A Constructive Introduction to Quantum Field Theory

Observables and field equations

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Joint work with Sarah-Jean Meyer, with the use of AI

SISSA lectures, Lecture 5 — notes: Lecture 5, §9, Appendices A–B

Where we are

Lecture 4. The interacting field at resolution t solves the FBSDE

$$\mathrm{d}\Phi_t = \dot{Q}_t F_t \mathrm{d}t + \dot{Q}_t^{1/2} \mathrm{d}W_t, \qquad F_t = -\mathrm{D}V_t(\Phi_t) = \mathbb{E}_t[-\mathrm{D}V_T(\Phi_T)],$$

$\Phi_t \sim \nu_t \propto \mathrm{e}^{-V_t}\mu^{Q_t}$. Filtering: $\Phi_t = J_t\Phi_T + \zeta_t$, $J_t = K\,Q_t$, $K = Q_T^{-1}$; forecast $\psi_t = \mathbb{E}_t\Phi_T = \Phi_t + S_t F_t$; regulator $\mathscr{R}_t = S_t^{-1} - K$, $S_t = Q_T - Q_t$; effective average action Γ_t; μ_t^ϕ the law of Φ_T given $\Phi_t = \phi$, and Cov_t its covariance.

Today.

1. From Minkowski to Euclidean: the field equation and **Nelson's Euclidean noise**.
2. The **field equation at every resolution**; DS equations, also pathwise.
3. The noise is **sub-Gaussian**, whatever the interaction.
4. Infinite volume; appendices: algebra of the flow, counterterms, localisation.

Observables and field equations

Lecture 5

Observables of the interacting theory

For G a functional of the terminal field, $\mathbb{E}_t^{\mathbb{P}}[G(\Phi_T)] = G_t(\Phi_t)$, $G_t(\phi) = \mu_t^{\phi}(G)$, and

$$\partial_t G_t + \tfrac{1}{2}\mathrm{Tr}(\dot{Q}_t \mathrm{D}^2 G_t) - \langle \dot{Q}_t \mathrm{D}V_t, \mathrm{D}G_t \rangle = 0, \qquad G_T = G:$$

the backward Kolmogorov equation, a Doob transform of the Gaussian heat equation. Any solution gives an observable, closed at finite T, not necessarily as $T \to \infty$. The force F_t is the observable of $G = -\mathrm{D}V_T$.

Tangent observables. V_T^{θ} depending on a coupling, $U_t = \partial_\theta V_t^{\theta}$:

$$U_t(\Phi_t) = \mathbb{E}_t^{\theta}[\partial_\theta V_T^{\theta}(\Phi_T)], \qquad \partial_\theta \mathbb{E}^{\nu_T^{\theta}}[G] = -\mathrm{Cov}^{\nu_T^{\theta}}(G, \partial_\theta V_T^{\theta}).$$

E.g. the running $:\varphi^4:$ for $\theta = \lambda$: observables **conjugate to the coordinates of theory space**.

Field equations: Minkowski and Euclidean

Minkowski (Lecture 1): $(\Box + m^2)\,\varphi + V'(\varphi) = 0$ as an **operator** equation, with the CCR. On time-ordered functions ∂_t^2 also hits the step functions:

$$(\Box_x + m^2)\,\tau(x, x_1, \dots) = -\tau(V'(\varphi)(x), x_1, \dots) - \mathrm{i}\sum_j \delta(x - x_j)\,\tau(\dots, \hat{x}_j, \dots).$$

Contact terms: the canonical commutator, a derivative, $\mathrm{i}[\partial_t \varphi(\mathbf{x}), \cdot] = \delta / \delta\varphi(\mathbf{x})$. Rotate, $t = -\mathrm{i}\,\tau$: $\Box + m^2 \to A$, $-\mathrm{i}\,\delta(t) \to \delta(\tau)$, and we get the DS equation of Lecture 2:

$$\mathbb{E}^\nu[G\,\langle h, \omega\rangle] = \mathbb{E}^\nu[\mathrm{D}_h G], \qquad \omega := A\,\varphi + V'(\varphi),$$

nonzero: the Euclidean field does not solve $A\,\varphi + V'(\varphi) = 0$.

The noise is the transpose of the derivative: $\mathrm{D}_h^\top = -\mathrm{D}_h + \langle h, \omega\rangle$, so $\langle h, \omega\rangle = \mathrm{D}_h^\top 1$, minus the score of ν. Not an independent input: a functional of the field, fixed by its law. Free field: $\omega = A^{1/2}\xi$, one derivative of white noise.

Characterising the noise: two attempts

Nelson (1975): condition in real space. On the lattice $\varepsilon\mathbb{Z}^d$, $\omega := A\varphi + V'(\varphi)$. By the Markov property $\varepsilon^d\omega(x)$ is minus the score of the law of $\varphi(x)$ given its neighbours:

$$\mathbb{E}[\omega(x)|\varphi(y), y \neq x] = 0, \qquad \mathbb{E}[\omega(x)\,\omega(y)] = (-\Delta + m^2 + \mathbb{E}\,V''(\varphi))\,\delta(x-y),$$

not Gaussian. A random variable iff $\mathbb{E}\,V''(\varphi) < \infty$ in the limit (heuristically): unlikely in $d = 3$ (cf. Albeverio–Liang–Zegarlinski 2006). ω: **Nelson's Euclidean noise** (we keep his letter); as far as we know, not studied since as a noise.

