A Constructive Introduction to Quantum Field Theory

Φ^4_3 on the torus I: the FBSDE in force coordinates and renormalisation

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Joint work with Sarah-Jean Meyer, with the use of AI

SISSA lectures, Lecture 6 — notes: Lecture 6, §§10–12

Where we are

Lecture 4. Interacting field at resolution t = solution of the FBSDE

$$\Phi_t = \int_0^t \dot{Q}_s F_s \, \mathrm{d}s + \int_0^t \dot{Q}_s^{1/2} \mathrm{d}W_s, \qquad F_t = \mathbb{E}_t\big[-D\,V_T(\Phi_T)\big].$$

In the **forecast coordinate** $\psi_t = \mathbb{E}_t \Phi_T$ the forward equation has no drift and the trees are integrated exactly; the backward half is the **force equation**.

Lectures 6–7. Construct Φ^4_3 on $\mathbb{T}^3$ with this FBSDE:

1. **deterministic renormalisation**: an explicit approximate force $\bar{\mathscr{F}}_q$ and its defect (today);
2. **a priori estimates** for the exact FBSDE in the coordinates of $\bar{\mathscr{F}}_q$, uniform in the lattice (Lecture 7);
3. **removing the cutoffs** and identifying the limit (Lecture 7).

Notes: Lectures 6–7 and the Supplement; Bailleul–Chevyrev–G. (in prep.)

What is proved

For every coupling $\lambda > 0$, every reference mass $q_0 > 0$ and every finite part r of the counterterm, on a fixed torus, with no smallness, and an explicit counterterm with arbitrary finite part:

- the whole lattice sequence converges to a measure ν, an explicit density over an explicit SDE;
- the forecasts converge to the solution of an FBSDE from $\Upsilon_{q_0} = 0$, driven by the limit of the lattice innovations, and $\psi_q = m_q(\Phi_q)$;
- uniqueness in law for its scalar form, a variational formula with no infinite constant, quartic tails, integration by parts.

Not proved: the vector FBSDE alone, strong solutions, OS axioms in full, infinite volume.

Guiding principle. Do not estimate Φ and F. Estimate the error of an explicit approximate forecast.

The results of Lectures 6–7 are new: joint work with S.-J. Meyer + AI.

The problem and the exact FBSDE

Lecture 6

Lattice and the mass-regulator profile

$\Lambda_N = (N^{-1}\mathbb{Z}/\mathbb{Z})^3$, $\int f = N^{-3}\sum_x f(x)$, $\Delta = \Delta_N$ nearest neighbour. Resolution parameter $q = q_0 + t^2 \in [q_0, \infty)$:

$$K_q = q - \Delta, \qquad C_q = K_q^{-1}, \qquad C = C_{q_0} = (q_0 - \Delta)^{-1}.$$

In Lecture 3 notation ($T = \infty$, and $\dot{Q} = \partial_q Q_q$):

$$Q_q = C - C_q, \quad \dot{Q} = C_q^2, \quad \dot{Q}^{1/2} = C_q, \quad S_q = C_q, \quad \mathscr{R}_q = q - q_0.$$

Free resolvents (uniform in N)

$C_q \geqslant 0$ as a kernel, $C_q 1 = q^{-1} 1$, $\|C_q\|_{p\to p} = q^{-1}$,

$$a_2(q) = (C_q^2)(x,x) \lesssim q^{-1/2} \wedge N^3 q^{-2}, \qquad \mathrm{Tr} C \asymp N.$$

The **bubble** $a_2(q)$ is the coincident-point variance of the shell at resolution q.

The cutoff measure

$$U(\varphi)=\tfrac{\lambda}{12}\int\varphi^4,\ V_N=U+\tfrac{\mu_N}{2}\int\varphi^2,\ \nu_N(\mathrm{d}\varphi)=\mathcal{Z}_N^{-1}\mathrm{e}^{-V_N(\varphi)}\mu^C(\mathrm{d}\varphi).$$

So $D^2U(y)=\lambda y^2$: the coupling is the curvature, the same λ as in Lecture 4.

The Wick ordering is now part of μ_N.

Reference mass $q_0>0$ (constants $\in\ker\Delta$ on the torus; r may be negative). Only $q_0+\mu_N$ matters: fix q_0, vary r.

Main result, informal. Let

$$\mu_N=r-\lambda\,\mathrm{Tr}\,C+\frac{2}{3}\lambda^2\int C^3.$$

Then (a) $\nu_N\to\nu$ in H^{-s}, $s>\frac{1}{2}$, along the whole sequence; (b) the forecasts converge in law to a solution of a limiting FBSDE; (c) ν has a variational formula, an integration by parts formula and quartic tails.

These come from bounds on the error Z of an approximate forecast, uniform in N: **UV stability**.

The Polchinski process as a filter

By Lecture 4, in the profile $S_q = C_q$, $\mathscr{R}_q = q - q_0$; at fixed N everything is a theorem. The Polchinski process solves

$$\mathrm{d}\Phi_q = C_q^2 F_q \,\mathrm{d}q + C_q \,\mathrm{d}W_q, \quad F_q = \mathbb{E}_q[-D\,V_N(\Phi_\infty)], \quad \Phi_{q_0} = 0,$$

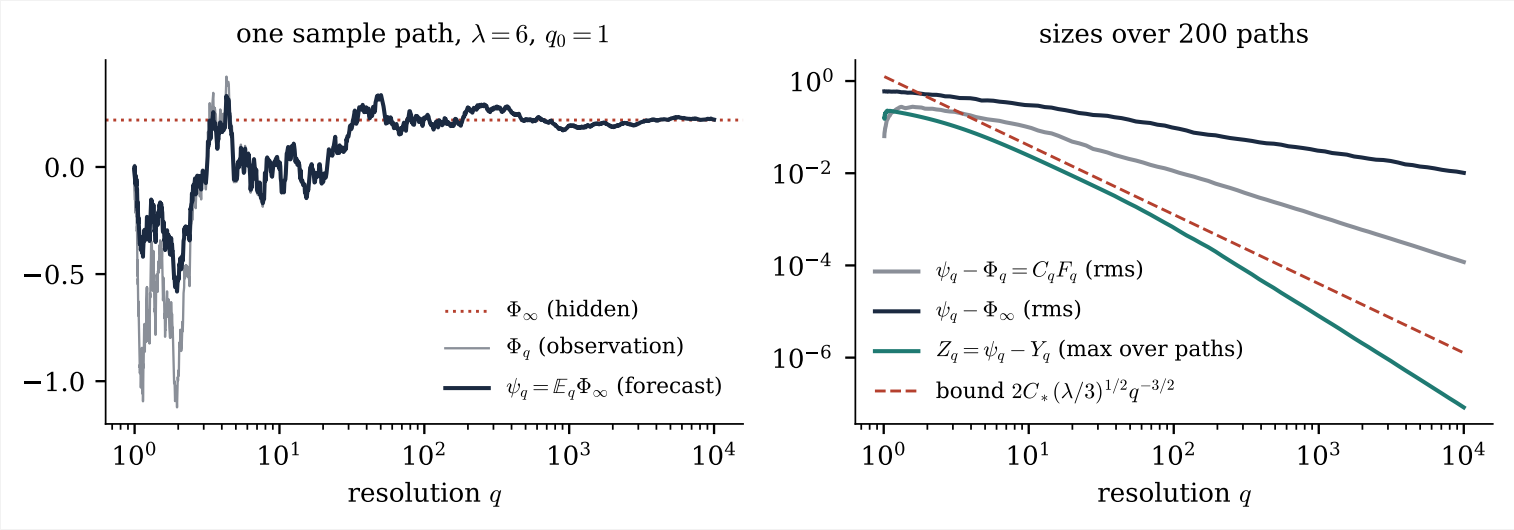
and $\Phi_q \to \Phi_\infty \sim \nu_N$.

