A Constructive Introduction to Quantum Field Theory

Forward–backward equations

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Joint work with Sarah-Jean Meyer, with the use of AI

SISSA lectures, Lecture 4 — notes: Lecture 4, §§7–8

Where we are

So far. Stochastic quantisation of the GFF; the field at resolution t is a martingale of smooth fields; **observables** are martingales in the filtration of finite-resolution measurements.

Today. Interactions, still indexed by the **resolution** $t \in [0, T]$.

1. Generate the measure **three ways**: renormalisation, optimal control, filtering.
2. They meet in one system for a **field** and a **force**: the **force coordinates**.
3. Potentials: the value function generates them; Legendre; **Wetterich**.

Next time (Lecture 5): observables, and the **field equation** with its sub-Gaussian noise.

Caveat. Statements marked (formal) are exact in finite dimensions, with a terminal resolution $T < \infty$; the limits $T \to \infty$, $L \to \infty$ are the real problems. Bailleul–Chevyrev–G. (in prep.); Polchinski 1984; Bauerschmidt–Bodineau–Dagallier 2024

One measure, three constructions

Fix V_T and $\nu_T \propto \mathrm{e}^{-V_T}\mu^{Q_T}$. Let $S_t = Q_T - Q_t$, the covariance still unresolved at resolution t.

- **Renormalisation.** Split the covariance into shells and integrate them out. At every resolution the field is again a Gibbs measure, driven by an effective force.
- **Optimal control.** Steer the Gaussian field by a drift, at the least cost. The effective potential is the value function, and the Polchinski equation is the dynamic programming equation.
- **Filtering.** The field at resolution t is a noisy observation of the terminal field. The conditional law is the posterior, the force is what the observation reveals, and backward in resolution its dynamics is linear and Gaussian, started from ν_T.

They produce the same process.

Three ways to generate the interacting measure

Lecture 4

Effective potentials

Finite volume $\mathbb{T}_L^d$, terminal resolution $T < \infty$ (UV cutoff), interaction V_T bounded below, e.g.

$$V_T(\varphi) = \int_{\mathbb{T}_L^d} \left[\frac{\lambda}{12} H_4(\varphi; c_T) + \gamma_T H_2(\varphi; c_T) \right], \qquad \nu_T(\mathrm{d}\varphi) = Z_T^{-1} \mathrm{e}^{-V_T(\varphi)} \mu^{Q_T}(\mathrm{d}\varphi).$$

Definition. (Wilson effective action)

$$\mathrm{e}^{-V_t(\varphi)} := \int \mathrm{e}^{-V_T(\varphi+\zeta)} \, \mu^{Q_{T,t}}(\mathrm{d}\zeta), \qquad t \in [0, T].$$

The interaction felt at resolution t once the scales $[t, T]$ are integrated out; $V_0(0) = -\log Z_T$ is the free energy.

$$\mathrm{e}^{-V_t(X_t)} = \mathbb{E}_t^{\mathbb{P}_0}[\mathrm{e}^{-V_T(X_T)}]$$

Marginals and stochastic localisation

Proposition. (Marginals) $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{P}_0 = Z_T^{-1}\mathrm{e}^{-V_T(X_T)}$. *Then* $\mathrm{Law}_{\mathbb{P}}(X_t) = \nu_t \propto \mathrm{e}^{-V_t}\mu^{Q_t}$, *and* $\mathrm{Law}_{\mathbb{P}}(X_T|\mathscr{F}_t) = \mu_t^{X_t}$, *the tilt of* $\mu^{Q_{T,t}}(\cdot - X_t)$ *by* e^{-V_T}.

- At every resolution the field is again a **Gibbs measure**: reference Q_t, potential V_t.
- The measure-valued martingale $t \mapsto \mu_t^{X_t}$ goes from ν_T ($t=0$) to δ_{X_T} ($t=T$): a **stochastic localisation** of ν_T (Eldan), adapted to the covariance profile.

$$\mu_t^{X_t}(\mathrm{d}\varphi) = \frac{\mathrm{e}^{-V_T(\varphi)}}{Z}\mu^{Q_{T,t}}(\mathrm{d}\varphi - X_t)$$

Proof of the marginals. Under $\mathbb{P}_0$, $X_T = X_t + (X_T - X_t)$ with independent increment $\sim\mu^{Q_{T,t}}$, so

$$\mathbb{E}_{\mathbb{P}}[G(X_t)] = Z_T^{-1}\,\mathbb{E}_0[G(X_t)\mathrm{e}^{-V_T(X_T)}] = Z_T^{-1}\,\mathbb{E}_0[G(X_t)\mathrm{e}^{-V_t(X_t)}].$$

The Polchinski process

Theorem. (FBSDE for the interacting field) (formal) *Under* $\mathbb{P}$, *the* ***interacting field at resolution*** t, $\Phi_t := X_t$, *solves*

$$\mathrm{d}\Phi_t = \dot{Q}_t\, F_t \mathrm{d}t + \dot{Q}_t^{1/2}\mathrm{d}W_t^{\mathbb{P}}, \qquad F_t = -\mathrm{D}V_t(\Phi_t) = \mathbb{E}_t^{\mathbb{P}}\left[-\mathrm{D}V_T(\Phi_T)\right].$$

The ***effective force*** F *is a* $\mathbb{P}$*-martingale: an observable field of the interacting theory.*

Proof. Density process $M_t = Z_T^{-1}\mathrm{e}^{-V_t(X_t)}$. It is a martingale, so by Itô it has no drift, $\mathrm{d}M_t = -M_t\langle \mathrm{D}V_t(X_t), \dot{Q}_t^{1/2}\mathrm{d}W_t\rangle$. Girsanov: $W_t^{\mathbb{P}} = W_t + \int_0^t \dot{Q}_s^{1/2}\mathrm{D}V_s(X_s)\,\mathrm{d}s$. Finally $\mathrm{D}V_t(\phi) = \mu_t^{\phi}(\mathrm{D}V_T)$.

$$\mathrm{D}V_t(X_t) = -\frac{\mathbb{E}_t^{\mathbb{P}_0}[(-\mathrm{D}V_T(X_T))\mathrm{e}^{-V_T(X_T)}]}{\mathrm{e}^{-V_t(X_t)}} = \mathbb{E}_t^{\mathbb{P}}[-\mathrm{D}V_T(\Phi_T)]$$

A Doob h-transform

Under $\mathbb{P}_0$, X is the **GFF martingale**, with generator $L_t = \frac{1}{2}\mathrm{Tr}(\dot{Q}_t \mathrm{D}^2)$. The function

$$h_t(\phi) := Z_T^{-1} \mathrm{e}^{-V_t(\phi)} = \mathbb{E}_0[Z_T^{-1} \mathrm{e}^{-V_T(X_T)} | X_t = \phi], \qquad (\partial_t + L_t)\, h_t = 0,$$

is space-time harmonic: this is the **Polchinski equation**. Since $\mathrm{d}\mathbb{P} / \mathrm{d}\mathbb{P}_0|_{\mathscr{F}_t} = h_t(X_t)$,

$$L_t^h G = h_t^{-1} (\partial_t + L_t)(h_t G) - \partial_t G = L_t G + \langle \dot{Q}_t \mathrm{D}\log h_t, \mathrm{D}G \rangle, \qquad F_t = \mathrm{D}\log h_t(\Phi_t).$$

Simplest case: a Brownian motion conditioned on its endpoint, $h_t =$ Gaussian density of the remaining increment, and the drift of the bridge.

The Polchinski process is the **Doob h-transform** of the GFF martingale by e^{-V_T}: the free field conditioned to end with the law ν_T.