Albeverio–Yoshida (2002): prescribe a Gaussian noise. Solve $A\psi + V'(\psi) = A^{1/2}\xi$: $\psi + A^{-1}V'(\psi)$ is a free field, and formally

$$\mathrm{Law}(\psi) \propto |\det(1 + A^{-1}V'')|\,\mathrm{e}^{-\langle\psi, V'\rangle - \frac{1}{2}\langle V', A^{-1}V'\rangle}\,\mu^{A^{-1}} \neq \mathrm{e}^{-V}\,\mu^{A^{-1}}:$$

nonlocal, not Markov, **no reflection positivity** (only on functionals of $\psi + A^{-1}V'(\psi)$, a free field).

Here: condition on the scales. The noise at resolution t is the conditional expectation of ω given the coarse field: not Gaussian, but a genuine random variable at each resolution, with Gaussian tails.

Dyson–Schwinger equations at every resolution

All operators are functions of A (so they commute); $K = Q_T^{-1}$, $J_t = K Q_t$ (heat profile: $J_t = \mathrm{e}^{-A(t^{-2}-T^{-2})} \to \mathrm{e}^{-A/t^2}$ as $T \to \infty$).

The noise observable is the **filtered bridge noise**: with $\Phi_t = J_t \Phi_T + \zeta_t$, $\omega_t = K\Phi_t - J_t F_t = S_t^{-1} \mathbb{E}_t \zeta_t$.

Theorem. (formal) *The **noise observable** $\omega_t = K\,\Phi_t - J_t F_t$ is a centred martingale, $\mathrm{d}\omega_t = K \dot{Q}_t^{1/2} \mathrm{d}W_t^{\mathbb{P}} - J_t\,\mathrm{d}F_t$, and*

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{D}_{J_t h} G(\Phi_t)\right] = \mathbb{E}^{\mathbb{P}}\left[G(\Phi_t)\,\langle h, K\,\Phi_t - J_t F_t\rangle\right].$$

At $t = T$: the DS equation $\mathbb{E}^{\nu_T}[\mathrm{D}_h G] = \mathbb{E}^{\nu_T}[G\langle h, K\varphi + \mathrm{D}V_T\rangle]$.

Proof. $\dot{J}_t = K\dot{Q}_t$: drifts cancel. IBP for μ^{Q_t} in direction $J_t h$, using $\Phi_t \sim \nu_t$.

At $t = T$: $\omega_T = K\,\Phi_T + \mathrm{D}V_T(\Phi_T)$, the Euclidean noise, and $\omega_t = \mathbb{E}_t[\omega_T]$: **what an observer at resolution t knows of it**.

Dyson–Schwinger, pathwise

G a terminal functional, $G_t(\Phi_t) = \mathbb{E}_t[G(\Phi_T)]$ its observable. Then

$$N_t := N_t(\Phi_t) = (\mathrm{D}_{J_t h} G_t)(\Phi_t) - G_t(\Phi_t)\,\langle h, \omega_t\rangle \quad \text{is a martingale, } N_0 = 0.$$

- **Itô**: the drift of $\mathrm{D}_{J_t h} G_t$ is $\langle \mathrm{D}G_t, \dot{Q}_t(K - \mathrm{D}F_t J_t)h\rangle \mathrm{d}t$, exactly the covariation of the two martingales $G_t(\Phi_t)$ and $\langle h, \omega_t\rangle$ ($\mathrm{D}F_t$ symmetric). $N_0 = 0$ since $J_0 = 0$, $\omega_0 = 0$.
- N is the **observable of the DS defect** $\mathrm{D}_h G - G\langle h, \omega_T\rangle$; $\mathbb{E}\,N_t = 0$ is DS at resolution t, and at T the DS equation of ν_T.
- N_t uses only resolution-t objects: G_t, its derivative along $J_t h$, ω_t. No terminal value needed.

For Φ^4_3, ω_T is not expected to have a limit, but DS survives as an **identity between observables**: N is a martingale null at 0, for every G, h.

The field equation at resolution t

As $T \to \infty$, $K \to A$, and $\omega_t := K\Phi_t - J_t F_t$ becomes

$$A\Phi_t = AQ_t F_t + \omega_t \qquad (AQ_t = \mathrm{e}^{-A/t^2} \text{ for the heat profile}):$$

the Euclidean field equation $A\varphi = -V'(\varphi) + \text{noise}$, **seen at resolution t**. No products of distributions: the nonlinearity enters only through the force **observable**, smeared at scale $1/t$. For Φ^4_3, with F_t in the role of the renormalised cube, $t \to \infty$ corresponds to the renormalised DS equation (Gubinelli–Hofmanová 2021); for the limit see Lecture 7.