- a noisy observation of its endpoint, $\Phi_q = (q - q_0) C_q \Phi_\infty + C_q B_{q-q_0}$, with B independent of Φ_∞;
- forecast

  $$\psi_q = \mathbb{E}_q \Phi_\infty = \Phi_q + C_q F_q = m_q(\Phi_q),$$

  covariance $\mathrm{Cov}_q > 0$;
- W is the innovation: $\mathrm{d}\psi_q = \mathrm{Cov}_q \,\mathrm{d}W_q$;
- noise: $\omega_q = K_q \Phi_q - (q - q_0)\psi_q = \mathbb{E}_q B_{q-q_0}$, sub-Gaussian whatever the interaction (L5).

One site



One site, $\nu \propto e^{-\lambda\varphi^4/12}\mu^{1/q_0}$, $\lambda = 6$, $q_0 = 1$. Left: one path of the observation Φ_q and of the forecast ψ_q, both converging to the hidden Φ_∞. Right, over 200 paths: the forecast error $\sim q^{-1/2}$, the force term $\psi_q - \Phi_q = C_q F_q \sim q^{-1}$, and the error Z_q of the approximate forecast from $\bar{\mathscr{F}} = \frac{\lambda}{3} y^3$, below the bound $2C_*(\lambda/3)^{1/2} q^{-3/2}$ of Theorem 10.12 (see "Integrating the cube" below).

The exact force in the forecast coordinate

Since $\Phi_q \mapsto \psi_q$ is invertible, define $\mathscr{F}_q$ by

$$\Phi_q = \psi_q + C_q \mathscr{F}_q(\psi_q)$$

and $H_q^{\mathrm{ex}}(y) = (K_q + D\mathscr{F}_q(y))^{-1}$. It is the $\mathscr{F}$ of Lecture 4 ($x = \psi + S\mathscr{F}(\psi)$), and $F_q = -\mathscr{F}_q(\psi_q)$.

The force equation with a mass regulator

$$\partial_q \mathscr{F}_q + \frac{1}{2}\mathrm{Ctr}_{(H_q^{\mathrm{ex}})^2} D^2 \mathscr{F}_q = 0, \quad \mathrm{Cov}_q = H_q^{\mathrm{ex}}(\psi_q).$$

Itô on $\Phi = \psi + C\mathscr{F}(\psi)$. Its integrated form is Wetterich's equation, which we do not need. An identity about the unknown ν_N, not a construction.

Strategy: an approximate force $\bar{\mathscr{F}}$ gives approximate forecast coordinates; the error should be driven by the failure of $\bar{\mathscr{F}}$ to solve this equation.

Approximate forecast and error

$\bar{\mathscr{F}}_q$ odd, $D\bar{\mathscr{F}}_q$ symmetric, $K_q + D\bar{\mathscr{F}}_q \geqslant \bar{c}_q I$, $\bar{c}_q > 0$. Recall $\psi_q = \mathbb{E}_q[\Phi_\infty]$, $\mathrm{Cov}_q = \mathrm{Var}_q(\Phi_\infty)$,

$$H_q = (K_q + D\bar{\mathscr{F}}_q)^{-1}, \quad \mathscr{D}_q = \partial_q \bar{\mathscr{F}}_q + \frac{1}{2}\mathrm{C}\mathrm{tr}_{H_q^2} D^2 \bar{\mathscr{F}}_q, \quad b_q = -H_q \mathscr{D}_q.$$

$$\Upsilon_q + C_q \bar{\mathscr{F}}_q(\Upsilon_q) = \Phi_q, \quad Z_q = \psi_q - \Upsilon_q, \quad \Xi_q = \mathrm{Cov}_q - H_q(\Upsilon_q).$$

Exact, at fixed N

$$\mathrm{d}\Upsilon_q = [b_q(\Upsilon_q) + H_q(\Upsilon_q) Z_q]\,\mathrm{d}q + H_q(\Upsilon_q)\,\mathrm{d}W_q, \qquad \Upsilon_{q_0} = 0$$

$$\mathrm{d}Z_q = -[b_q(\Upsilon_q) + H_q(\Upsilon_q) Z_q]\,\mathrm{d}q + \Xi_q\,\mathrm{d}W_q, \quad Z_\infty = \Phi_\infty - \Upsilon_\infty.$$

Proof: Itô on the coordinate equation, with the Polchinski process and $\mathrm{d}C_q = -C_q^2 \mathrm{d}q$: the martingale part gives H, the drift gives $HZ + b$.

Reading the system

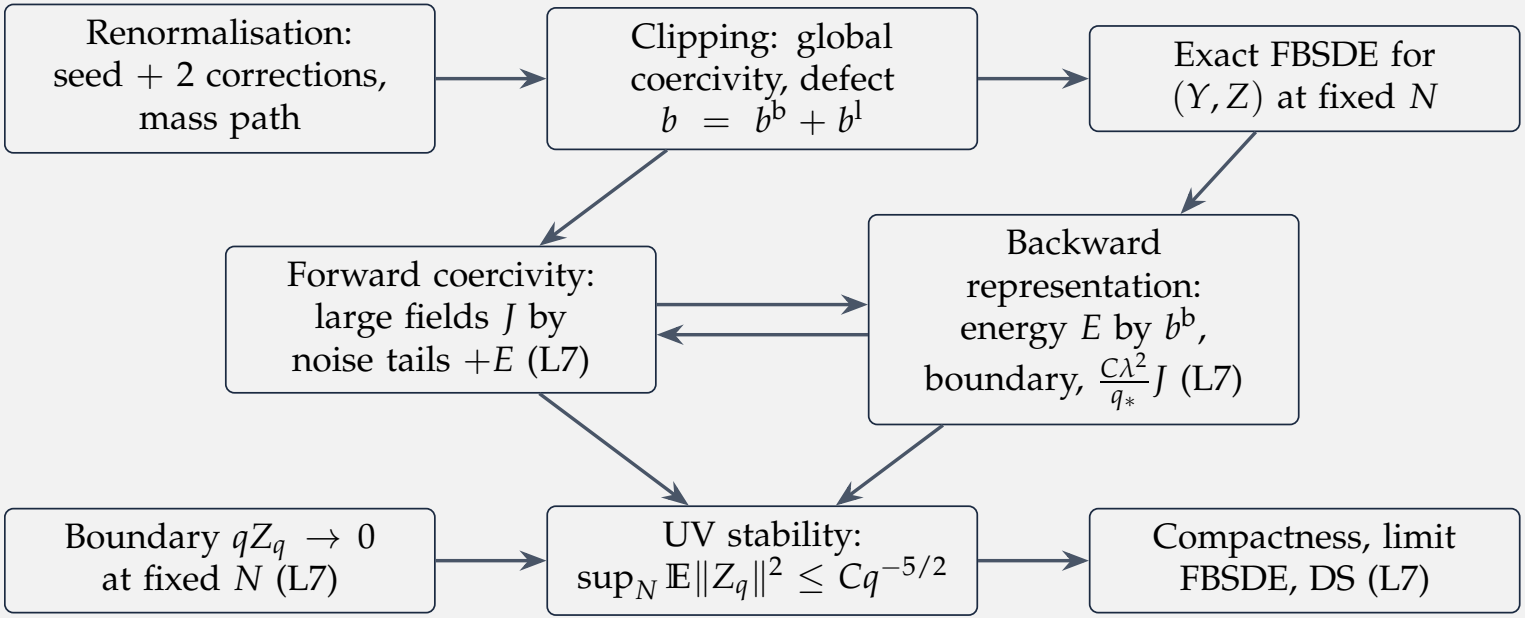
- Y: forward and explicit once Z is known; coefficients b, H built from $\bar{\mathscr{F}}$.
- Z: backward. The terminal condition $q\,Z_q \to 0$ is **proved** at fixed N (Lecture 7), because $\bar{\mathscr{F}}_q \to$ bare force.
- The only source of Z is the drift b, the force defect, transported by $\exp \int H$.