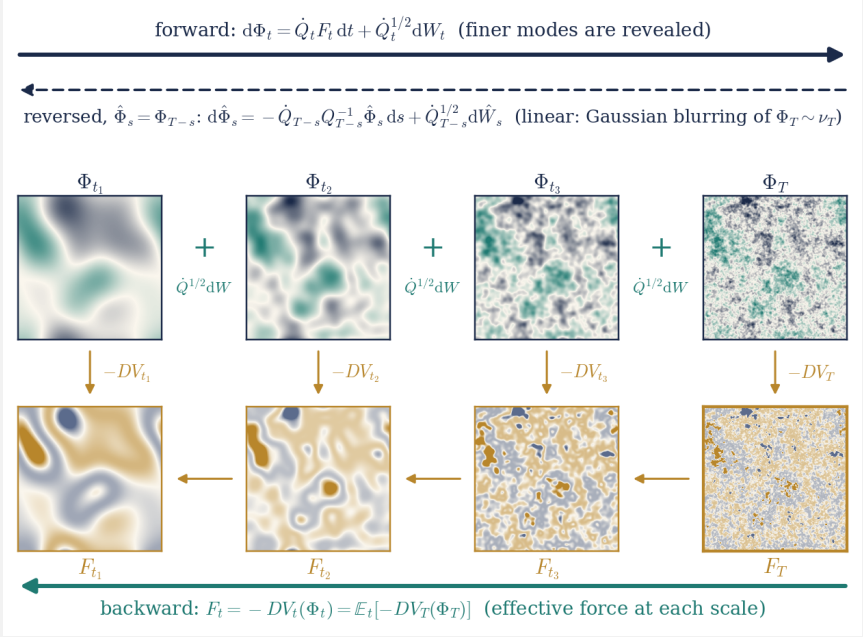
A forward–backward system

Read as an equation for an unknown pair (Φ, F):

$$\Phi_t = \int_0^t \dot{Q}_s F_s \, \mathrm{d}s + \int_0^t \dot{Q}_s^{1/2} \mathrm{d}W_s, \qquad F_t = \mathbb{E}_t\left[-\mathrm{D}V_T(\Phi_T)\right], \qquad t \in [0,T].$$

- The field is driven **forward** by a force determined **backward** from the terminal field.
- No Polchinski equation needed; $\nu_T = \mathrm{Law}(\Phi_T)$ from the solution.
- **Strategy**: solve with renormalised V_T, estimates uniform in T and volume, take limits.
- Φ is **not** an observable (a semimartingale); observables are built from the noise and from F.

Forward field, backward force



Vertical: $F_t = -DV_t(\Phi_t)$. Dashed: backward the process is **linear** (Gaussian blurring). Force to first order, $-(\lambda/3)\,H_3(\Phi_t; c_t)$.

The Polchinski equation

Theorem. (Polchinski 1984) (formal)

$$\partial_t V_t = \tfrac{1}{2}\langle \mathrm{D}V_t, \dot{Q}_t\, \mathrm{D}V_t\rangle - \tfrac{1}{2}\mathrm{Tr}(\dot{Q}_t\, \mathrm{D}^2 V_t), \qquad V_T \text{ given.}$$

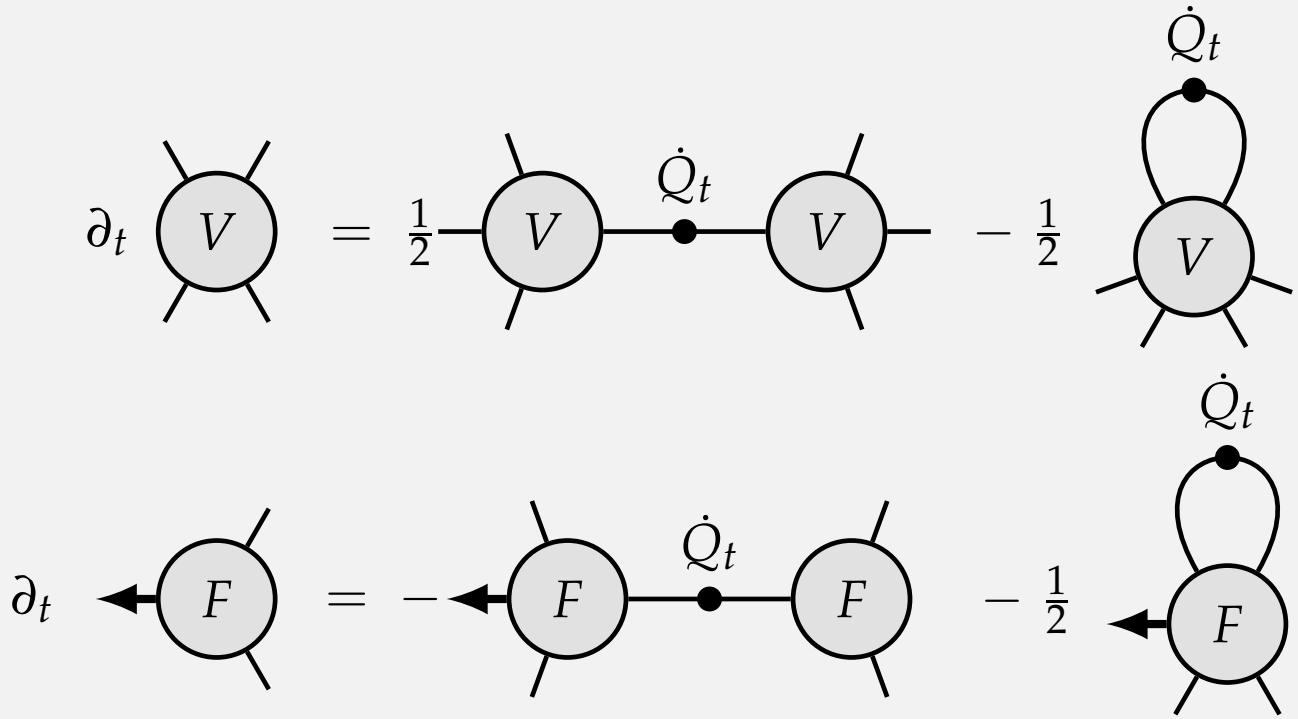
Proof. $u_t = \mathrm{e}^{-V_t} = \mathrm{e}^{\frac{1}{2}\Delta_{Q_T,t}} u_T$, $\Delta_Q = \mathrm{Tr}(Q\,\mathrm{D}^2)$, solves the backward heat equation $\partial_t u_t = -\frac{1}{2}\mathrm{Tr}(\dot{Q}_t \mathrm{D}^2 u_t)$. Insert $\mathrm{D}^2 u = (\mathrm{D}V \otimes \mathrm{D}V - \mathrm{D}^2 V)\, u$.

Nonlinear term: generates new, nonlocal interactions (**trees**).

Trace term: contracts pairs of fields with the shell covariance $\dot{Q}_t\,\mathrm{d}t$: **loops**.

A backward Hamilton–Jacobi equation; $h \propto \mathrm{e}^{-V}$ is a Cole–Hopf transform.

The Polchinski equation in graphs



Blob: derivatives of V_t or of the force. Dotted line: $\dot{Q}_t$. Tree term and loop term; iterating gives the connected Feynman graphs.

The Boué–Dupuis formula

$\mathbb{H}_a$: adapted drifts with $\mathbb{E}\int_0^T \|u_s\|^2 < \infty$; $X^u = X + \int_0^\cdot \dot{Q}_s^{1/2} u_s \, \mathrm{d}s$.

Theorem. (Polchinski = HJB) (formal)

$$-\log \mathbb{E}_0[\mathrm{e}^{-V_T(X_T)}] = V_0(0) = \inf_{u \in \mathbb{H}_a} \mathbb{E}_0\left[V_T(X_T^u) + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s \right].$$

Optimal feedback $u_s^* = -\dot{Q}_s^{1/2} \mathrm{D}V_s(X_s^{u^*})$, $\mathrm{Law}(X_T^{u^*}) = \nu_T$. *$V_t$ is the* ***value function****; Polchinski is the* ***HJB equation****.*