Compare: stochastic quantisation (Lecture 3)

There the field equation is imposed in an **auxiliary time**; here it holds at **each resolution**, and the auxiliary parameter is the resolution itself.

Conditioning an SPDE solution on a scale filtration gives identities of the same shape but with a different noise: the martingale $J_t \mathrm{d}F_t$ in $\mathrm{d}\omega_t$ is essential.

The field equation for the forecast

Substitute $\Phi_t = \psi_t - S_t F_t$; since $K S_t = 1 - J_t$, the smoothing disappears: $K\psi_t = F_t + \omega_t$. As $T \to \infty$, with $F_t = -\mathrm{D}\Gamma_t(\psi_t) = -\mathrm{D}V_t(\Phi_t)$:

$$\underbrace{A\,\Phi_t + \mathrm{e}^{-A/t^2}\mathrm{D}V_t(\Phi_t) = \omega_t}_{\text{Polchinski process}} \qquad \underbrace{A\,\psi_t + \mathrm{D}\Gamma_t(\psi_t) = \omega_t}_{\text{forecast}}$$

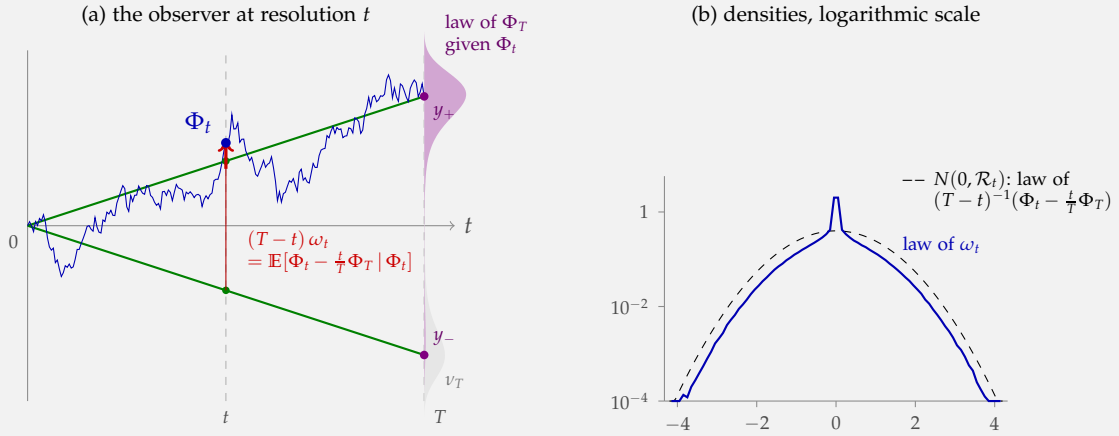
- For the forecast the force acts on the same variable, with no smoothing: the Euler–Lagrange equation of $\frac{1}{2}\langle\psi, A\,\psi\rangle + \Gamma_t(\psi)$, forced by ω_t.
- Φ^4, one loop (finite T, first order in λ): $\Gamma_t \approx V_T + \frac{1}{2}\mathrm{Tr}(S_t \mathrm{D}^2 V_T)$ moves the Wick ordering to c_t:

$$\langle\psi_t, K\psi_t\rangle + \frac{\lambda}{3}\int (\psi_t^4 - 3\,c_t\,\psi_t^2) = \langle\psi_t, \omega_t\rangle + \ldots,$$

  and $a^4 - 3c_t a^2 \geqslant \frac{1}{2}a^4 - \frac{9}{2}c_t^2$ (since $(a^2 - 3c_t)^2 \geqslant 0$): the **coercivity of the cube** is visible (heuristically; in $d = 3$ the mass counterterm still diverges). For Φ_t it is hidden behind e^{-A/t^2}.
- IBP: $\mathbb{E}[G(\psi_t)\langle h, K\psi_t - F_t\rangle] = \mathbb{E}[\mathrm{D}_{\mathrm{Cov}_t \mathscr{R}_t h} G(\psi_t)]$.

The noise as a posterior mean

One mode, $Q_t = t$. Known endpoint y: a Brownian bridge, $\Phi_t = \frac{t}{T} y + \zeta_t$ and $\omega_t = \zeta_t / (T-t) \sim N(0, \mathcal{R}_t)$. Random endpoint $\Phi_T \sim \nu_T$: $\omega_t = \mathbb{E}[(\Phi_t - \frac{t}{T}\Phi_T) / (T-t) | \Phi_t]$.



$T = 1$, $t = 1/2$, $\nu_T(\mathrm{d}y) \propto \mathrm{e}^{-2y^4+6y^2} N(0,1)(\mathrm{d}y)$ (grey). Green: bridges to the endpoints $y_\pm$; violet: posterior. The peak at 0 in (b): a fold of $\Phi_t \mapsto \omega_t$ between the wells.

