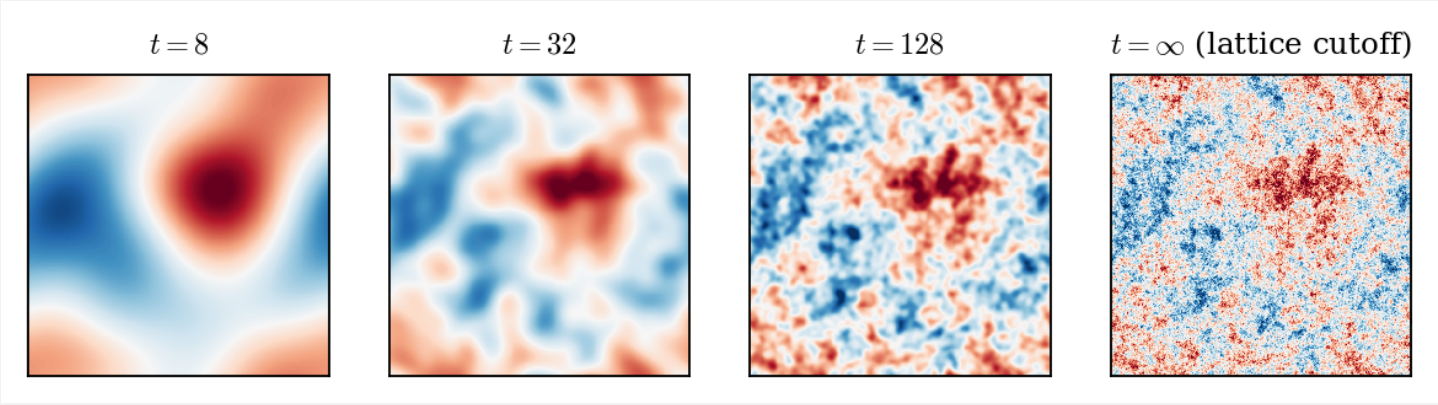
$$X_t := \int_0^t \dot{Q}_s^{1/2} \, \mathrm{d}W_s.$$

Proposition. *X_t is a smooth Gaussian field,* $\mathrm{Law}(X_t) = \mu^{Q_t}$*; $(X_t(x))_t$ is a continuous martingale with* $\mathrm{d}\langle X(x), X(y)\rangle_t = \dot{Q}_t(x,y)\,\mathrm{d}t$*; $X_t \to X_\infty$ in H^{-s}_{loc}, $s > \frac{d}{2} - 1$, with* $\mathrm{Law}(X_\infty) = \mu_m$*; $X_T - X_t \perp \mathscr{F}_t$, with law $\mu^{Q_{T,t}}$.*

The GFF is the **terminal value of a Gaussian martingale of smooth fields**.

Proof (torus). $X_t = \sum_n \left(\int_0^t \dot{q}_n(s)^{1/2} \mathrm{d}B_s^n\right) e_n$: independent Gaussian martingales, variance $q_n(t) \uparrow q_n(\infty)$; then as in Lecture 2.

The field at resolution t: a picture



Heat profile on $\mathbb{T}^2$, $m = 4$: $t = 8, 32, 128$ and the lattice cutoff, same white noise.

The martingale reveals finer and finer details; the large scales are fixed at low resolution.

Stochastic observables

$\mathbb{E}_t = \mathbb{E}[\,\cdot\,|\mathscr{F}_t]$.

Definition. (Bailleul–Chevyrev–G.) *An **observable** is an $(\mathscr{F}_t)$-martingale $O = (O_t)_{t\geqslant 0}$. Its expectation is $\langle O\rangle = \mathbb{E}\,O_t$. An observable field is a family $(O(x))_{x\in\mathscr{M}}$.*

- $Y \in L^1(\mathscr{F}_\infty)$ gives a **closed** observable $O_t = \mathbb{E}_t Y$.
- The point: observables need **not** be closed. $X_t(x)$, $d \geqslant 2$, diverges, yet is a perfectly good family of estimates.