UV stability

The explicit force $\bar{\mathscr{F}}$ computes the forecast up to an error Z which is small uniformly in N.

Needed: (i) a defect integrable against the backward propagator (**renormalisation**); (ii) global coercivity of $\bar{\mathscr{F}}$ (**large fields**); (iii) a priori estimates (Lecture 7).

The structure of the construction

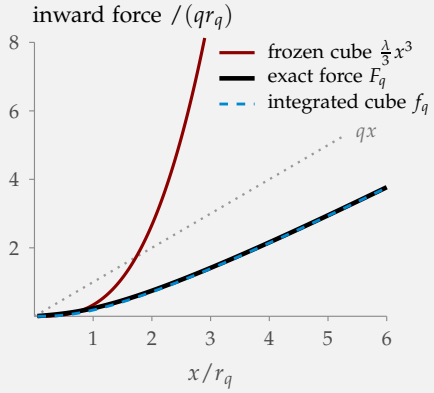


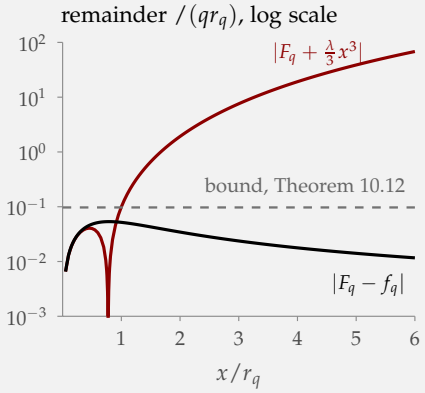
J: large-field functional, E: energy (both Lecture 7); $b^{\mathrm{b}}, b^{\mathrm{l}}$: bounded- and large-field parts of the drift.

Why not Polchinski with a frozen cube?

Programme: keep $-\frac{\lambda}{3}\varphi^3$ unclipped, clip the higher orders (result: f), pay in the source of $R = F - f$. Test it on one site: $K_q = q$, $S = 1/q$.

$$F_q(x) = -\frac{\lambda}{3}\mathbb{E}_x\Phi_\infty^3 = q(m_q(x) - x), \quad |m_q(x)| \leqslant 2(\lambda/3)^{-1/4} + 2(3q|x|/\lambda)^{1/3}.$$





So $3R/(\lambda x^3) \to 1$: the remainder **cancels the cube** (the dip: R changes sign, Wick term below, cube above). Field equation: $q\,x = (q - q_0)m_q + \omega_q$, only linear confinement.

The price of clipping

$f^c = f_1 + \mathscr{C} f_2$, with $f_1 = -\frac{\lambda}{3}(x^3 + 3Sx)$, f_2 the first tree, κ_q a smooth clip at $|x| \sim r_q$, $\mathscr{C} f_2 = \kappa_q' \cdot f_2 \circ \kappa_q$. For $|x| \geqslant 2r_q$, the Burgers defect is exactly

$$\mathscr{S}_q = S^2 f_1 f_1' \approx \frac{\lambda^2}{3} S^2 x^5,$$

of order $(q/\lambda)^{1/2}$ at the edge of the clip: not integrable in q.

$$\mathrm{d}R_q = -[\mathscr{S}_q + S^2(f^c)\,R_q]\,\mathrm{d}q + \Xi_q^R\,\mathrm{d}W_q, \quad \text{bracket} \approx \frac{\lambda^2}{3} S^2 x^5 - \lambda S^2 x^2 R.$$

The source and the dissipation cancel for the true $R \sim \frac{\lambda}{3} x^3$: separate estimates give nothing.

Tails independent of R: on one site from the posterior moment, but not uniform on the lattice (negative bare mass).

More orders do not help: the tree series has radius $\frac{2}{3} r_q$.

Integrating the cube

Solve the transport part of the flow exactly:

$$\partial_q f + S^2 f f' = 0, \quad f_\infty = -\frac{\lambda}{3} x^3 \quad \Rightarrow \quad f_q(x) = -\frac{\lambda}{3} y^3, \quad q\,y + \frac{\lambda}{3} y^3 = q\,x.$$

$$\sup_x |F_q(x) - f_q(x)| \leqslant 2C_*(\frac{\lambda}{3q})^{1/2}, \quad C_* < \frac{1}{2}.$$

- $y_q(\Phi_q) = Y_q$: **this is the forecast coordinate**. $F_q - f_q = K_q Z_q$, and Z is the posterior mean minus the mode.
- The defect is $q\,b_q(y)$: bounded for all fields. The trees are in $x \mapsto y$; the cube does not flow in y, so only loops are clipped.
- The cube is coercive in y: $q\,y + \frac{\lambda}{3} y^3 = q\,\Phi_q$.

Renormalisation: why two of them

Lecture 6

Approximating the exact force

We seek an explicit force with a small **defect**, not the full solution of the nonlinear flow.

$$\partial_q \mathscr{F}_q = -\frac{1}{2}\mathrm{C}\mathrm{tr}_{(H_q^{\mathrm{ex}})^2} D^2 \mathscr{F}_q, \qquad H_q^{\mathrm{ex}} = (K_q + D\mathscr{F}_q)^{-1}.$$

At infinite resolution: $\mathscr{F}_\infty = U' + \mu_N y$, with $U'(y) = \frac{\lambda}{3} y^3$.

1. Insert the bare cube into the response and the vertex: a **one-loop seed**.
2. Feed each approximation back into the equation: **two correction steps**.
3. Estimate the remaining defect in the backward equation for Z.

First separate the linear flow: it produces the divergent mass counterterm and cannot be treated as a small nonlinear error.

Why the linear term must be separated

Insert U' into the loop, with

$$G_q(y) = (K_q + \lambda y^2)^{-1}:$$

$$-\frac{1}{2}\mathrm{C}\mathrm{tr}_{G_q^2} D^2 U'(y)(x) = -\lambda y(x) G_q(y)^2(x,x).$$

Its linear Taylor coefficient is

$$-\lambda a_2(q) y(x), \qquad a_2(q) = \mathrm{C}_q^2(x,x).$$

In three dimensions $a_2(q)$ has continuum size $q^{-1/2}$: its integral to infinity diverges. At fixed lattice cutoff it is finite, but grows with N.