General field: $\mathrm{Cov}(\omega_t) = \mathcal{R}_t - S_t^{-1} J_t \, \mathbb{E}[\mathrm{Cov}_t] \, J_t \, S_t^{-1} = K J_t + J_t \, \mathbb{E}[\mathrm{D}^2 V_t(\Phi_t)] \, J_t$: the free bridge noise **minus what the observer cannot resolve**; Nelson's mass $m^2 + \mathbb{E}\, V''$, at resolution t, and finite.

The noise is sub-Gaussian

ω_t is minus the score of the law of Φ_t (density p_t), seen through J_t: $\omega_t = -J_t \mathrm{D}\log p_t(\Phi_t)$. Given Φ_T, the interacting field is still the Gaussian bridge:

$$\Phi_t = J_t \Phi_T + \zeta_t, \quad \zeta_t \sim N(0, Q_t(1-J_t)) \quad \Rightarrow \quad \omega_t = \mathbb{E}[S_t^{-1}\zeta_t | \Phi_t].$$

Proposition. *ω_t is dominated in convex order by $N(0, \mathscr{R}_t)$, with $\mathscr{R}_t = S_t^{-1} - Q_T^{-1}$ the **Wetterich regulator**; in particular*

$$\mathbb{E}\, \mathrm{e}^{u\langle h, \omega_t\rangle} \leqslant \exp\left(\frac{u^2}{2}\langle h, \mathscr{R}_t h\rangle\right),$$

uniformly in the interaction, with no convexity (conditional Jensen).

- Heat profile, finite spatial cutoff: $\mathscr{R}_t = A/(\mathrm{e}^{A/t^2}-1) \leqslant t^2$ if $T=\infty$; $\mathscr{R}_t \leqslant \frac{4}{3}t^2$ if $T \geqslant 2t$.
- With $\mathrm{d}F_s = \Upsilon_s \mathrm{d}W_s$ (Lecture 4) and Itô: $\mathbb{E}\int_0^t \|\Upsilon_s^* J_s h\|^2 \mathrm{d}s \leqslant 2(1+\mathrm{e}^{-1})\, t^2 \|h\|^2$: a smoothed derivative of the force; no terminal force needed.
- Φ^4_3: the noise in $A\Phi_t = AQ_t F_t + \omega_t$ is controlled a priori; what remains is coercivity of the renormalised force.

And the force?

$F_t = -\mathrm{D}V_t(\Phi_t) = -\mathrm{D}\Gamma_t(\psi_t)$. Read it from the measure at resolution t, $\nu_t \propto \mathrm{e}^{-V_t}\mu^{Q_t}$.

Proposition. *If V_T is even and Γ_t is convex,*

$$\mathbb{E}\,\mathrm{e}^{u\langle h, F_t\rangle} \leqslant \exp\left(\frac{u^2}{2}\langle h, S_t^{-1}h\rangle\right).$$

1. **Always:** $\mathrm{D}^2 V_t \leqslant S_t^{-1}$ ($\mathrm{Cov}_t \geqslant 0$), then Cameron–Martin:

$$\mathbb{E}\,\mathrm{e}^{u\langle h, F_t\rangle} \leqslant \mathrm{e}^{\frac{u^2}{2}\langle h, (S_t^{-1}-Q_t^{-1})h\rangle}\ \mathbb{E}\,\mathrm{e}^{u\langle Q_t^{-1}h, \Phi_t\rangle}.$$

2. **Convexity:** Γ_t convex $\Rightarrow$ V_t convex $\Rightarrow$ (Prékopa) Φ_t no wider than the free field: $\mathbb{E}\,\mathrm{e}^{\langle k, \Phi_t\rangle} \leqslant \mathrm{e}^{\frac{1}{2}\langle k, Q_t k\rangle}$.

- Without convexity: $\mathrm{e}^{\frac{u^2}{2}\langle h, \mathscr{R}_t h\rangle}\,\mathbb{E}_{\nu_T}\mathrm{e}^{u\langle Kh, \Phi_T\rangle}$: the noise plus the terminal field seen through K.
- $\mathrm{D}\Gamma_t^{\mathsf{EAA}}(\psi_t) = \omega_t$, sub-Gaussian for free. Γ_t^{EAA} convex $\Leftrightarrow$ ν_t log-concave: not enough (negative mass: $\mathrm{Var} F_t \sim 1/\varepsilon$).

How rough is the noise?

Spectral densities, heat profile, $T=\infty$ (X_t: the free field, $V=0$), $A=|k|^2+m^2$:

white noise	GFF	free noise AX_t	bound $\mathscr{R}_t$
1	A^{-1}	$A\,\mathrm{e}^{-A/t^2}$	$\dfrac{A}{\mathrm{e}^{A/t^2}-1}$