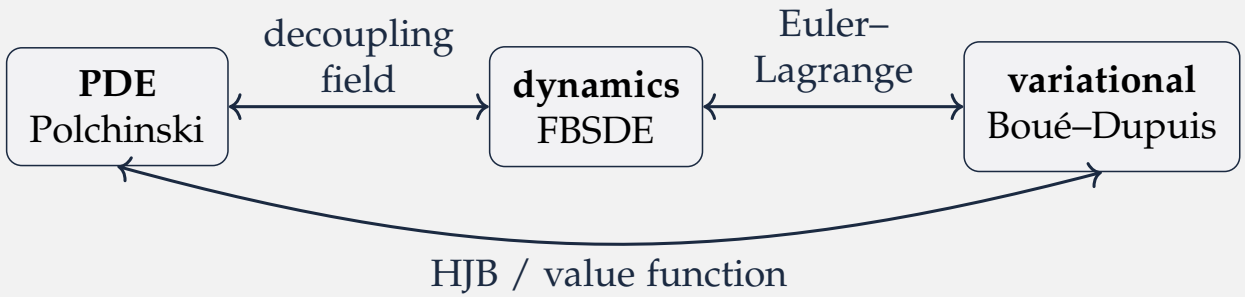
Proof (verification). Itô + Polchinski:

$$\mathbb{E}_0\left[V_T(X_T^u) + \tfrac{1}{2}\int_0^T \|u_t\|^2 \mathrm{d}t \right] = V_0(0) + \tfrac{1}{2}\mathbb{E}_0 \int_0^T \left\| u_t + \dot{Q}_t^{1/2} \mathrm{D}V_t(X_t^u) \right\|^2 \mathrm{d}t.$$

The control problem: PDE, dynamics, variational

Euler–Lagrange. Perturb $u \to u + \varepsilon v$:

$$u_s = -\dot{Q}_s^{1/2}\, \mathbb{E}_s\left[\mathrm{D}V_T(X_T^u)\right] = \dot{Q}_s^{1/2} F_s: \quad \text{the FBSDE.}$$



Entropy. $\frac{1}{2}\mathbb{E}_0 \int \|u\|^2 \geqslant \mathsf{Ent}(\mathsf{Law}(X^u)|\mathsf{Law}(X))$: a pathwise lift of the Gibbs principle

$$-\log Z_T = \inf_{\nu}\left\{\int V_T \,\mathrm{d}\nu + \mathsf{Ent}(\nu|\mu^{Q_T})\right\}.$$

Useful because adapted drifts can be **built scale by scale**, as functionals of X.

Renormalisation in the variational picture

Generating functional:

$$\mathscr{W}_T(f) = -\log \int \mathrm{e}^{-f}\,\mathrm{d}\nu_T = \inf_u \mathbb{E}_0\Big[(f+V_T)(X_T^u) + \tfrac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s\Big] - V_0(0).$$

Bounds uniform in $T \Rightarrow$ tightness, moments, identification of limits.

Φ^4_3. The leading drift $u_s \approx -\frac{\lambda}{3}\dot{Q}_s^{1/2}{:}X_s^3{:}$ has **divergent cost**,

$$\tfrac{1}{2}\mathbb{E}\int_0^T \|\tfrac{\lambda}{3}\dot{Q}_s^{1/2}{:}X_s^3{:}\|^2 \mathrm{d}s = \frac{\lambda^2}{12} L^3 \int Q_T(z)^4\,\mathrm{d}z \sim T.$$

Change variables $u = -\frac{\lambda}{3}\dot{Q}^{1/2}{:}X^3{:} + r$, then paracontrolled changes of variables: the singular terms become explicit polynomials in Wick powers of X, **cancelled by the counterterms**, and the remainder is coercive. Uniform control in Besov/paracontrolled norms = construction of Φ^4_3. Barashkov–Gubinelli, Duke 2020; EJP 2021

Pontryagin's maximum principle

Minimise

$$J(u) = g(x_T) + \int_t^T \ell(x_s, u_s)\,\mathrm{d}s$$

subject to $\dot{x} = b(x,u)$, $x_t = x$. **Hamiltonian:**

$$\mathscr{H}(x,p,u) = \langle p, b(x,u)\rangle - \ell(x,u).$$

Theorem. (Pontryagin) *Along an optimal pair there is a* ***costate*** p *with*

$$\dot{x} = \partial_p \mathscr{H}, \qquad \dot{p} = -\partial_x \mathscr{H}, \qquad p_T = -Dg(x_T), \qquad u \in \arg\max \mathscr{H}(x,p,\cdot).$$

- State forward, costate backward: a **two-point boundary problem**.
- The costate is the Lagrange multiplier of the constraint $\dot{x} = b$; an optimal control is a critical point of the Hamiltonian.
- Next: why this holds, and what p is.

Pontryagin–Boltyanskii–Gamkrelidze–Mishchenko 1962.

Pontryagin: first variation and HJB

First variation. Perturb $u \to u + \varepsilon v$; the state varies by $\xi = \partial_\varepsilon x$,

$$\dot{\xi} = \partial_x b\, \xi + \partial_u b\, v, \qquad \xi_t = 0.$$

The costate equation $\dot{p} = -\partial_x \mathcal{H}$ is exactly what makes

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle p, \xi \rangle = \langle \partial_x \ell, \xi \rangle + \langle p, \partial_u b\, v \rangle.$$

Integrating, with $p_T = -Dg(x_T)$:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon v)|_{\varepsilon=0} = -\int_t^T \langle \partial_u \mathcal{H}(x_s, p_s, u_s), v_s \rangle \, \mathrm{d}s.$$

HJB. Dynamic programming over $[t, t + \mathrm{d}t]$:

$$V(t,x) = \min_u \left[\ell \, \mathrm{d}t + V(t + \mathrm{d}t, x + b\, \mathrm{d}t) \right] \quad \Longrightarrow \quad \partial_t V = \mathcal{H}_*(x, -\mathrm{D}V), \quad \mathcal{H}_* = \max_u \mathcal{H}.$$

Hamilton's equations for $\mathcal{H}_*$ are its characteristics, and along them $p_s = -\mathrm{D}V(s, x_s)$: the costate is the force the future exerts on the present.

The Hamiltonian of the field

State = field, $b = \dot{Q}_s^{1/2} u$, $\ell = \frac{1}{2}\|u\|^2$, $g = V_T$. $\mathscr{H}$ is maximised at $u = \dot{Q}_s^{1/2} p$:

$$\mathscr{H}_*(p) = \frac{1}{2}\langle p, \dot{Q}_s p\rangle:$$

a free particle with mass matrix $\dot{Q}_s^{-1}$, and no dependence on the state x.

- **Without noise:** $\dot{p} = 0$, $\dot{x} = \dot{Q} p$: the **shear** Σ_t: $(x,p) \mapsto (x + S_t p, p)$, with $S_t := Q_T - Q_t$ the covariance still unresolved. All interaction is in $p = -\mathrm{D}V_T(x + S_T p)$: **trees**. HJB = Polchinski without the trace.
- **With noise** (Bismut 1973, Peng 1990): the costate solves a BSDE

$$\mathrm{d}p_s = -\partial_x \mathscr{H}\,\mathrm{d}s + \Upsilon_s \mathrm{d}W_s,$$

  with $\Upsilon\,\mathrm{d}W$ the price of adaptedness. Here $\partial_x \mathscr{H} = 0$: the costate is a **martingale**, $p_t = -\mathrm{D}V_t(X_t) = F_t$.
- The Itô term turns HJB into **Polchinski**: quadratic part = trees, trace term = loops.

The FBSDE is the stochastic Hamiltonian system: field = state, force = costate.

State and costate

Here $\dot{S}_t = -\dot{Q}_t$, $S_T = 0$. Optimal control $u_t = \dot{Q}_t^{1/2} p_t$, with costate $p_t = F_t$:

$$\mathrm{d}F_t = \Upsilon_t \, \mathrm{d}W_t, \qquad \Upsilon_t = \mathrm{D}F_t(\Phi_t)\, \dot{Q}_t^{1/2}.$$

- $F_t(\cdot)$ is the **decoupling field**, $F_t = F_t(\Phi_t)$; Burgers is its quasilinear PDE, and Υ the martingale representation integrand of the costate.
- In Burgers: transport term $-\mathrm{D}F\,[\dot{Q}\,F] \leftarrow$ drift of the state; Itô term $-\frac{1}{2}\mathrm{Tr}(\dot{Q}\,\mathrm{D}^2 F) \leftarrow$ noise.
- The shear Σ_t is the flow of the transport part: its characteristics.