This is the **tadpole**. Further insertions also produce the logarithmic **sunset**. Keep the linear flow in a running mass; iterate only the nonlinear remainder.

The projections Π and Π'

For a smooth odd force f, and a translation-invariant operator $L = \langle L \rangle I + L^\circ$ (constant mode plus centred part):

$$(\Pi f)(y) = f(y) - \mathrm{Df}(0)y, \qquad (\Pi' f)(y) = f(y) - \langle \mathrm{Df}(0) \rangle y.$$

- Πf starts at **cubic order**; $\Pi' f = \Pi f + (\mathrm{Df}(0))^\circ y$ removes only the **scalar** part of the linear jet.
- The divergences (tadpole, sunset) are **constant modes**. The centred part of the linear flow is integrable at infinity.

$$\mathscr{F}_q = U' + M_q y + \ell_q, \qquad \Pi \ell_q = \ell_q, \quad M_q = \mu_q I + \Theta_q.$$

ℓ and the centred kernel Θ: integrate from ∞, by the same finite iteration. Scalar mass μ: solve exactly, fixed at an auxiliary large scale q_* (with g_{q_*} small).

Scales and power counting

$$g_q = \lambda q^{-1/2} (\text{dimensionless coupling}),$$

$$r_q = (q/\lambda)^{1/2} (\lambda r_q^2 = q : \text{nonlinearity} \approx \text{regulator}).$$

Free field at resolution q: free field X_q, $\mathrm{Var} X_q(x) \lesssim q^{1/2}$, typical size $q^{1/4} = g_q^{1/2} r_q \ll r_q$.

Rooted norm of a vertex $T(x_1, \dots, x_k)$ and field ball:

$$s(T) = \max_i \sup_{x_i} \int |T| \prod_{j \neq i} \mathrm{d}x_j, \qquad \mathsf{S}_k(f) = \sup_{\mathcal{B}_q} s\,(D^{k-1} f), \qquad \mathcal{B}_q = \{\|y\|_\infty < 2 r_q\}.$$

Natural size of the k-leg vertex of the **loop corrections**:

$$\mathsf{S}_k \lesssim g_q\, q\, r_q^{2-k} \qquad (k = 4 : \lambda\, g_q).$$

The loop estimate

Root one leg of a closed loop and follow the loop: two of its propagators are estimated in L^2 (this gives the bubble $a_2(q)$), all the others in operator norm (each q^{-1}).

Loop estimate

A closed loop with p vertices:

$$s(\text{loop}) \leqslant C a_2(q)\; q^{-(p-1)} \prod_i s(T_i)$$

(two propagators in L^2, the others in operator norm). After the q integration a one-loop term is of size $g_q q r_q^{2-k}$; every further loop gains a factor g_q.

Computing the one-loop seed

Project the bare loop before integrating:

$$\Pi[-\lambda y(x) G_q(y)^2(x,x)] = -\lambda y(x)[G_q(y)^2(x,x) - a_2(q)].$$

Use $\partial_q G_q = -G_q^2$ and $\partial_q C_q = -C_q^2$. With zero boundary value at infinity:

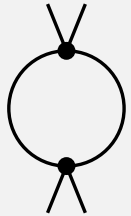
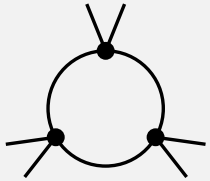
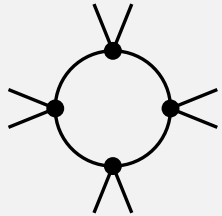
$$\ell_q^{[1]}(y)(x) = \lambda y(x)[G_q(y)(x,x) - C_q(x,x)].$$

The leading term is the bubble:

$$-\lambda^2 y(x) \int C_q(x,z)^2 y(z)^2 \mathrm{d}z.$$

Four-leg size: $\mathsf{S}_4(\ell_q^{[1]}) \lesssim \lambda^2 q^{-1/2} = \lambda g_q$. This is the seed $\ell^{[1]}$: the superscript counts loops.

The seed in graphs

$$\ell_q^{[1]}(y) \;=\; D\Big[-\tfrac{1}{4}$$  $$+\tfrac{1}{6}$$  $$-\tfrac{1}{8}$$  $$+\;\cdots\;\Big]$$

The seed as one-loop graphs: lines C_q, a vertex λy^2 with its two external fields at each corner. The j-th ring has $2j$ legs and coefficient $(-1)^{j+1}/(2j)$; D marks one leg, giving the force. The series sums to a logarithm, with radius of convergence r_q. The quartic ring is the bubble.

Splitting the flow and deriving the iteration

For $\mathscr{F} = U' + My + \ell$, the loop map

$$\mathscr{T}_q[M,\ell] = -\frac{1}{2}\mathrm{C}\mathrm{tr}_{R^2} D^2(U'+\ell), \quad R = (K_q + M_q + D(U'+\ell))^{-1}.$$

Three equations of different nature:

$$\partial_q \ell = \Pi\mathscr{T}[M,\ell], \quad \partial_q \Theta = (D\mathscr{T}[M,\ell](0))^\circ, \quad \partial_q \mu = \langle D\mathscr{T}[M,\ell](0)\rangle.$$

Fix a trial scalar path μ (determined at the end as a fixed point, "Closing the mass path"). Level masses $M^{[j]} = \mu I + \Theta^{[j]}$, sources $\mathfrak{S}^{[j]} = D\mathscr{T}[M^{[j]}, \ell^{[j]}](0)$, zero-field response $R_{0,q}^{[j]} = (K_q + M_q^{[j]})^{-1}$:

$$\ell_q^{[j+1]} = -\int_q^\infty \Pi\mathscr{T}_s[M^{[j]},\ell^{[j]}]\mathrm{d}s, \quad \Theta_q^{[j+1]} = -\int_q^\infty (\mathfrak{S}_s^{[j]})^\circ \mathrm{d}s, \quad j = 1,2.$$

With $\ell^{[0]} = 0$, $M^{[0]} = 0$, the case $j = 0$ is the seed step without mass, $\partial_q \ell^{[1]} = \Pi\mathscr{T}[0,0]$, and $\Theta^{[1]} = 0$ (the tadpole is local). Output: $(\ell^{[3]}, \Theta^{[3]})[\mu]$.

Centre before taking absolute values

A centred kernel $\Theta(z) = \Theta(0, z)$ is measured by its first moment $v_1(\Theta) = \int d(0,z)|\Theta(z)|\mathrm{d}z$, d the torus distance. For an even kernel, $\Theta 1 = 0$ means that the diagonal cancels the off-diagonal mass. With $(\sigma_z f)(x) = f(x+z)$, $\delta_z^2 = \sigma_z + \sigma_{-z} - 2I$,

$$C_q\Theta = \frac{1}{2}\int \Theta(z)\delta_z^2 C_q \mathrm{d}z.$$

The absolute row norm discards the cancellation. For a lattice step h in direction e, take

$$\Theta_h = \frac{a}{2h}(\sigma_{\mathrm{he}} + \sigma_{-\mathrm{he}} - 2I), \qquad v_1(\Theta_h) = a, \quad \|\Theta_h\|_{\mathrm{row}} = \frac{2a}{h}.$$

The row norm diverges as $h \to 0$; the first moment stays bounded.