Lemma. (Asymptotic uniqueness) *(a) $\|O^1_t - O^2_t\|_{L^1} \to 0 \Rightarrow O^1 = O^2$. (b) N adapted, $\|N_t - O_t\|_{L^1} \to 0 \Rightarrow O_s = \lim_t \mathbb{E}_s N_t$.*

Proof. $\mathbb{E}_s$ is an L^1 contraction.

Observables as equivalence classes

Two integrable adapted processes are **asymptotically equivalent** if

$$N \sim N' \quad \Longleftrightarrow \quad \|N_t - N'_t\|_{L^1} \longrightarrow 0 \quad (t \to \infty).$$

- A class contains **at most one martingale**: if $O \sim O'$ are martingales, then $\|O_s - O'_s\|_{L^1} \leqslant \|O_t - O'_t\|_{L^1} \to 0$.
- When it exists, it is recovered by looking back: $O_s = \lim_{t\to\infty} \mathbb{E}_s N_t$ in L^1, for any N in the class.
- An **observable** = a class containing a martingale; its members are *descriptions*, the martingale is the canonical representative.

Random variables sit inside. For $Y \in L^1(\mathscr{F}_\infty)$,

$$Y \quad \longmapsto \quad (\mathbb{E}_t Y)_{t\geqslant 0}$$

is injective ($\mathbb{E}_t Y \to Y$ in L^1), with image the uniformly integrable martingales: the closed observables.

Observables generalise L^1 random variables. Removing uniform integrability enlarges the space; an almost sure limit alone need not determine the observable.

Brownian observables

For a standard Brownian motion B and the Hermite polynomials $H_n(a;c)$ of variance c, the martingales

$$B_t, \qquad B_t^2 - t, \qquad B_t^3 - 3tB_t, \qquad \dots, \qquad H_n(B_t;t)$$

are observables without a terminal value, and not even bounded in L^1:

$$\mathbb{E}|B_t^2 - t| = t\,\mathbb{E}|Z^2 - 1| \longrightarrow \infty, \qquad Z \sim N(0,1).$$

They sum up to the exponential martingale ($\lambda \in \mathbb{R}, \lambda \neq 0$):

$$Z_t = \mathrm{e}^{\lambda B_t - \lambda^2 t/2} = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} H_n(B_t;t), \qquad Z_t \to 0 \text{ a.s.}, \qquad \mathbb{E}\, Z_t = 1,$$

since $\lambda B_t - \lambda^2 t/2 \sim -\lambda^2 t/2 \to -\infty$. The mass escapes to rarer and rarer paths.

The limiting random variable is zero; the observable is not. A point value $X_t(x)$ of the GFF is a Brownian motion run with the clock c_t: these are the prototypes of the Wick powers $H_n(X_t(x);c_t)$.

Repair by the future drift

Proposition. *Let $N_t = N_0 + \int_0^t \beta_s\,ds + M_t$, with $N_0 \in L^1$, M a martingale and $\mathbb{E}\int_0^\infty |\beta_s|\,ds < \infty$. The only observable with $\|N_t - O_t\|_{L^1} \to 0$ is*

$$O_t = N_t + \mathbb{E}_t\Big[\int_t^\infty \beta_s\,ds\Big] \qquad \textit{(the expected drift still to come).}$$

Proof. $O_t = N_0 + M_t + \mathbb{E}_t\int_0^\infty \beta_s\,ds$ is a martingale, and $\|O_t - N_t\|_{L^1} \leqslant \mathbb{E}\int_t^\infty |\beta_s|\,ds \to 0$; uniqueness is asymptotic uniqueness.