Filtering: the field observes its future

All covariance operators are functions of A; $K = Q_T^{-1}$, $J_t = KQ_t$. Since $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{P}_0$ depends on X_T only, given Φ_T the interacting field is the **Gaussian bridge**:

$$\Phi_t = J_t\Phi_T + \zeta_t, \qquad \mathrm{d}\Phi_t = \dot{Q}_t S_t^{-1}(\Phi_T - \Phi_t)\,\mathrm{d}t + \dot{Q}_t^{1/2}\,\mathrm{d}B_t,$$

with ζ, B independent of Φ_T. **Signal** Φ_T, with prior ν_T; **observation** $(\Phi_s)_{s\leqslant t}$, pulled towards the hidden field.

Proposition. (Filtering) (formal)

- *Posterior:* $\mu_t^{\Phi_t}(\mathrm{d}y) \propto \mathrm{e}^{\langle S_t^{-1}\Phi_t, y\rangle - \frac{1}{2}\langle y, \mathscr{R}_t y\rangle}\nu_T(\mathrm{d}y)$, $\mathscr{R}_t = S_t^{-1} - Q_T^{-1}$ *the precision gained: the* ***regulator****.*
- $F_t = \mathbb{E}_t[S_t^{-1}(\Phi_T - \Phi_t)]$*:* $\dot{Q}_t F_t$ *is the posterior mean of the pull,* $\psi_t = \Phi_t + S_t F_t$*, and* W *is the* ***innovation****. The Polchinski SDE is the observation equation in its own filtration.*
- *Filter equation:* $\mathrm{d}\mathbb{E}_t[G(\Phi_T)] = \langle \mathrm{Cov}_t(G, \Phi_T), S_t^{-1}\dot{Q}_t^{1/2}\mathrm{d}W_t\rangle$.

Backward in resolution: Gaussian blurring

Reverse the resolution, $\hat{\Phi}_s = \Phi_{T-s}$:

$$\mathrm{d}\hat{\Phi}_s = -\dot{Q}_{T-s} Q_{T-s}^{-1} \hat{\Phi}_s \, \mathrm{d}s + \dot{Q}_{T-s}^{1/2} \, \mathrm{d}\hat{W}_s, \qquad \hat{\Phi}_0 = \Phi_T \sim \nu_T.$$

- $\mathrm{d}\mathbb{P} / \mathrm{d}\mathbb{P}_0$ depends only on the *starting point* X_T: the transition kernels are those of the Gaussian reference, only the initial law changes. A linear process with Gaussian transitions, not a Gaussian process.
- **Backward**: blur the interacting field with Gaussian noise, $\Phi_t = J_t \Phi_T + \zeta_t$; this does not know the interaction. **Forward**: Polchinski undoes the blurring, all the interaction is in the drift.
- Time reversal ($t > 0$, finite cutoff): forward drift $b_t = \dot{Q}_t F_t$, backward drift $\beta_t = -\dot{Q}_t Q_t^{-1} x$, and $b_t + \beta_t = \dot{Q}_t \mathrm{D}\log p_t$: reversal **compensates the drift up to the score** of the density at that scale (Nelson's osmotic velocity). So F is the score of ν_t relative to μ^{Q_t}, ψ_t is the denoiser (Tweedie), and the noise is $\omega_t = -J_t \mathrm{D}\log p_t(\Phi_t)$.

The Polchinski process is a **denoising diffusion** whose data distribution is the interacting measure.

The measurement coordinate

In the stochastic localisation literature the observation is written as a linear measurement: with $j_t = S_t^{-1}\Phi_t$,

$$j_t = \mathscr{R}_t \Phi_T + N_t, \quad N_t = S_t^{-1}\zeta_t, \quad \mathbb{E}[N_s \otimes N_t] = \mathscr{R}_{s\wedge t},$$

signal-to-noise ratio $\mathscr{R}_t$. The posterior is an exponential family in j, with log-partition function $\mathscr{K}_t(j) = \log \int \mathrm{e}^{\langle j,y\rangle - \frac{1}{2}\langle y, \mathscr{R}_t y\rangle} \nu_T(\mathrm{d}y)$:

- $\psi_t = \mathrm{D}\mathscr{K}_t(j_t)$, $\mathrm{Cov}_t = \mathrm{D}^2\mathscr{K}_t(j_t)$, $\mathrm{d}j_t = \dot{\mathscr{R}}_t \psi_t \,\mathrm{d}t + \dot{\mathscr{R}}_t^{1/2}\,\mathrm{d}W_t$;
- $\partial_t \mathscr{K} + \frac{1}{2}\mathrm{Tr}(\dot{\mathscr{R}}\mathrm{D}^2\mathscr{K}) + \frac{1}{2}\langle \mathrm{D}\mathscr{K}, \dot{\mathscr{R}}\mathrm{D}\mathscr{K}\rangle = 0$: Polchinski in the variable j.

Eldan's basic scheme is the profile with $\mathscr{R}_t = t$, N a Brownian motion, run to $t = \infty$: the profile of Lectures 6–7.

Eldan 2013; El Alaoui–Montanari 2022; Bauerschmidt–Bodineau–Dagallier 2024

Where the three views meet

	Renormalisation	Control	Filtering
Φ_t	field at resolution t	state	observation; posterior mean, Gaussian prior
F_t	effective force	costate	posterior mean of the terminal force
W	fresh fluctuation of the shell	noise of the controlled field	innovation
S_t	unresolved covariance	free motion from t to T	posterior covariance, Gaussian prior
$\psi_t = \Phi_t + S_t F_t$	mean of the conditional law	terminal state, free prediction	posterior mean
Cov_t	covariance of the conditional law	$S_t - S_t \mathrm{D}^2 V_t S_t$	posterior covariance
V_t	effective potential	value function	$-$log-likelihood ratio of the observation, up to $\log Z_T$

$\psi_t - \Phi_t = S_t F_t$: **the force is what the interaction adds to the estimate**, in units of the free propagator S_t, and Cov_t replaces S_t as the propagator.

Closed versus local

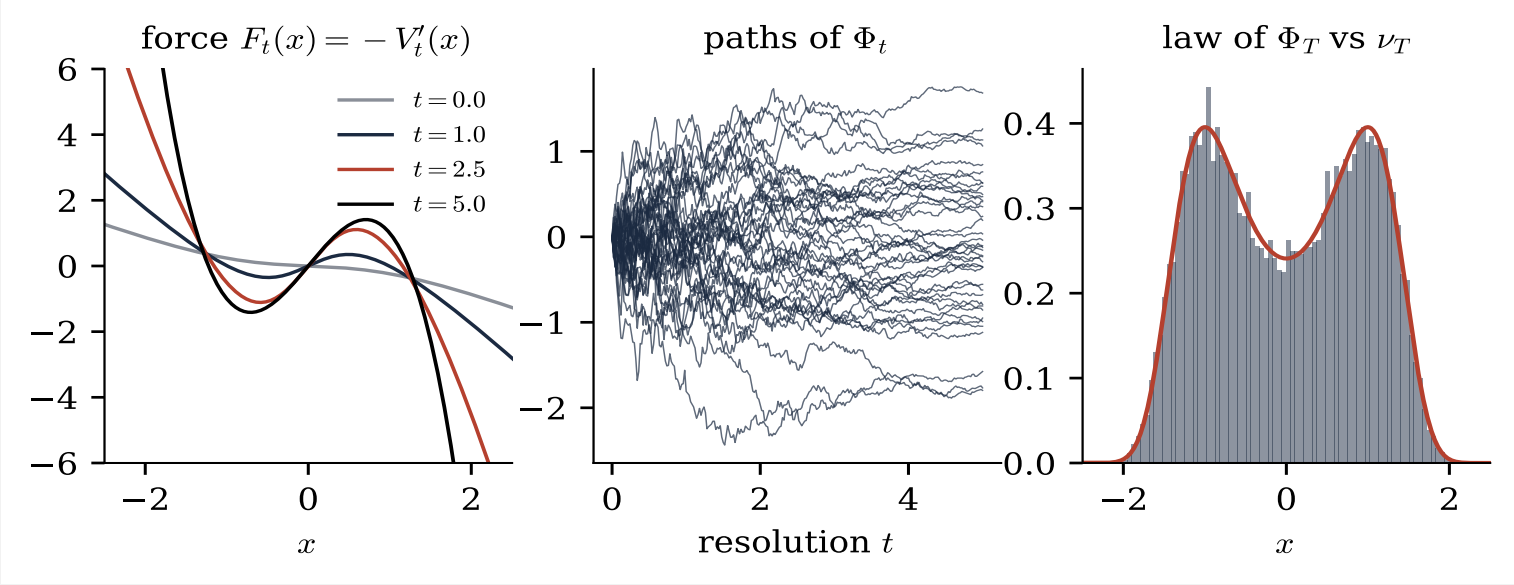
At finite T the force is a **closed** observable, $F_t = \mathbb{E}_t[F_T]$, and for Φ^4 it is **local**:

$$F_T(x) = -\frac{\lambda}{3} H_3(\Phi_T(x); c_T) - 2\gamma_T H_1(\Phi_T(x); c_T).$$