Estimate the signed product $C_q\Theta$ first. The quantity needed by the loop is the kernel after resolvent smoothing, not the absolute mass of Θ itself.

Smoothing turns cancellation into a moment bound

The resolvent smooths on length scale $q^{-1/2}$:

$$\|\delta_z^2 C_q\|_{\mathrm{row}} \leqslant C q^{-1/2} d(0,z).$$

Scale check: integrate the heat-kernel translation bound $C d(0,z) t^{-1/2}$ against $\mathrm{e}^{-qt}\mathrm{d}t$.

$$\|C_q\Theta\|_{\mathrm{row}} \leqslant C q^{-1/2} \int d(0,z)|\Theta(z)|\mathrm{d}z = C q^{-1/2} v_1(\Theta).$$

The iteration (next slides) supplies exactly the compensating moment:

$$v_1(\Theta_q) \lesssim \lambda^2 q^{-1/2} \quad \Longrightarrow \quad q^{-1/2} v_1(\Theta_q) \lesssim \lambda^2 q^{-1} = g_q^2.$$

A bound by $d(0,z)^2$ loses a logarithm; one distance power, with $q^{-1/2}$, suffices (Lemma 19.3 of the Supplement).

The small factor in a resolvent comparison

The field-dependent response also receives this cancellation. With $V = \lambda y^2$ and $G = (K_q + V)^{-1}$,

$$G\Theta = (I - \mathrm{GV})C_q\Theta, \qquad \|I - \mathrm{GV}\|_{\infty\to\infty} \leqslant 2.$$

The bounded Neumann series for the full R then gives

$$\|R\delta M\|_{\infty\to\infty} \leqslant C\rho_q, \quad \rho_q = \frac{|\delta\mu_q|}{q} + q^{-1/2}v_1(\delta\Theta_q).$$

Holding the nonlinear correction fixed, $\delta R = -R\delta M\hat{R}$ ($\hat{R}$ the response at mass $M - \delta M$) has row norm $\lesssim \rho_q/q$ and row L^2 norm $\sup_x \|\delta R(x,\cdot)\|_{L^2} \lesssim \rho_q\sqrt{a_2(q)}$.

A marked propagator gains the relative factor ρ_q: for $\Theta^{[2]}$ in the second correction, $\rho_q \lesssim g_q^2$; for two scalar paths, see "Mass differences" below.

Centring the source removes local terms

Separate the source at level j into tadpole and correction vertex:

$$\mathfrak{S}_q^{[j]} = -\lambda \mathsf{Tr}((R_{0,q}^{[j]})^2) I + E_q, \quad E_q = -\frac{1}{2} \mathsf{C}\mathrm{tr}_{(R_{0,q}^{[j]})^2} T_q^{[j]}, \quad T_q^{[j]} = D^3 \ell_q^{[j]}(0).$$

Write $E_q 1 = e_q 1$. The constant mode contains the whole local contribution:

$$\langle \mathfrak{S}_q^{[j]} \rangle = -\lambda \mathsf{Tr}((R_{0,q}^{[j]})^2) + e_q, \qquad (\mathfrak{S}_q^{[j]})^\circ = E_q - e_q I.$$

The tadpole cancels; so does any term $c_q I$. Off-diagonal entries stay unchanged. Since $d(0,0) = 0$,

$$v_1((\mathfrak{S}_q^{[j]})^\circ) = v_1(E_q).$$

Centring removes the local obstruction. The remaining spatial estimate must come from the first moment of the nonlinear vertex.

The iteration produces the first-moment bound

Four-leg vertex $T_q^{[j]}$ of $\ell^{[j]}$, proved below level by level ($\Theta^{[2]}$ needs only the bubble $T^{[1]}$, $\Theta^{[3]}$ only $T^{[2]}$, at $M^{[1]}=\mu I$):

$$s(T_q^{[j]})+\sqrt{q}\,v_1(T_q^{[j]})\lesssim\lambda^2 q^{-1/2},\quad\text{hence}\quad v_1(T_q^{[j]})\lesssim\lambda^2 q^{-1}.$$

Keep the distance between the uncontracted legs of $T_q^{[j]}$; since $\sup_{u,v}|(R_{0,q}^{[j]})^2(u,v)|\lesssim q^{-1/2}$,

$$v_1((\mathfrak{S}_q^{[j]})^\circ)=v_1(E_q)\lesssim q^{-1/2}v_1(T_q^{[j]})\lesssim\lambda^2 q^{-3/2}.$$

Integrate with zero boundary value at infinity:

$$v_1(\Theta_q^{[j+1]})\leqslant\int_q^\infty v_1((\mathfrak{S}_s^{[j]})^\circ)\mathrm{d}s\leqslant C\lambda^2 q^{-1/2}.$$

Uniformly in N: $\eta_q(\Theta^{[2]})\lesssim g_q^2$, and one more g for the difference, $\eta_q(\Theta^{[3]}-\Theta^{[2]})\lesssim g_q^3$, with $\eta_q(\Theta)=q^{-1/2}v_1(\Theta)$.

First correction: insert the seed

Set $R^{[j]} = (K + M^{[j]} + \mathrm{D}U' + D\ell^{[j]})^{-1}$ and $\delta^{[2]} = \ell^{[2]} - \ell^{[1]}$. Subtract the bare loop from the loop containing the seed:

$$\delta_q^{[2]} = \frac{1}{2}\int_q^\infty \Pi[\mathrm{Ctr}_{(R^{[1]})^2 - G^2} D^2 U' + \mathrm{Ctr}_{(R^{[1]})^2} D^2 \ell^{[1]}]_s \mathrm{d}s.$$

Two changes: the propagators and the vertex.

$$R^{[1]} - G = -R^{[1]}(\mu + D\ell^{[1]})G, \qquad M^{[1]} = \mu I.$$

The insertion $(\mu + D\ell^{[1]})/q$ is $O(g_q)$. One extra loop gives

$$\mathsf{S}_4(\delta_q^{[2]}) \lesssim \int_q^\infty g_s s^{1/2} r_s^{-4} \mathrm{d}s = \lambda^3 q^{-1} = \lambda g_q^2.$$

All factors inside the integral are evaluated at scale s.

Second correction: insert the first correction

Put $\delta^{[3]} = \ell^{[3]} - \ell^{[2]}$. Subtract the first two Picard equations:

$$\delta_q^{[3]} = \frac{1}{2}\int_q^\infty \Pi[\mathrm{Ctr}_{(R^{[2]})^2-(R^{[1]})^2} D^2(U' + \ell^{[1]}) + \mathrm{Ctr}_{(R^{[2]})^2} D^2\delta^{[2]}]_s \mathrm{d}s.$$

The marked insertion is now the first correction itself, nonlinear and centred:

$$R^{[2]} - R^{[1]} = -R^{[2]}(\Theta^{[2]} + D\delta^{[2]})R^{[1]},$$

where $\Theta^{[2]}$ costs $\eta_q(\Theta^{[2]}) \lesssim g_q^2$ (above).

$$\mathsf{S}_4(\delta_q^{[3]}) \lesssim \int_q^\infty g_s^2 s^{1/2} r_s^{-4} \mathrm{d}s = \frac{2}{3}\lambda^4 q^{-3/2}.$$

Four-leg sizes: seed λg_q, first correction λg_q^2, second correction λg_q^3.