- $\mathscr{R}_t \leqslant t^2\,\mathrm{e}^{-A/2t^2}$: at resolution t, $\omega_t \lesssim t\times$ white noise smoothed at scale $1/t$, so $\omega_t(x)=O(t^{1+d/2})$.
- At scale $1/t$ this is the size of regularity $-1-d/2$ ($-5/2$ in $d=3$): one derivative rougher than white noise, like the free noise $A^{1/2}\xi$, **whatever the interaction**.
- Not a limit distribution: on a fixed h only $\langle h,\omega_t\rangle=O(t)$, while the free noise stays bounded.
- On resolved modes ($|k|\ll t$) the bound exceeds the free noise by t^2/A; it is attained by a pinning potential $\kappa\|\varphi\|^2/2$, $\kappa\to\infty$.
- **Pathwise** (convex domination of the norm, Doob): a.s. for $t\geqslant 1$ ($t\leqslant T/2$ if $T<\infty$), at finite spatial cutoff, with a random constant $\tilde{C}$,

$$\sup_x\,(1+|x|)^{-\delta}|\omega_t(x)| \leqslant \tilde{C}\,t^{1+d/2}\sqrt{\log\,(2+t)}.$$

The noise as a stochastic observable

- $(\omega_t)_{t \leqslant T}$ is a **martingale**, with terminal value Nelson's Euclidean noise $K\,\Phi_T + \mathrm{D}V_T(\Phi_T)$.
- At each resolution a **smooth random field**; every $\langle h, \omega_t \rangle$ is **sub-Gaussian** with variance $\leqslant \langle h, \mathscr{R}_t h \rangle$, uniformly in the interaction and the cutoff.
- Paired with functionals of the coarse field it is the **transpose of the smoothed derivative** $\mathrm{D}_{J_t h}$ (DS at every scale).
- Expected **local** in the sense of Lecture 3: germs $A\,\Phi_t(x) + n_t(\Phi_t(x))$, n_t a renormalised cube; local iff the force is. For Φ^4_3: local but not closed (no terminal representative).
- For Φ^4_3, $A\,\varphi + V'(\varphi)$ is not expected to be a random variable (Nelson; Albeverio–Liang–Zegarlinski), but its conditional expectations are, consistently: $\omega_s = \mathbb{E}_s\,\omega_t$.

The Euclidean field equation of Φ^4_3 has a well-defined sub-Gaussian noise: an **observable without a terminal representative**.

Infinite volume

On $\mathbb{T}_L^d$, the free energy $V_0(0)$ and V_t are **extensive** ($\sim L^d$). Three **intensive** objects survive:

i. **the force.** For $V_T = \int v_T(\varphi)$, on $\mathbb{R}^d$:

$$\Phi_t = \int_0^t \dot{Q}_s F_s \, \mathrm{d}s + \int_0^t \dot{Q}_s^{1/2} \mathrm{d}W_s, \qquad F_t(x) = -\mathbb{E}_t[v_T'(\Phi_T(x))].$$

Everything is local; infinite-volume limits should solve this. Uniqueness of solutions $\leftrightarrow$ uniqueness of the theory; non-uniqueness expected at phase transitions (Φ_2^4, deep double well: Glimm–Jaffe–Spencer 1975).

ii. **the Burgers flow** of the force (Lecture 4);

iii. **differences of free energies** $\mathscr{W}_T^{(L)}(f) = -\log \int \mathrm{e}^{-f} \mathrm{d}\nu_T^{(L)}$, f local: bounded as $L \to \infty$; basis of the infinite-volume analysis of Φ_2^4 and $\exp(\Phi)_2$.

Estimates must be **local** (weighted norms, volume-independent constants): the analogue of "coming down from infinity" for the parabolic dynamics.

Barashkov–Gubinelli, PMP 2023

Differences of free energies

Perturb the optimal drift u^* of the Boué–Dupuis formula (Lecture 4), $X^{u^*} = \Phi$: $u = u^* + w$, $X^u = \Phi + I(w)$, $I_t(w) = \int_0^t \dot{Q}_s^{1/2} w_s \mathrm{d}s$. Euler–Lagrange gives the cross term

$$\mathbb{E} \int \langle u^*, w \rangle = -\mathbb{E} \langle \mathrm{D}V_T(\Phi_T), I_T(w) \rangle.$$

Proposition. (Relative variational principle) (formal) *For f bounded below,*

$$-\log \int \mathrm{e}^{-f} \, \mathrm{d}\nu_T = \inf_w \mathbb{E}\left[f(\Phi_T + I_T(w)) + B_{V_T}(\Phi_T, I_T(w)) + \frac{1}{2} \int_0^T \|w\|^2 \right],$$

over adapted w with $\mathbb{E} \int \|w_s\|^2 \mathrm{d}s < \infty$: the two extensive infima cancel exactly.

$B_V(\phi, h) = V(\phi + h) - V(\phi) - \langle \mathrm{D}V(\phi), h \rangle$: the **first-order Taylor remainder**.