As $T \to \infty$

F should converge **as an observable**. In $d = 2$ it is closed, $F_\infty = -\frac{\lambda}{3} {:}\Phi^3_\infty{:}$. In $d = 3$ a closed limit $F_t = \mathbb{E}_t F_\infty$ would let the DS identity pass to $t \to \infty$, and conditioning on $\sigma(\Phi_\infty)$ would give an integrable logarithmic derivative of Φ^4_3 as a function of the field: not expected (Albeverio–Liang–Zegarlinski 2006: the regularised ones do not converge in L^2). Locality of the limiting force is the crux of the construction.

Zero dimensions



$Q_t = 1 - \mathrm{e}^{-t}$, $T = 5$, $V_T(x) = \frac{1}{2}x^4 - \frac{3}{2}x^2$. Left: $F_t = -V_t'$, almost linear at low resolution, double well as $t \to T$. Centre: FBSDE paths split between the wells (stochastic localisation). Right: histogram of Φ_T vs. the density of ν_T.

Force coordinates and potentials

Lecture 4

The force: a Burgers equation

Differentiating Polchinski, $F_t = -\mathrm{D}V_t$ solves the backward viscous **Burgers equation**

$$\partial_t F_t = -\mathrm{D}F_t[\dot{Q}_t F_t] - \tfrac{1}{2}\mathrm{Tr}(\dot{Q}_t \mathrm{D}^2 F_t), \qquad F_T = -\mathrm{D}V_T.$$

- Cole–Hopf $F = \mathrm{D}\log h$ linearises it ($h \propto \mathrm{e}^{-V}$ solves a heat equation).
- Uses only the force: **intensive**, while V_t is extensive ($\sim$ volume). Matters in infinite volume.
- Flows for different covariances commute when $\mathrm{D}F$ is symmetric, i.e. F a gradient. Non-gradient forces (general Wilson–Itô diffusions): curvature.

Removing the predictable drift: the forecast

Proposition. (formal) *The conditional mean of the terminal field is*

$$\psi_t := \mathbb{E}_t[\Phi_T] = \Phi_t + S_t F_t, \qquad (\psi_t, F_t) = \Sigma_t(\Phi_t, F_t),$$

a martingale with

$$\mathrm{d}\psi_t = \dot{Q}_t^{1/2}\mathrm{d}W_t + S_t\,\mathrm{d}F_t = (1 + S_t\,\mathrm{D}F_t(\Phi_t))\,\dot{Q}_t^{1/2}\mathrm{d}W_t.$$

Proof. F martingale $\Rightarrow \mathbb{E}_t \int_t^T \dot{Q}_s F_s\,\mathrm{d}s = S_t F_t$. In the differential, $\dot{Q}_t F_t\,\mathrm{d}t$ cancels with $\dot{S}_t F_t\,\mathrm{d}t$.

- $\Phi_t = \psi_t - S_t F_t$: a martingale (the **field observable**) minus the drift still expected on the unresolved scales.
- Noise of ψ: fresh shell fluctuation + **revision of the forecast**. The forward–backward coupling moves from the drift to the volatility.

ψ is the **interaction picture** w.r.t. the classical dynamics.

The FBSDE in forecast coordinates

Invert $x \mapsto \psi = x + S_t F_t(x)$: $x = \psi + S_t \mathscr{F}_t(\psi)$, $\mathscr{F}_t(\psi) := -F_t(x)$. Response and regulator:

$$H_t(\psi) = (S_t^{-1} + \mathrm{D}\mathscr{F}_t(\psi))^{-1}, \qquad \mathscr{R}_t = S_t^{-1} - Q_T^{-1}, \qquad \dot{\mathscr{R}}_t = S_t^{-1}\dot{Q}_t S_t^{-1} \geqslant 0.$$

Theorem. (formal)

$$\mathrm{d}\psi_t = H_t(\psi_t)\; S_t^{-1}\dot{Q}_t^{1/2}\,\mathrm{d}W_t, \qquad \mathscr{F}_t(\psi_t) = \mathbb{E}_t\left[\mathrm{D}V_T(\psi_T)\right],$$

$\psi_T = \Phi_T$. ***No drift***; *covariance* $\mathscr{C}_t = H_t \dot{\mathscr{R}}_t H_t^*$. *The decoupling field solves the* ***pure diffusion*** *equation*

$$\partial_t \mathscr{F}_t + \tfrac{1}{2}\mathrm{Tr}(\mathscr{C}_t\,\mathrm{D}^2\mathscr{F}_t) = 0, \qquad \mathscr{F}_T = \mathrm{D}V_T.$$

Proof. $1 + S_t \mathrm{D}F_t = H_t S_t^{-1}$; $\mathscr{F}_t(\psi_t)$ is a martingale and ψ has no drift: Itô.

What the change of coordinates did

Compare with Burgers:

$$\underbrace{\partial_t F = -\mathrm{D}F\,[\dot{Q}F] - \tfrac{1}{2}\mathrm{Tr}(\dot{Q}\,\mathrm{D}^2 F)}_{\text{field coordinate}} \quad \longrightarrow \quad \underbrace{\partial_t \mathscr{F} = -\tfrac{1}{2}\mathrm{Tr}(\mathscr{C}\ \mathrm{D}^2\,\mathscr{F})}_{\text{forecast coordinate}}$$

- The transport term is **absorbed exactly**: dropping the Itô term, $\mathscr{F}_t = \mathrm{D}V_T$ does not move. All **trees** live in the coordinate change $x = \psi + S_t\,\mathscr{F}_t(\psi)$.
- **Loops** come only from the quadratic variation $\mathscr{C}_t$ of the forecast; the nonlinearity has moved into its field dependence.
- No gradient structure, no Gaussian convolution used: valid for any FBSDE with a Markov martingale force, whenever the coordinate change is invertible.

Burgers in one dimension: Lagrangian coordinates

$\partial_s u + u\,\partial_x u = 0$, $u(0,\cdot) = g'$. Along the **characteristics** $x = y + s\,g'(y)$, u is constant: in the Lagrangian label y nothing moves. With $u = \partial_x G$, the **Lax–Oleinik formula**

$$G(s,x) = \min_y \left\{ g(y) + \frac{(x-y)^2}{2s} \right\}, \qquad u(s,x) = \frac{x - y(s,x)}{s},$$

the minimiser being the label of x; characteristics cross (shocks) once $1 + s\,g''$ vanishes somewhere. Viscous case: Cole–Hopf. Dictionary:

time s	S_t	characteristics	shear $x = \psi + S_t \mathrm{D}V_T(\psi)$
Eulerian x	field x	Lax–Oleinik	tree level (Hopf–Lax)
label y	forecast ψ	Cole–Hopf, $\varepsilon = \frac{1}{2}$	Polchinski
$u = \partial_x G$	$-F_t = \mathrm{D}V_t$	datum g	V_T

With noise the label is the **posterior mean** of y under the Cole–Hopf measure, a martingale; the Moreau formula is Lax–Oleinik with $V_T \to \Gamma_t$, and at finite cutoff no shocks form.