For squared propagators use $A^2 - B^2 = A(A-B) + (A-B)B$; no commutativity is assumed.

Following a spatial distance through a loop

At zero field, set $W_q(T) = s(T) + \sqrt{q}\, v_1(T)$. Mark two external legs and follow their distance around the loop.

- A displacement inside a vertex is controlled by that vertex's first moment.
- A displacement along a propagator is controlled by its weighted L^1 or L^2 bound. The triangle inequality distributes the external distance over these pieces.

For p vertices and p propagators:

$$W_q(\text{loop}) \lesssim q^{-1/2} q^{-(p-1)} \prod_{i=1}^{p} W_q(T_i).$$

The weight $\sqrt{q}$ makes a distance of order $q^{-1/2}$ dimensionless: the weighted loop has the same scale powers as the unweighted loop.

Keep within-vertex moments: smoothing the propagators cannot supply spatial localisation for an arbitrary nonlocal vertex.

Integrating preserves the moment weight

A loop difference with relative size ε_s has a marked vertex or propagator. With $\delta\mathcal{T}_s = \mathcal{T}_s[M,\ell] - \mathcal{T}_s[M,\hat{\ell}]$, the weighted comparison gives

$$W_s(\mathrm{D}^{k-1}\delta\mathcal{T}_s(0)) \lesssim \varepsilon_s s^{1/2} r_s^{-k}.$$

For $s \geqslant q$, $\sqrt{q} \leqslant \sqrt{s}$:

$$W_q\Big(\int_q^\infty A_s \mathrm{d}s\Big) \leqslant \int_q^\infty (s(A_s) + \sqrt{q} v_1(A_s))\mathrm{d}s \leqslant \int_q^\infty W_s(A_s)\mathrm{d}s.$$

First correction: $\varepsilon_s \lesssim g_s$. Second correction: $\varepsilon_s \lesssim g_s^2$.

Π subtracts only a linear force, so it leaves all $k \geqslant 4$ vertices unchanged. The field supremum also propagates, since $\mathcal{B}_q \subseteq \mathcal{B}_s$.

The four-leg moment after two corrections

Write $T_q^{[j]} = \mathrm{D}^3 \ell_q^{[j]}(0)$. The seed is the bubble:

$$W_q(T_q^{[1]}) \lesssim \lambda^2 \int (1 + \sqrt{q} d(0,z)) C_q(z)^2 \mathrm{d}z \lesssim \lambda^2 q^{-1/2}.$$

For the increments, integrate the weighted loop bounds:

$$W_q(T_q^{[2]} - T_q^{[1]}) \lesssim \lambda^3 \int_q^\infty s^{-2} \mathrm{d}s = \lambda^3 q^{-1},$$

$$W_q(T_q^{[3]} - T_q^{[2]}) \lesssim \lambda^4 \int_q^\infty s^{-5/2} \mathrm{d}s = \frac{2}{3} \lambda^4 q^{-3/2}.$$

For $j = 1,2,3$, $s(T_q^{[j]}) + \sqrt{q} v_1(T_q^{[j]}) \lesssim \lambda^2 q^{-1/2}(1 + g_q + g_q^2) \lesssim \lambda^2 q^{-1/2}$.

The $10 \to 8 \to 6$ leg budget supplies the higher vertices needed by both comparisons. Thus $v_1(T_q^{[j]}) \lesssim \lambda^2 q^{-1}$.

The mass

Impose the **scalar** part of the linear flow for the final approximation $M = M^{[3]} = \mu I + \Theta^{[3]}$, with $R^{[3]}_{0,q} = (K_q + M_q)^{-1}$:

$$\partial_q \mu_q = \beta_q = \langle \mathfrak{S}^{[3]}_q \rangle, \quad \mathfrak{S}^{[3]}_q = -\lambda \mathrm{Tr}((R^{[3]}_{0,q})^2) I - \frac{1}{2} \mathrm{Ctr}_{(R^{[3]}_{0,q})^2} D^3 \ell^{[3]}_q(0),$$

with $\mu_{q_*} = b_*$, $|b_*| \leqslant B_0 \lambda q_*^{1/2}$, at an **auxiliary scale** q_*. The centred part of $\mathfrak{S}^{[3]}$ is not imposed. Bare mass $\mu_N = \lim_{q \to \infty} \mu_q$: an output of the flow. β depends on the whole path μ through $\ell^{[3]}[\mu]$: a fixed point ("Closing the mass path").

The defect after the second correction

The bounded-field approximation:

$$\mathscr{F}_q^{o} = U' + M_q y + \ell_q^{[3]}, \qquad \partial_q \mu = \langle D\mathscr{T}_q[M, \ell^{[3]}](0) \rangle.$$

The approximations use the preceding level, $\partial_q \ell^{[3]} = \Pi \mathscr{T}[M^{[2]}, \ell^{[2]}]$, $\partial_q \Theta^{[3]} = (\mathfrak{S}^{[2]})^{\circ}$. Hence

$$\mathscr{D}_q^{o} = \Pi'(\mathscr{T}_q[M^{[2]}, \ell^{[2]}] - \mathscr{T}_q[M^{[3]}, \ell^{[3]}]):$$

nonlinear part and a **centred linear part** $\mathscr{E}_q^{c} y$; the scalar part vanishes by the mass equation.

One loop with $\delta^{[3]}$ and $\Theta^{[3]} - \Theta^{[2]}$ inserted, **without a scale integration**:

$$\mathsf{S}_4(\mathscr{D}_q^{o}) \lesssim \lambda^5 q^{-3}, \qquad \|\mathscr{E}_q^{c}\|_{\mathrm{row}} \lesssim g_q^4.$$

Why stop after two corrections?

The constant and scalar linear jets of the defect vanish, so on the field ball

$$|b_s| \lesssim s^{-1}(\mathsf{S}_4(\mathscr{D}_s)r_s^3 + \|\mathscr{E}_s^c\|_{\mathrm{row}} r_s), \qquad Z_q \approx \int_q^\infty \frac{s}{q} b_s \mathrm{d}s.$$

Both terms have the same size; s/q is the backward transport. The cubic source:

corrections	$\mathsf{S}_4(\mathscr{D}_s)$	backward source integral
0: seed	$\lambda^3 s^{-2}$	$\lambda^{3/2} q^{-1} \int_q^\infty s^{-1/2} \mathrm{d}s = \infty$
1	$\lambda^4 s^{-5/2}$	$\lambda^{5/2} q^{-1} \int_q^\infty s^{-1} \mathrm{d}s = \infty$
2	$\lambda^5 s^{-3}$	$\lambda^{7/2} q^{-1} \int_q^\infty s^{-3/2} \mathrm{d}s = 2\lambda^{7/2} q^{-3/2}$

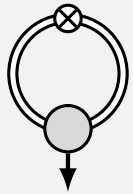
One correction is logarithmically borderline. **Two corrections beyond the seed suffice** for the bounded-field source estimate.

Each step costs two field derivatives: keep the iteration finite. Full estimates: Lecture 7.