Example. $N_t = B_t + e^{-t}\sin B_t$. Itô: $\beta_t = -\frac{3}{2}e^{-t}\sin B_t$, and $\mathbb{E}_t \sin B_s = e^{-(s-t)/2}\sin B_t$, so

$$\mathbb{E}_t\int_t^\infty \beta_s\,ds = -\frac{3}{2}\sin B_t \int_t^\infty e^{-s-(s-t)/2}\,ds = -e^{-t}\sin B_t, \qquad O_t = B_t.$$

This is the germ of the forward–backward equations: in Lecture 4 the forecast $\psi_t = \mathbb{E}_t\,\Phi_T$ is the field plus the expected drift still to come.

A delta at a Brownian endpoint

Fix $T>0$, $y\in\mathbb{R}$, the heat kernel $p_s(w)=(2\pi s)^{-1/2}\mathrm{e}^{-w^2/2s}$, and

$$\Pi_t^y := p_{T-t}(B_t - y), \qquad 0\leqslant t<T.$$

- **Martingale:** $\mathbb{E}_s\,\Pi_t^y=\Pi_s^y$ for $s\leqslant t$, by Chapman–Kolmogorov $p_{t-s}*p_{T-t}=p_{T-s}$.
- **Density of B_T:** $\int g(y)\,\Pi_t^y\,\mathrm{d}y=\mathbb{E}_t\,g(B_T)$ for bounded g, by the Markov property.
- **Singular:** at fixed y, $\Pi_t^y\to 0$ almost surely as $t\uparrow T$, since $B_T\neq y$ a.s.; but $\mathbb{E}\,\Pi_t^y=p_T(y)>0$.
- **Mass one:** $y\mapsto\Pi_t^y$ is a probability density for every $t<T$, so $(\Pi^y)_y$ is a measure-valued observable, although each Π_t^y tends to zero.

$\delta(B_T-y)$ is not a random variable, but it is an observable: the family of its conditional densities. Its derivatives $\partial_y^k\Pi^y$ are observables too.

High resolution means $t\uparrow T$; reparametrise by $(T-t)^{-1}$ to get $[T^{-1},\infty)$. The same works for any Markov process with a transition density.

Germs and locality

An observable is determined by its **asymptotics**.

- A **germ** of O: functionals $\mathring{O}_t$ of smooth fields with $\|O_t - \mathring{O}_t(X_t)\|_{L^1} \to 0$. A germ determines O.
- $O(x)$ is **local** if it has germs

$$\mathring{O}_t(X_t)(x) = \mathring{\mathcal{O}}_t(X_t(x), \nabla X_t(x), \dots, \nabla^k X_t(x)).$$

- Germs are not unique: this freedom **is** the freedom of choosing counterterms.

Example. $:X_t^n:(x) = H_n(X_t(x); c_t)$ is a local observable in **every** d (more later), although $:\varphi^n:$ does not exist as a random distribution when $n(d-2) \geqslant d$.

Observables and their products

Lecture 3

Wick observables in every dimension

Recall $\partial_c H_n(a;c) = -\frac{1}{2}\partial_a^2 H_n$, $\partial_a H_n = n H_{n-1}$.

Proposition. *For every d, n, x: $:X_t^n:(x) := H_n(X_t(x);c_t)$ is a continuous martingale, $\mathrm{d}:X_t^n:(x) = n:X_t^{n-1}:(x)\,\mathrm{d}X_t(x)$, and*

$$\mathbb{E}\left[:X_t^n:(x):X_t^k:(y)\right] = \delta_{nk}\, n!\, Q_t(x,y)^n \xrightarrow{t\to\infty} \delta_{nk}\, n!\, G(x-y)^n \quad (x \neq y).$$

Proof. Itô: the drift is $(\frac{1}{2}\partial_a^2 + \partial_c) H_n\, \dot{c}_t = 0$.

- Lecture 2: $:\varphi^n:$ exists iff $n(d-2) < d$. As **observables**, Wick powers exist in every d, unsmeared.
- For $n(d-2) \geqslant d$, $0 \leqslant f \not\equiv 0$, $:X_t^n:(f)$ is unbounded in L^2 and (Paley–Zygmund) not tight: **no terminal value**; the observable is the whole family.
- The singularity has moved to **products at coincident points**.