Conditional moments

Proposition. (formal) *If* $F_t = -\mathrm{D}V_t$, *then*

$$H_t(\psi_t) = \mathsf{Cov}_t := \mathsf{Cov}^{\mathbb{P}}(\Phi_T \mid \Phi_t) = S_t - S_t\,\mathrm{D}^2 V_t(\Phi_t)\,S_t.$$

So $H_t > 0$, $1 + S_t \mathrm{D}F_t$ *is always invertible, and* $H_t \leqslant S_t$ *for convex* V_T *(Brascamp–Lieb).*

The forecast coordinates are the first two conditional moments of the stochastic localisation: $(\psi_t, \mathsf{Cov}_t) = (\mathbb{E}[\Phi_T \mid \Phi_t], \mathsf{Cov}(\Phi_T \mid \Phi_t))$. The second moment

$$\mathbb{E}_t[\Phi_T \otimes \Phi_T] = \psi_t \otimes \psi_t + \mathsf{Cov}_t$$

is a martingale, hence

$$\mathrm{d}\langle \psi, \psi \rangle_t = \mathsf{Cov}_t \dot{\mathscr{R}}_t \mathsf{Cov}_t\,\mathrm{d}t, \qquad \mathrm{d}\mathsf{Cov}_t = -\mathsf{Cov}_t \dot{\mathscr{R}}_t \mathsf{Cov}_t\,\mathrm{d}t + \mathrm{d}(\text{mart.}).$$

Gaussian reference: $S_t\,\dot{\mathscr{R}}_t\,S_t = \dot{Q}_t$.

Brascamp–Lieb bounds on the conditional covariance

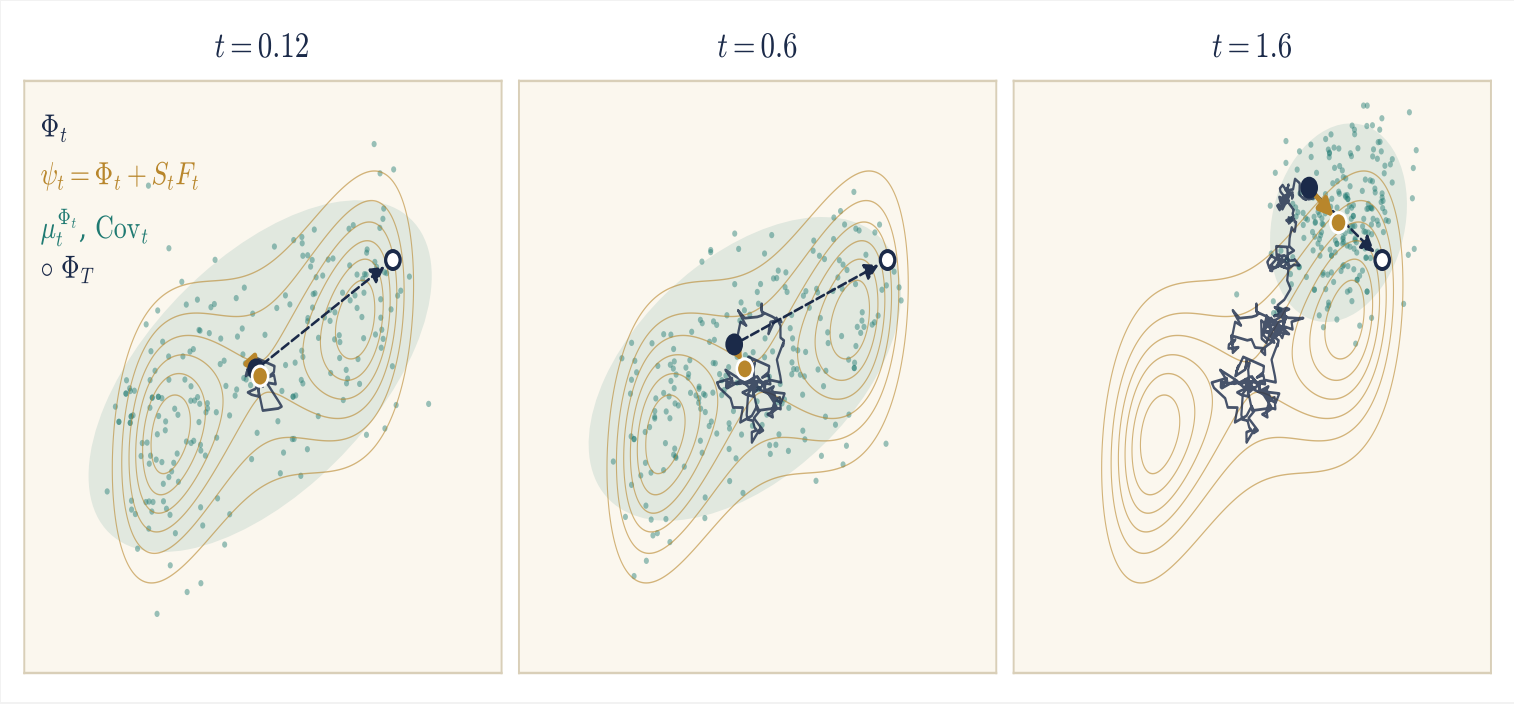
Conditional law $\mu_t^\phi \propto \mathrm{e}^{-\mathscr{U}}$, $\mathrm{D}^2\mathscr{U} = S_t^{-1} + \mathrm{D}^2 V_T$. Then

$$(\mathbb{E}_t[S_t^{-1} + \mathrm{D}^2 V_T])^{-1} \leqslant \mathrm{Cov}_t \leqslant \mathbb{E}_t[(S_t^{-1} + \mathrm{D}^2 V_T)^{-1}].$$

- **Lower:** Cramér–Rao, no convexity needed: $\mathbb{E}[(y-m)\otimes \mathrm{D}\mathscr{U}] = 1$ and Cauchy–Schwarz.
- **Upper:** Brascamp–Lieb (1976), for $\mathrm{D}^2\mathscr{U} > 0$: $\mathrm{Var}(f) \leqslant \mathbb{E}\langle \mathrm{D}f, (\mathrm{D}^2\mathscr{U})^{-1}\mathrm{D}f\rangle$.
- Equal for quadratic V_T. Via $\mathrm{Cov}_t = S_t - S_t \mathrm{D}^2 V_t S_t$: two-sided bounds on $\mathrm{D}^2 V_t$, i.e. on the Wetterich propagator H_t.
- V_T convex $\Rightarrow \mathrm{Cov}_t \leqslant S_t$, $\mathrm{D}^2 V_t \geqslant 0$: convexity propagates along Polchinski. $\mathrm{D}^2 V_T \geqslant \kappa \Rightarrow \mathrm{Cov}_t \leqslant (S_t^{-1} + \kappa)^{-1}$.

Φ_3^4: Wick ordering gives V_T the curvature $-\lambda c_T + 2\gamma_T \to -\infty$ at $\varphi = 0$ (already for Φ_2^4). The upper bound is lost, the conditional law may spread over several wells: the **large-field problem** (clipping and global coercivity, Lecture 6).

Stochastic localisation: a picture



Two-component field, V_T a tilted double well (contours: ν_T). Path Φ_t, forecast $\psi_t = \Phi_t + S_t F_t$, conditional law $\mu_t^{\Phi_t}$ and 2σ ellipse of Cov_t; open circle: Φ_T. The conditional law localises on one well.

Phase space and Lagrangian graphs

Phase space of pairs (field, force) $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, with the symplectic form

$$\Omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}p_i.$$

- L (dimension n) is **Lagrangian** if $\Omega|_L = 0$.
- A graph $p = F(x)$ is Lagrangian $\Leftrightarrow$ $\mathrm{D}F$ symmetric $\Leftrightarrow$ F is a gradient. **Gradient sector = Lagrangian graphs.**
- Symplectic maps send Lagrangian submanifolds to Lagrangian ones. The shear $\Sigma_t(x,p) = (x + S_t p, p)$ is symplectic since S_t is symmetric: $\Sigma_t^* \Omega = \Omega + \sum S_{ij}\, \mathrm{d}p_j \wedge \mathrm{d}p_i = \Omega$.

So Σ_t maps the graph of $F_t = -\mathrm{D}V_t$ to a Lagrangian submanifold; if it is a graph over the forecast ψ, it is the graph of a gradient.