The corrections in graphs

(a) one step of the iteration

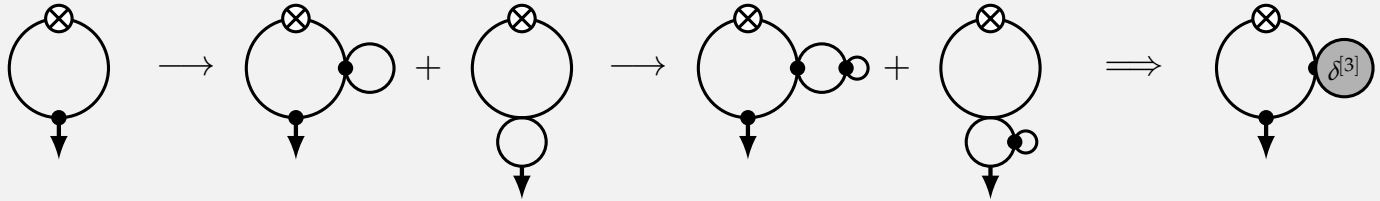
$$\partial_q \ell_q^{[j+1]} \;=\; -\tfrac{1}{2}\,\Pi$$



- vertex $D^2(U' + \ell^{[j]})$
- line $R = (K_q + M_q^{[j]} + D(U' + \ell^{[j]}))^{-1}$
- $\partial_q K_q = 1$, the derivative of the scale
- root: the output slot of the force

$\ell^{[j]}$, $j \geq 1$, enters twice: in the lines and in the vertex

(b) expanded in bare lines $G_q = (K_q + DU')^{-1}$ (single lines; external field legs not drawn)



seed $\delta^{[1]} = \ell^{[1]}$
one loop, λg_q

$\delta^{[2]}$: $\ell^{[1]}$ inserted
two loops, λg_q^2

$\delta^{[3]}$: $\delta^{[2]}$ inserted
three loops, λg_q^3

defect $\mathcal{D}_q^{\mathrm{o}}$, no q-integral
$\mathsf{S}_4(\mathcal{D}_q^{\mathrm{o}}) \lesssim \lambda^5 q^{-3}$

Each step inserts the previous correction into the loop: one more loop, one more g_q. Two steps give a defect of order g_q^3.

Closing the mass path

Given a trial **scalar** path μ, compute the two steps of the iteration, corrections and centred kernels, then

$$\beta_q[\mu] = \langle D\mathcal{T}_q[M^{[3]}[\mu], \ell^{[3]}[\mu]](0)\rangle, \qquad \mathcal{A}[\mu]_q = b_* + \int_{q_*}^{q} \beta_s[\mu]\mathrm{d}s.$$

The mass is the **fixed point** $\mu = \mathcal{A}[\mu]$: a scalar function of q, not an operator-valued path. It is the only fixed point we solve; the centred kernels are explicit outputs of two Picard steps.

This is an equation for the whole path: the corrections and centred kernels at q depend on μ at later scales.

What the fixed point gives

For q_* large, uniformly in N:

- a unique scalar path in the ball $\|\mu\|_* \leqslant R$ (norm on the next slide), so $|\mu_q| \lesssim \lambda q^{1/2}$;
- centred kernels: $v_1(\Theta_q^{[j]}) \lesssim \lambda^2 q^{-1/2}$;
- increments: $\mathcal{N}_{k,q}(\ell^{[j]} - \ell^{[j-1]}) \lesssim g_q^j q\, r_q^{2-k}$, $j = 1,2,3$, $4 \leqslant k \leqslant 12-2j$, where

$$\mathcal{N}_{k,q}(f) = \mathsf{S}_k(f) + \sqrt{q}\, v_1(D^{k-1} f(0));$$

- defect: $\mathsf{S}_4(\mathscr{D}_q^o) \lesssim \lambda^5 q^{-3}$, $\|\mathscr{E}_q^c\|_{\mathsf{row}} \lesssim g_q^4$.

Explicit counterterm. $\mu_N = r - \lambda \mathrm{Tr}\, C + \frac{2}{3}\lambda^2 \int C^3$ is reached by a unique boundary value $b_{*,N}$, and $b_{*,N}$ converges as $N \to \infty$. The measure does not depend on q_*.

Two distinct iterations

Inner recursion: for each trial scalar path μ, start from the seed and compute exactly two steps.

$$\mu \longmapsto (\ell^{[2]}, \Theta^{[2]})[\mu] \longmapsto (\ell^{[3]}, \Theta^{[3]})[\mu] \longmapsto \beta[\mu] \longmapsto \mathcal{A}[\mu].$$

Outer contraction: start at $\mu_q^{(0)} = b_*$; iterate $\mu^{(n+1)} = \mathcal{A}[\mu^{(n)}]$, in the norm $\|\mu\|_* = \sup_{q \geqslant q_*} |\mu_q| / (\lambda \sqrt{q})$:

$$\|\mathcal{A}[\mu] - \mathcal{A}[\hat{\mu}]\|_* \leqslant C g_{q_*} \|\mu - \hat{\mu}\|_*.$$

Every outer step recomputes the same two-level force. Its depth stays 2, so no additional field derivatives are consumed. At the fixed point the scalar linear flow is exact; the defect, nonlinear and centred linear, remains.

Mass differences gain a small insertion

Compare two scalar paths with the same b_*, $d = \|\mu - \hat{\mu}\|_*$; δ denotes output differences, $\delta M^{[j]} = \delta\mu I + \delta\Theta^{[j]}$.

$$R^{[j]} - \hat{R}^{[j]} = -R^{[j]}(\delta M^{[j]} + D\delta\ell^{[j]})\hat{R}^{[j]}.$$

The relative scalar and centred insertions obey

$$\frac{|\delta\mu_q|}{q} \leqslant g_q d, \qquad \eta_q(\delta\Theta^{[j]}) \lesssim g_q^3 d,$$

the latter by integrating the differences of the centred sources from infinity. Loop comparisons through the two levels give $\mathcal{N}_{k,q}(\delta\ell^{[j]}) \lesssim g_q^2 q\, r_q^{2-k} d$, $j = 2,3$, $4 \leqslant k \leqslant 12 - 2j$ (the seed does not depend on μ).

The resulting source differences are

$$|\delta\beta_q| \lesssim \lambda^2 q^{-1} d, \qquad v_1(\delta(\mathfrak{S}_q^{[j]})^\circ) \lesssim \lambda^3 q^{-2} d.$$

Integration leaves the factor g_{q_*}

Scalar update: the common boundary value b_* cancels. With $x = q/q_* \geqslant 1$,

$$|\delta\mathcal{A}[\mu]_q| \leqslant C\lambda^2 d \int_{q_*}^{q} \frac{\mathrm{d}s}{s} = C\lambda^2 \log x \, d,$$

$$\frac{|\delta\mathcal{A}[\mu]_q|}{\lambda\sqrt{q}} \leqslant C g_{q_*} \frac{\log x}{\sqrt{x}} d \leqslant \frac{2C}{\mathrm{e}} g_{q_*} d.$$

The centred kernels are not part of the unknown: their variation is already contained in $\delta\beta$.

Contraction constant $C g_{q_*}$. Fix an invariant ball, then choose q_* so that $C g_{q_*} < 1/2$, uniformly in N.