- Minimiser: $\Phi^f = \Phi + I(w^*)$ solves the FBSDE with terminal potential $V_T + f$, a perturbation of Φ:

$$F_s^f - F_s = -\mathbb{E}_s[\mathrm{D}f(\Phi_T^f) + \mathrm{D}V_T(\Phi_T^f) - \mathrm{D}V_T(\Phi_T)].$$

Infinite-volume measures

Local interaction $V_T = \int v_T(\varphi(x))\,\mathrm{d}x$:

$$B_{V_T}(\phi, h) = \int b_T(\phi(x), h(x))\,\mathrm{d}x, \qquad b_T(a,h) = v_T(a+h) - v_T(a) - v_T'(a)\,h.$$

- $b_T(a,0) = 0$: only the region where the perturbation acts counts. Every term is **intensive**: no free energy, only Φ and its perturbation.
- f local $\Rightarrow$ the optimal w should decay away from its support, at the rate of decay of correlations.
- Several solutions of the FBSDE (phase transitions) $\Rightarrow$ several measures, each with its own relative variational principle.
- Φ^4_3: the remainder term carries the counterterms; the change of variables of Barashkov–Gubinelli (Duke 2020) is still needed.

Formal statement (at fixed UV cutoff T). An infinite-volume measure is the law of Φ_T for a solution of the infinite-volume FBSDE satisfying the relative variational principle for all local f.

Appendices: additional material

Lecture 5

A semigroup of renormalisation maps

$\mathfrak{R}_S(V) = -\log\,(\mu^S * \mathrm{e}^{-V})$. Since $\mu^S * \mu^{S'} = \mu^{S+S'}$,

$$\mathfrak{R}_S \circ \mathfrak{R}_{S'} = \mathfrak{R}_{S+S'} = \mathfrak{R}_{S'} \circ \mathfrak{R}_S, \qquad V_t = \mathfrak{R}_{S_t}(V_T):$$

an orbit of an **abelian** semigroup indexed by the cone of positive covariances, nonlinear on V, linear on e^{-V}. Generators:

$$\mathscr{P}_B(V) = \tfrac{1}{2}\mathrm{Tr}(B\,\mathrm{D}^2 V) - \tfrac{1}{2}\langle \mathrm{D}V, B\,\mathrm{D}V\rangle, \qquad \partial_t V_t = -\mathscr{P}_{\dot{Q}_t}(V_t).$$

Proposition. (Flatness) (formal) $[\mathscr{P}_B, \mathscr{P}_{B'}] = 0$. *$V_t$ depends on the profile only through S_t; $V_0(0)$ and ν_T only through Q_T.*

Different profiles with the same endpoints are different **coordinate systems** on the same RG trajectory: they change Φ and intermediate observables, not the terminal theory. **No holonomy.**

Perturbation theory and counterterms

Gaussian convolution **lowers the Wick ordering**: $V_t = \int [\frac{\lambda}{12} {:}\phi^4{:}_t + \gamma_T {:}\phi^2{:}_t] + O(\lambda^2)$ ($\gamma_T = O(\lambda^2)$, kept for later use; ${:}\phi^p{:}_t = H_p(\phi; c_t)$, $c_t = Q_t(x,x)$): dual to $\int H_4(X_t; c_t)$ being an observable (Lecture 3). At second order, $S = S_t$:

Proposition. (formal) *To second order in λ:*

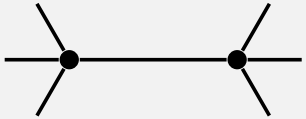
$$V_t = \int \left[\frac{\lambda}{12} {:}\phi^4{:}_t + \gamma_T {:}\phi^2{:}_t \right] - \frac{\lambda^2}{288} \sum_{k=1}^{4} \binom{4}{k}^2 k! \iint S(x,y)^k {:}\phi^{4-k}{:}_t(x) {:}\phi^{4-k}{:}_t(y) + O(\lambda^3).$$

$T \to \infty$ at fixed t, $d = 3$, $G(z) \sim (4\pi|z|)^{-1}$ (the kernel of A^{-1}):

- $k = 1, 2$: finite, nonlocal.
- $k = 3$ (**sunset**): $S^3 \sim |z|^{-3}$; local divergence $-\frac{\lambda^2}{3}(\int S^3) \int {:}\phi^2{:}_t$ (+ finite), cancelled by $\gamma_T = \frac{\lambda^2}{3} \int Q_T^3 = \frac{\lambda^2}{48\pi^2} \log T + O(1)$.
- $k = 4$: vacuum energy $\sim T$; cancels in ν_T.

Renormalisation = choosing V_T so that V_t at **fixed** resolution converges. $\nu_t \ll \mu^{Q_t}$ expected for $t < \infty$; the terminal Φ^4_3 is singular w.r.t. the GFF.

Second order in graphs



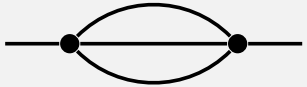
$:\varphi^6:$

tree, finite



$:\varphi^4:$

bubble, finite in $d = 3$



$:\varphi^2:$

sunset, $\sim \lambda^2 \log T$



1

vacuum, a constant

k lines S between two Wick-ordered vertices. $k = 1, \ldots, 4$ from left to right; Wick ordering removes self-loops. Only the sunset needs a further counterterm, γ_T.