Products of observables

Definition. *$O^1 \star O^2$ is the observable with $\|(O^1 \star O^2)_t - O^1_t\, O^2_t\|_{L^1} \to 0$; unique, and $(O^1 \star O^2)_s = \lim_t \mathbb{E}_s[O^1_t\, O^2_t]$.*

Proposition. *If O^1, O^2 are continuous, square integrable on bounded intervals, and $\mathbb{E}\int_0^\infty |\mathrm{d}\langle O^1, O^2\rangle_s| < \infty$, then*

$$(O^1 \star O^2)_t = O^1_t\, O^2_t + \mathbb{E}_t\Big[\int_t^\infty \mathrm{d}\langle O^1, O^2\rangle_s\Big].$$

Proof. $O^1\, O^2 - \langle O^1, O^2\rangle$ and $\mathbb{E}_t\langle O^1, O^2\rangle_\infty$ are martingales.

Brownian deltas at different targets

For $y \neq z$, $t < T$, $\bar{y} = \frac{y+z}{2}$ and $\kappa_s = (4\pi s)^{-1/2}$, the product is exact:

$$\Pi_t^y \Pi_t^z = \kappa_{T-t} \mathrm{e}^{-(y-z)^2/4(T-t)} p_{(T-t)/2}(B_t - \bar{y}).$$

Proof. Complete the square: $(B_t - y)^2 + (B_t - z)^2 = 2(B_t - \bar{y})^2 + \frac{(y-z)^2}{2}$, and use $(2\pi(T-t))^{-1} = \kappa_{T-t}(\pi(T-t))^{-1/2}$.

The product is $\geqslant 0$, so its expectation is its L^1 norm ($B_t \sim N(0,t)$):

$$\|\Pi_t^y \Pi_t^z\|_{L^1} = \kappa_{T-t} \mathrm{e}^{-(y-z)^2/4(T-t)} p_{(T+t)/2}(\bar{y}) \longrightarrow 0 \quad (t \uparrow T).$$

So $\Pi_t^y \Pi_t^z \sim 0$: the zero observable is the martingale in its class, hence $\Pi^y \star \Pi^z = 0$ for $y \neq z$. At $y = z$ the exponential suppression disappears, $\mathbb{E}(\Pi_t^y)^2 = \kappa_{T-t} p_{(T+t)/2}(y) \to \infty$, and no martingale is in the class: its constant expectation would have to be this divergent limit.

The product depends dramatically on whether the singular supports meet: separated supports multiply to zero, coincident ones call for renormalisation.

Brownian OPE: resolve the delta square

Fine resolution $u = T - \varepsilon$, coarse resolution $s < u$, $\kappa_\varepsilon = (4\pi\varepsilon)^{-1/2}$. Since $p_\varepsilon(w)^2 = \kappa_\varepsilon p_{\varepsilon/2}(w)$, Chapman–Kolmogorov gives the exact identity

$$\mathbb{E}_s\big[(\Pi^y_{T-\varepsilon})^2\big] = \kappa_\varepsilon\, p_{T-s-\varepsilon/2}(B_s - y).$$

Taylor expand in the time variable, with $\partial_\tau p_\tau = \frac{1}{2}\partial_y^2 p_\tau$:

$$\mathbb{E}_s\big[(\Pi^y_{T-\varepsilon})^2\big] = \kappa_\varepsilon\,\Pi^y_s - \frac{\varepsilon\,\kappa_\varepsilon}{4}\partial_y^2\Pi^y_s + \frac{\varepsilon^2\kappa_\varepsilon}{32}\partial_y^4\Pi^y_s + O_{L^1,s}(\varepsilon^{5/2}).$$

- Singular coefficients ($\kappa_\varepsilon \to \infty$; $\varepsilon\,\kappa_\varepsilon, \varepsilon^2\kappa_\varepsilon \to 0$) times observables $\Pi^y, \partial_y^2\Pi^y, \ldots$: an **operator product expansion**.
- The order of limits matters: fixed coarse resolution s, then $\varepsilon \downarrow 0$. Without conditioning, $\|(\Pi^y_{T-\varepsilon})^2 - \kappa_\varepsilon\,\Pi^y_{T-\varepsilon}\|_{L^1} \asymp \varepsilon^{-1/2}$.