The free action generates the shear

The cheapest way to reach ψ at T from x at t, without interaction, costs the **free action**

$$\mathscr{A}_t(x,\psi) = \frac{1}{2}\langle \psi - x, S_t^{-1}(\psi - x)\rangle, \quad p = S_t^{-1}(\psi - x) \text{ constant.}$$

- $\mathrm{D}_x \mathscr{A}_t = -p$, $\mathrm{D}_\psi \mathscr{A}_t = p$: so $\mathscr{A}_t$ is the **generating function** of the shear Σ_t.
- $\partial_t \mathscr{A}_t = \frac{1}{2}\langle \mathrm{D}_x \mathscr{A}_t, \dot{Q}_t \mathrm{D}_x \mathscr{A}_t\rangle$: a complete integral of the noiseless Polchinski equation.
- Jacobi: in the new coordinates the Hamiltonian vanishes, and ψ, p are constants of motion.

(ψ_t, F_t) are the Jacobi coordinates of the free Hamiltonian of the field. With noise, the constants of motion become **martingales**.

The gradient sector

$F_t = -\mathrm{D}V_t$. The map $x \mapsto x + S_t F_t(x)$ is S_t times the gradient of

$$\frac{1}{2}\langle x, S_t^{-1} x\rangle - V_t(x), \qquad \text{with Hessian } S_t^{-1}\mathrm{Cov}_t S_t^{-1} > 0,$$

so it is injective: the image of the graph of F_t is a graph over ψ, that of $-\mathscr{F}_t$ with

$$\mathscr{F}_t = \mathrm{D}\Gamma_t, \qquad H_t = (S_t^{-1} + \mathrm{D}^2\Gamma_t)^{-1}, \qquad \Gamma_T = V_T.$$

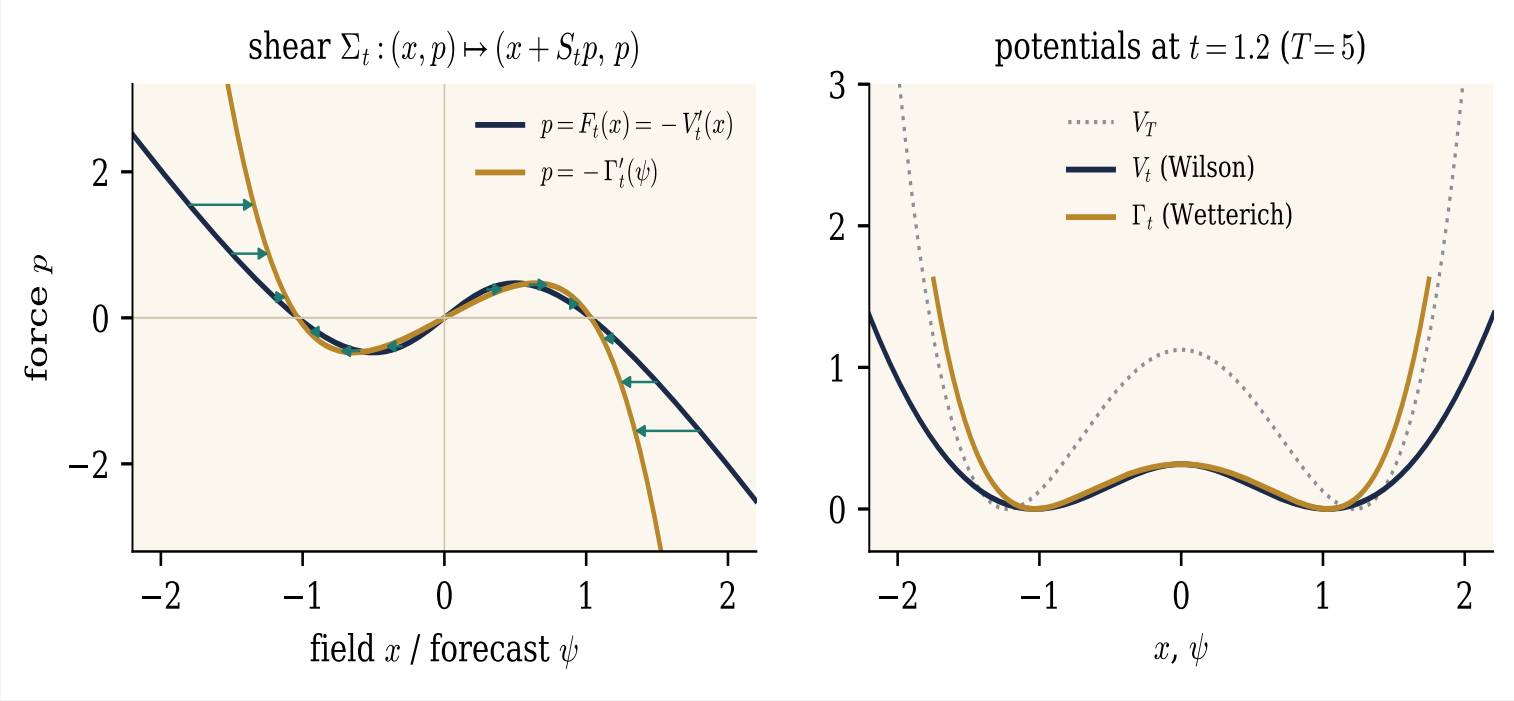
Generating function: the **Moreau transform**

$$V_t(x) = \inf_{\psi}\left\{\Gamma_t(\psi) + \tfrac{1}{2}\langle x - \psi, S_t^{-1}(x-\psi)\rangle\right\},$$

optimality: $x = \psi + S_t \mathrm{D}\Gamma_t(\psi)$, $\mathrm{D}V_t(x) = \mathrm{D}\Gamma_t(\psi)$.

No convexity of V_T or Γ_t is used: only $S_t^{-1} + \mathrm{D}^2\Gamma_t = \mathrm{Cov}_t^{-1} > 0$. Without noise, caustics and shocks.

The shear in zero dimensions



Left: Σ_t maps the graph of $F_t = -V_t'$ to the graph of $-\Gamma_t'$ over the forecast.
Right: V_T, V_t and Γ_t ($t = 1.2$, $T = 5$).

The Wetterich equation

Theorem. (formal)

$$\partial_t \Gamma_t = \tfrac{1}{2}\mathrm{Tr}\big[\dot{\mathscr{R}}_t\,((S_t^{-1} + \mathrm{D}^2\Gamma_t)^{-1} - S_t)\big], \qquad \Gamma_T = V_T.$$

Equivalently, for the ***effective average action*** $\Gamma_t^{\mathsf{EAA}} = \Gamma_t + \tfrac{1}{2}\langle\psi, Q_T^{-1}\psi\rangle$,

$$\partial_t \Gamma_t^{\mathsf{EAA}} = \tfrac{1}{2}\mathrm{Tr}\big[\dot{\mathscr{R}}_t\,(\mathrm{D}^2\Gamma_t^{\mathsf{EAA}} + \mathscr{R}_t)^{-1}\big] + c(t),$$

from the bare action ($t = T$, $\mathscr{R}_T = \infty$) to the 1PI effective action of ν_T ($t = 0$, $\mathscr{R}_0 = 0$).

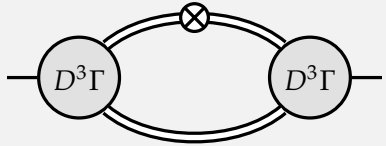
Proof. $\mathrm{D}_v\mathrm{Tr}(\dot{\mathscr{R}}H) = -\mathrm{Tr}(\mathscr{C}\,\mathrm{D}^2\mathscr{F})[v]$ by symmetry of $\mathrm{D}^3\Gamma$, so the force equation reads $D\left(\partial_t\Gamma - \tfrac{1}{2}\mathrm{Tr}(\dot{\mathscr{R}}H)\right) = 0$. The constant: envelope theorem + Polchinski.

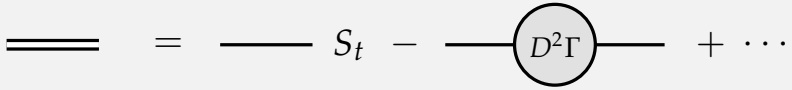
The Wetterich equation in graphs

$$\partial_t \Gamma_t^{\mathrm{EAA}} = \tfrac{1}{2}$$  $\dot{\mathcal{R}}_t$

$$\partial_t D\Gamma_t^{\mathrm{EAA}} = -\tfrac{1}{2}$$ 

$$\partial_t D^2\Gamma_t^{\mathrm{EAA}} = -\tfrac{1}{2}$$ 

$+$ 



Double line: full response. Cross: $\dot{\mathcal{R}}_t$. Exactly one loop: the Legendre transform has absorbed the tree term.