Stochastic localisation: what and why

Eldan 2013: write a measure as a mixture of simpler ones, by a measure-valued martingale. Here

$$\mu_t^{\phi}(\mathrm{d}y) \propto \mathrm{e}^{\langle S_t^{-1}\phi, y\rangle - \frac{1}{2}\langle y, \mathscr{R}_t y\rangle} \nu_T(\mathrm{d}y), \qquad \Phi_t = J_t \Phi_T + \zeta_t:$$

Φ_t is a noisy **observation** of Φ_T, with precision $\mathscr{R}_t$ gained, and μ_t the **posterior**. Eldan's identity-control case: $\mathscr{R}_t = t \cdot 1$.

- **Martingale:** Itô splits variance and entropy into revealed and hidden parts.
- **Simpler pieces:** ν_T log-concave $\Rightarrow$ μ_t is $\mathscr{R}_t$-uniformly log-concave.
- **One quantity:** the loss is governed by Cov_t, with explicit dynamics. An inequality reduces to: *the localisation is not too fast*.
- **Algorithms:** sampling (El Alaoui–Montanari–Sellke), denoising diffusions (Montanari); ω_t is minus the score, seen through J_t.

Entropy along the localisation

$\rho \geqslant 0$, $\mathbb{E}_{\nu_T} \rho = 1$, $M_t = \mathbb{E}_t \rho(\Phi_T) = G_t(\Phi_t)$, $G_t(\phi) = \int \rho \, \mathrm{d}\mu_t^\phi$.

- $M_0 = 1$, $M_T = \rho(\Phi_T)$, and $\mathrm{d}M_t = \langle \mathrm{D}G_t(\Phi_t), \dot{Q}_t^{1/2} \mathrm{d}W_t \rangle$.
- Itô for $M \log M$:

$$\mathrm{Ent}_{\nu_T}(\rho) = \frac{1}{2} \mathbb{E} \int_0^T \frac{\mathrm{d}\langle M \rangle_t}{M_t} = 2 \, \mathbb{E} \int_0^T \left\| \dot{Q}_t^{1/2} \mathrm{D} \sqrt{G_t}(\Phi_t) \right\|^2 \mathrm{d}t.$$

- Stopping at t: **revealed + hidden** entropy,

$$\mathrm{Ent}_{\nu_T}(\rho) = \mathbb{E}[M_t \log M_t] + \mathbb{E} \, \mathrm{Ent}_{\mu_t^{\Phi_t}}(\rho).$$

- The increments see the conditional law: $\mathrm{D}G_t(\phi) = S_t^{-1} \mathrm{Cov}_{\mu_t^\phi}(\rho, y)$.
- **Entropic stability** (Chen–Eldan; $\dot{\mathscr{R}}_r = S_r^{-1} \dot{Q}_r S_r^{-1}$): if $\dot{\mathscr{R}}_r \mathrm{Cov}_{\mu_r^\phi} \dot{\mathscr{R}}_r \leqslant \alpha_r \dot{\mathscr{R}}_r$ for all ϕ and $r \leqslant t$, then $\mathrm{Ent}_{\nu_T}(\rho) \leqslant \mathrm{e}^{\int_0^t \alpha_r \mathrm{d}r} \, \mathbb{E} \, \mathrm{Ent}_{\mu_t^{\Phi_t}}(\rho)$: the low resolutions must not hide too much.

Bauerschmidt–Bodineau–Dagallier, Probab. Surveys 2024: their fluctuation measure is our stochastic localisation (scale reversed).

Multiscale Bakry–Émery

Theorem. (Bauerschmidt–Bodineau) (formal) *If for all t, ϕ, with $\kappa_t \in \mathbb{R}$ (possibly negative),*

$$\dot{Q}_t \, \mathrm{D}^2 V_t(\phi) \, \dot{Q}_t + \frac{1}{2} \ddot{Q}_t \geqslant \kappa_t \dot{Q}_t,$$

then $\mathrm{Ent}_{\nu_T}(\rho) \leqslant \frac{2}{\varrho} \mathbb{E}_{\nu_T} \| \dot{Q}_T^{1/2} \mathrm{D} \sqrt{\rho} \|^2$, $\frac{1}{\varrho} = \int_0^T \mathrm{e}^{-2\int_t^T \kappa_r \mathrm{d}r} \, \mathrm{d}t$.

- Gradient bound at each scale + entropy formula. Gaussian check: $V_T = 0$, $Q_t = t Q_T / T$, $\kappa = 0 \Rightarrow$ Gaussian LSI.
- Convexity of the **effective** potentials, not of V_T.
- By $\mathrm{D}^2 V_t = S_t^{-1} - S_t^{-1} \mathrm{Cov}_t \, S_t^{-1}$, for $\dot{Q}_t$ invertible the hypothesis is equivalent to the bound, uniform in ϕ,

$$\mathrm{Cov}_t \leqslant S_t - S_t \dot{Q}_t^{-1} \left(\kappa_t \dot{Q}_t - \frac{1}{2} \ddot{Q}_t \right) \dot{Q}_t^{-1} S_t :$$

  at most S_t iff $\kappa_t \dot{Q}_t \geqslant \frac{1}{2} \ddot{Q}_t$ (linear profile: $\kappa_t \geqslant 0$).