Products of point evaluations

$\langle \mathbf{X}(x), \mathbf{X}(y) \rangle_t = Q_t(x-y)$ is deterministic. Heat profile, $d \geqslant 2$: the condition holds iff $x \neq y$, and then

$$(\mathbf{X}(x) \star \mathbf{X}(y))_t = X_t(x)\, X_t(y) + G(x-y) - Q_t(x-y), \qquad \langle \mathbf{X}(x) \star \mathbf{X}(y) \rangle = G(x-y).$$

The correct two-point function, **without ever discussing the regularity of** X_∞. For $x = y$: no product, since $c_t \to \infty$.

OPE. At every fixed resolution, as $y \to x$,

$$(\mathbf{X}(x) \star \mathbf{X}(y))_t - G(x-y) \longrightarrow X_t(x)^2 - c_t =: X_t^2 : (x),$$

i.e. $\mathbf{X}(x) \star \mathbf{X}(y) = G(x-y)\, 1 + : \mathbf{X}^2 : (x) + o(1)$: the Wick square is the regular part of the product.

The operator product expansion

For $x \neq y$,

$$:\mathbf{X}^n:(x) \star :\mathbf{X}^k:(y) = \sum_{j \leqslant n \wedge k} j! \binom{n}{j} \binom{k}{j} G(x-y)^j :\mathbf{X}^{n-j}(x)\mathbf{X}^{k-j}(y):.$$

- Wick's theorem at the level of observables.
- The **coefficients** $G(x-y)^j$ carry all singularities; the observables on the right have regular limits as $y \to x$.