Wetterich = the backward half of the FBSDE

Every object in the Wetterich equation is an observable of the Polchinski process.

Wetterich	FBSDE
field argument ψ	forecast $\psi_t = \mathbb{E}_t[\Phi_T]$
$\mathrm{D}\Gamma_t(\psi_t)$	minus the costate, $\mathbb{E}_t[\mathrm{D}V_T(\Phi_T)]$
propagator $(\mathrm{D}^2\Gamma_t^{\mathsf{EAA}} + \mathscr{R}_t)^{-1}$	conditional covariance Cov_t
regulator $\mathscr{R}_t = S_t^{-1} - Q_T^{-1}$	shell precision minus prior precision
$\partial_t \Gamma_t$	$\frac{1}{2}\mathsf{Tr}[\dot{\mathscr{R}}_t(\mathsf{Cov}_t - S_t)]$

Driven by **interacting minus free** conditional covariance; **one-loop exact**, with the full propagator; in the Gaussian case only the constant flows. Wetterich 1993; Morris 1994; Berges–Tetradis–Wetterich 2002

One graph, three generators

Value function. At the optimal forecast, $V_t(x) = \Gamma_t(\psi) + \mathscr{A}_t(x, \psi)$: the cost attached to the forecast plus the cost of reaching it by free motion. This transports the generating function along Σ_t:

$$\mathrm{d}\, V_t = \langle \mathrm{D}\Gamma_t, \mathrm{d}\psi \rangle + \langle p, \mathrm{d}\psi \rangle - \langle p, \mathrm{d}x \rangle = -\langle F_t, \mathrm{d}x \rangle.$$

Legendre. With $\mathscr{H}_t(S_t^{-1}x) = \frac{1}{2}\langle x, S_t^{-1}x \rangle - V_t(x)$ up to a function of t,

$$\mathscr{H}_t^*(\psi) = \Gamma_t(\psi) + \frac{1}{2}\langle \psi, S_t^{-1}\psi \rangle.$$

- V_t in the field x; $\mathscr{H}_t$ in the coordinate $j = S_t^{-1}x$; Γ_t in the forecast ψ.
- Polchinski is the Hamilton–Jacobi equation of $\mathscr{H}$; Wetterich is its Legendre dual. The regulator is the information gained.

The regulator of the heat profile

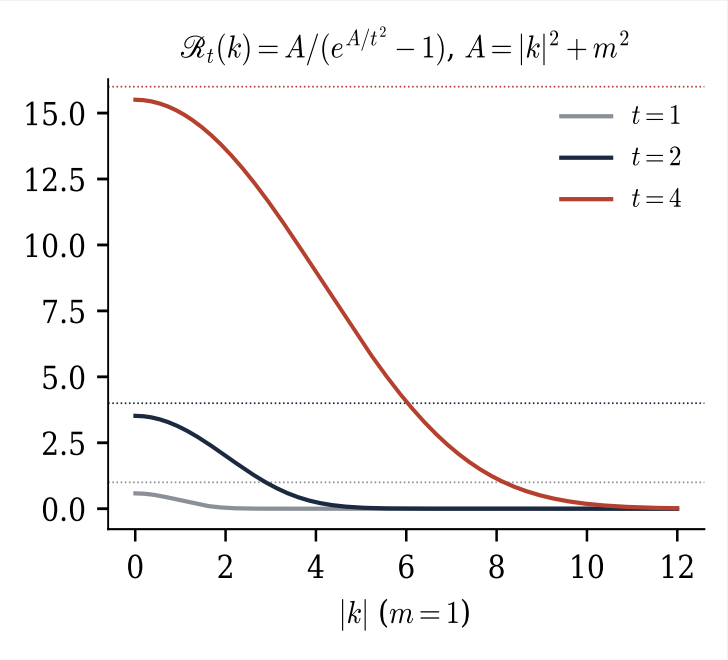
For $T=\infty$, $S_t = C\,(1-\mathrm{e}^{-A/t^2})$ and

$$\mathscr{R}_t = \frac{A}{\mathrm{e}^{A/t^2}-1}.$$

- $\approx t^2$ (a mass term) on the modes $|k| \ll t$;
- exponentially small on $|k| \gg t$.

The **exponential regulator** of the functional RG.

Other profiles give other regulators, e.g. the mass regulator $\mathscr{R}_t = t^2$.



Γ as a rate function

At $t=0$: the 1PI effective action of ν_T (Jona-Lasinio 1964),

$$\Gamma_0^{\mathsf{EAA}}(\psi) = V_0(0) + \sup_j \left\{ \langle j, \psi \rangle - \log \int \mathrm{e}^{\langle j, \varphi \rangle} \, \mathrm{d}\nu_T \right\}.$$

- **Cramér**: the mean of N independent samples is $\approx\psi$ with probability $\approx \mathrm{e}^{-N(\Gamma_0^{\mathsf{EAA}}(\psi) - V_0(0))}$: a **large-deviation rate function**.
- Convex even when V_T is not; minimal at the mean; Hessian there $=$ inverse covariance.
- Formally, the same for the field averaged over a large volume: Γ per unit volume (the effective potential of QFT, not V_t) is the rate function of the magnetisation; flat parts may appear (Maxwell).
- The renormalisation group as a probabilistic object: limit theorems for strongly dependent variables.

Jona-Lasinio, Nuovo Cimento 1964; Nuovo Cimento B 1975; Phys. Rep. 2001.

Variational meaning and the tree level

Constrained free energy. Gibbs variational principle for the conditional law:

$$\Gamma_t(\psi) = \inf\left\{\int V_T \,\mathrm{d}\nu + \mathsf{Ent}(\nu \,|\, \mu^{S_t}(\cdot - \psi)) \colon \int \varphi \,\mathrm{d}\nu = \psi\right\}:$$

the free energy of the unresolved shell, **constrained to the forecast**. V_t is the value function; Γ_t is its Moreau transform.

Tree level. Drop the Itô term: $\Gamma_t = V_T$, and the Moreau transform becomes the **Hopf–Lax formula**

$$V_t^{\mathsf{tree}}(x) = \inf_y \left\{ V_T(y) + \tfrac{1}{2}\langle y - x, S_t^{-1}(y-x)\rangle \right\},$$

which solves $\partial_t V = \frac{1}{2}\langle \mathrm{D}V, \dot{Q}\,\mathrm{D}V\rangle$: the saddle point of the Gaussian convolution. The minimiser solves the classical field equation $S_t^{-1}(y-x) + \mathrm{D}V_T(y) = 0$ in the shell $[t,T]$; expanding in V_T gives exactly the tree diagrams.

Summary

- **Three constructions** of the interacting measure, renormalisation, control and filtering, produce one forward–backward system. Backward in resolution it is a linear Gaussian blurring of ν_T.
- In **force coordinates** the system has no drift, trees are integrated exactly, and the propagator is the conditional covariance.
- **Potentials:** the value function generates the canonical transformation to these coordinates; the Legendre transform of the filter gives Γ_t and the **Wetterich** equation.

Next

Lecture 5: observables, the field equation at every resolution and its sub-Gaussian noise.

Exercises and reading

Exercises (numbering of the notes)

- 7.1 Quadratic interactions; 7.2 zero dimensions; 7.3 stochastic maximum principle; 7.5 cost of the leading drift.
- 8.1 Characteristics; 8.4 the constant; 8.6 1PI effective action.
- 8.7 Trees versus loops; 8.8 regulators; 8.9 Brascamp–Lieb; 8.10 the free action.

Reading

- Polchinski, Nucl. Phys. B 1984; Wetterich, Phys. Lett. B 1993; Jona-Lasinio, Phys. Rep. 2001; Eldan, ICM 2022.
- Bauerschmidt–Bodineau–Dagallier, *Stochastic dynamics and the Polchinski equation*, 2024.
- Barashkov–Gubinelli, *A variational method for* Φ^4_3, Duke 2020.
- Berges–Tetradis–Wetterich, Phys. Rep. 2002.
- Ma–Yong, *FBSDEs and their applications*, LNM 1702.