Poincaré from the localisation

Proposition. *If* $\dot{\mathscr{R}}_r \operatorname{Cov}_{\mu_r^\phi} \dot{\mathscr{R}}_r \leqslant \alpha_r \dot{\mathscr{R}}_r$ *for all* ϕ, $r \leqslant t$, *then*

$$\operatorname{Var}_{\nu_T}(f) \leqslant \mathrm{e}^{\int_0^t \alpha}\, \mathbb{E} \operatorname{Var}_{\mu_t}(f),$$

and, for ν_T *log-concave,* $\operatorname{Var}_{\nu_T}(f) \leqslant \mathrm{e}^{\int_0^t \alpha}\, \mathbb{E}\langle \mathrm{D}f, \mathscr{R}_t^{-1} \mathrm{D}f\rangle$.

Proof. $v(r) = \mathbb{E} \operatorname{Var}_{\mu_r}(f)$ has $v' = -\mathbb{E}\langle \chi, \dot{\mathscr{R}} \chi\rangle$, $\chi = \operatorname{Cov}(f, y)$; Cauchy–Schwarz gives $\langle \chi, \dot{\mathscr{R}} \chi\rangle \leqslant \alpha \operatorname{Var}(f)$. Then Brascamp–Lieb for μ_t.

- Gaussian, one mode: $\mathrm{e}^{\int \alpha} = Q_T / S_t$; equality for linear f.
- **KLS conjecture**: isotropic log-concave on $\mathbb{R}^n$, $\mathscr{R}_t = t$: $\operatorname{Var}(f) \leqslant t^{-1} \mathrm{e}^{\int \alpha}\, \mathbb{E}|\mathrm{D}f|^2$. Keep $\|\operatorname{Cov}_r\|$ bounded as long as possible (only with high probability). Eldan 2013, Chen 2021, Klartag–Lehec 2022, Klartag 2023: Poincaré constant $O(\log n)$.

Two views of one martingale

these notes	Bauerschmidt–Bodineau	localisation
ν_t	renormalised measure	law of the tilt
μ_t^ϕ	fluctuation measure	tilted measure
G_t	Polchinski semigroup	martingale $\int \rho \, \mathrm{d}\mu_t$
$\mathrm{D}^2 V_t$	Hessian of the potential	$S_t^{-1} - S_t^{-1} \mathrm{Cov}_t S_t^{-1}$: covariance of the tilted measure

- **Bridge:** $\mathrm{D}^2 V_t = S_t^{-1} - S_t^{-1} \mathrm{Cov}_t S_t^{-1}$: convex potentials = bounded conditional covariance.
- **Bauerschmidt–Bodineau:** Bakry–Émery for the Kolmogorov equation, curvature $\dot{Q} \mathrm{D}^2 V_t \dot{Q} + \frac{1}{2} \ddot{Q}$: gradient bound at each scale, LSI at the terminal scale, far from log-concavity. Uniform in ϕ: suits the RG (sine-Gordon, $\beta < 6\pi$).
- **Localisation:** bound what is lost on $[0, t]$, then use a known inequality for μ_t. Moments of the conditional law, possibly only with high probability (KLS).

Bauerschmidt–Bodineau, CPAM 2021; Bauerschmidt–Bodineau–Dagallier, Probab. Surveys 2024; Eldan, GAFA 2013; Chen–Eldan, FOCS 2022.

Summary

- The **Euclidean field equation** $A\varphi + V'(\varphi) = \omega$: Nelson's Euclidean noise is the transpose of the derivative, fixed by the law of the field, not Gaussian.
- **Field equation** at every resolution: its noise is a conditional expectation both of Nelson's noise and of a Gaussian bridge noise, hence sub-Gaussian whatever the interaction; for the forecast the force enters without smoothing.
- **DS, pathwise**: an identity between stochastic observables, with no terminal representative needed.

Next

Lectures 6–7: a construction of Φ^4_3 on the torus via the FBSDE.

Exercises and reading

Exercises (numbering of the notes)

- 9.1 Doob transform; 9.2 relative variational principle; 9.3 DS from the dynamics; 9.4 Gaussian check; 9.5 sub-Gaussian noise; 9.6 the noise of a two-point endpoint; 9.7 contact terms.
- A.1 Commuting flows; A.2 the sunset; A.3 power counting.
- B.1 Entropy along the localisation; B.2 Poincaré from the localisation.

Reading

- Nelson, *Remarks on Markov field equations*, 1975.
- Albeverio–Yoshida, J. Funct. Anal. 2002; Albeverio–Liang–Zegarlinski 2006.
- Gubinelli–Hofmanová, *A PDE construction of the Euclidean Φ^4_3 quantum field theory*, CMP 2021.
- Bauerschmidt–Bodineau, CPAM 2021 (log-Sobolev via the Polchinski flow); Eldan, GAFA 2013.
- Barashkov–Gubinelli, Duke 2020; PMP 2023 (infinite volume).