Locality and factorisation

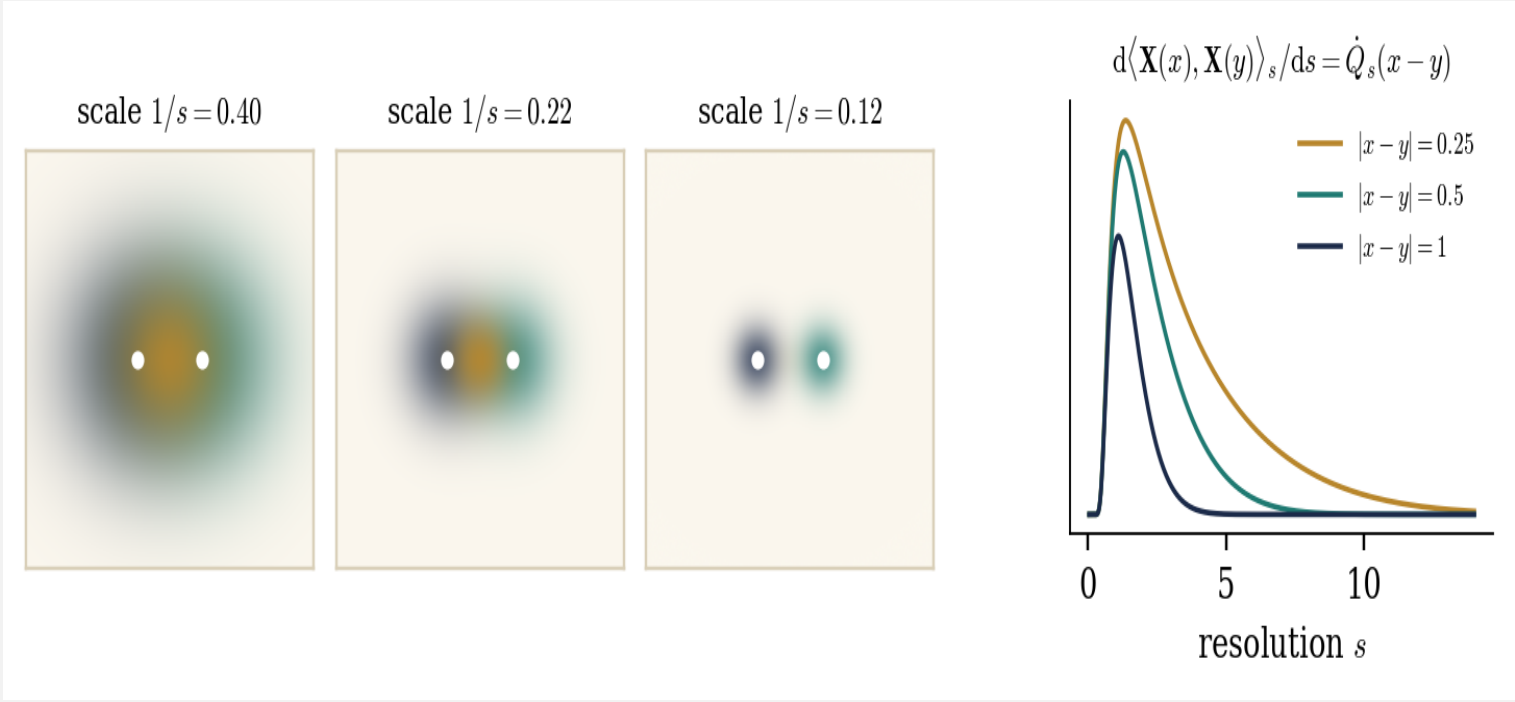
- O^1, O^2 with germs supported in $U^1_{1/t}$, $U^2_{1/t}$, $\operatorname{dist}(U^1, U^2) > 0$: the covariation $\langle D O^1_t, \dot{Q}_t \, D O^2_t \rangle \mathrm{d}t$ sees $\dot{Q}_t$ only between the supports.
- Finite range decomposition: it vanishes for large t, so the product exists and is asymptotically the pointwise product. Heat profile: exponentially small.
- Observables on **disjoint regions** multiply freely; products at the **same point** need renormalisation.

Factorisation algebra

This is the structure of a **factorisation algebra** of observables (precise axioms not established here).

Bauerschmidt–Brydges–Slade, LNM 2242 (finite range decompositions)

Locality: a picture



Left: kernels of $\dot{Q}_s^{1/2}$ at x and y; their overlap integrates to $\dot{Q}_s(x-y) = \mathrm{d}\langle \mathbf{X}(x), \mathbf{X}(y)\rangle_s / \mathrm{d}s$. Right: negligible for $s \gg 1/|x-y|$; total $G(x-y)$. As $y \to x$ the mass escapes to $s \to \infty$.

Summary

- **Stochastic quantisation**: the GFF is the stationary marginal of the parabolic Langevin dynamics and the one-point marginal of the elliptic SPDE.
- The GFF is the terminal value of a martingale X_t of smooth fields; **observables are martingales**, possibly without terminal value.
- Wick observables exist in every dimension; products and the OPE live at **finite resolution**.

Next

Lecture 4: interactions, the Polchinski equation and its FBSDE, the Wetterich equation.

Exercises and reading

Exercises (numbering of the notes)

- 4.1 Infinitesimal invariance characterises μ_m.
- 4.4 Slices of Gaussian fields.
- 5.1 The field at resolution t; 5.2 divergence of c_t.
- 5.3 Brownian observables; 5.4 Brownian deltas.
- 6.2 Products of Wick observables; 6.3 exponentials.
- 6.4 The Brownian OPE; 6.5 point splitting.

Reading

- Da Prato–Zabczyk, *Stochastic Equations in Infinite Dimensions*.
- Albeverio–De Vecchi–Gubinelli, *Elliptic stochastic quantization*, AoP 2020.
- Bailleul–Chevyrev–Gubinelli, *Stochastic observables and Wilson–Itô diffusions*, in prep.