Why the derivative budget stays fixed

A loop needs $k+2$ legs to control a k-leg vertex. Every evaluation of $\mathscr{A}$ uses the same budget:

$$\ell^{[1]}\colon 10 \longrightarrow \ell^{[2]}\colon 8 \longrightarrow \ell^{[3]}\colon 6 \longrightarrow \text{defect: } 4.$$

A k-leg force vertex has $k-1$ field derivatives. The mass source uses only the four-leg vertex of $\ell^{[3]}$, the centred sources those of $\ell^{[1]}$, $\ell^{[2]}$.

- A path difference inserts $\delta M^{[j]}$ in a resolvent. It is field-independent: $D_y \delta M^{[j]} = 0$. The comparison uses the same field-derivative ranges as the construction.
- $\mathscr{A}$ returns a **scalar path**. At the next outer step, restart from the original seed and perform the same two steps. We do not use the previous $\ell^{[3]}$ as a new seed.

The outer iteration contracts in one fixed space of scalar paths. The finite Lipschitz estimates transfer convergence to the corrections; their derivative budget does not shrink with the outer index.

Two mass renormalisations

The two divergent self-energies of Appendix A.2 of Lecture 5:

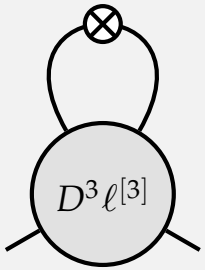
- **Tadpole**: $-\lambda \int_{q_*}^{\infty} \mathrm{Tr} R_{0,q}^2 \, \mathrm{d}q \approx -\lambda \mathrm{Tr} C \sim -\lambda N$. Wick ordering: $\frac{\lambda}{12}\varphi^4 - \frac{1}{2}\lambda c \varphi^2 = \frac{\lambda}{12} H_4(\varphi; c) + \text{const.}$
- **Sunset**: contracting the bubble in $D^3 \ell^{[3]}(0)$ with R_0^2 and integrating gives the two-loop self-energy,

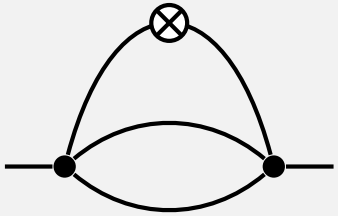
$$\frac{2}{3}\lambda^2 \int C^3 \approx \frac{\lambda^2}{24\pi^2} \log N \quad (= 2\gamma_T \text{ of Lecture 5}).$$

Power counting: nothing else. In the flow: the integrals from $q = \infty$ of all vertices with $k \geqslant 4$ legs converge; only the constant mode of the linear jet needs a boundary condition at a finite scale.

μ_q at fixed q has a limit as $N \to \infty$; μ_N does not. **Never interchange the two limits.**

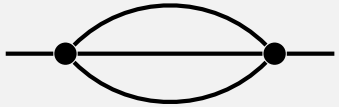
The two mass renormalisations in graphs

$$\partial_q \mu_q \;=\; -\tfrac{1}{2}$$  $$-\;\tfrac{1}{2}$$ 

$$\ni$$ 

tadpole, $D^4U = 2\lambda$

sunset

$$\mu_N \;\approx\; -$$  $$+$$ 

$\lambda \operatorname{Tr} C \sim \lambda N$

$\frac{2}{3}\lambda^2 \int C^3 \sim \lambda^2 \log N$

the q-integral removes
the cross: $\int C_q^2\, dq$ closes a line

The cross is $\partial_q K_q = 1$, as before; with the two lines through it, it gives $(R_{0,q}^{[3]})^2$. The bubble in $D^3\ell^{[3]}(0)$ produces the sunset.

The flow at large fields

Lecture 6

The large-field problem

All bounds hold on $\mathcal{B}_q$ only. The free field exceeds r_q with small, nonzero probability, and nothing a priori controls the interacting forecast. We need $\bar{\mathcal{F}}_q$ defined and uniformly coercive for all fields.

Scalar model

$$\ell_q^{[1]}(y) = \lambda y[\,(q+\lambda y^2)^{-1} - q^{-1}]$$

Its power series has radius exactly r_q (singularities at $\pm \mathrm{i} r_q$). Half of the truncations destroy monotonicity. The exact function is harmless, $|\ell_q^{[1]\prime}| \leqslant \frac{9}{8}\lambda/q$.

Perturbative control for $|y| \lesssim r_q$; outside, a non-perturbative replacement that keeps the structure the estimates use: coercivity, carried by the bare cube.

One site: $q y + \frac{\lambda}{3} y^3 = q\,\Phi_q$, noise coefficient $H_q(y) = (q+\lambda y^2)^{-1} \leqslant q^{-1} \wedge (\lambda y^2)^{-1}$. Large fields are more elliptic and less noisy.

Clipping

τ smooth odd, $\tau(u)=u$ on $[-1,1]$, constant outside $[-2,2]$, $0\leqslant\tau'\leqslant 1$;

$$\kappa_q(y)=r_q\tau(y/r_q),\quad \tilde{\ell}_q=D\kappa_q\cdot \ell_q^{[3]}\circ\kappa_q,\quad \bar{\mathscr{F}}_q=U'+M_qy+\tilde{\ell}_q.$$

The cubic and linear parts are not clipped, and $D\kappa_q$ keeps the force a gradient. Globally: $s(D^{j-1}\tilde{\ell}_q)\leqslant C g_q q\, r_q^{2-j}$ $(j=2,3,4)$, $s(D\partial_q\tilde{\ell}_q)\leqslant C g_q$.

Global coercivity

For all y: $c(K_q+\lambda y^2)\leqslant H_q(y)^{-1}\leqslant C(K_q+\lambda y^2)$,

$$\|H_q(y)\|_{p\to p}\leqslant \frac{1+C(g_q+g_q^2)}{q},\quad \operatorname{diag} H_q(y)^2\leqslant C a_2(q).$$

Neumann series for $H=(I+G(M+D\tilde{\ell}))^{-1}G$, $G=(K_q+\lambda y^2)^{-1}\leqslant C_q$.

The complete defect

$b = b^{\mathsf{b}} + b^{\mathsf{I}}$, $b^{\mathsf{b}} = -H\, D\kappa\, \mathscr{D}^{\circ}(\kappa y)$, $t_q(y) = y\, 1_{|y|>r_q}$.

Proposition. (uniformly in y, N, p)

$$\|b_q^{\mathsf{b}}\|_p \leqslant C\lambda^{7/2} q^{-5/2}, \qquad \|b_q^{\mathsf{I}}\|_p \leqslant C\lambda\, q^{-3/2}(\|t_q\|_p + r_q^{-1}\|y^2 1_{|y|>r_q}\|_p),$$

$$\|b_q^{\mathsf{I}}\|_2^2 \leqslant C\lambda^4 q^{-5}\|t_q\|_6^6.$$

Sources of b^{I} (four groups): the mass term $M' y - J M'\, \kappa y$ ($J = D\kappa$, $M' = \partial_q M$; linear part unclipped, flow computed at κy); the motion of the clip; cubic vertex at y vs correction at κy; chain-rule terms with κ'', κ'''. All vanish if $\|y\|_\infty \leqslant r_q$.

The point

b^{I} is as large as the whole drift, but **supported on large fields**. It is controlled by how often the actual dynamics visits them: $\int q^{-3/2}\lambda \|t_q(Y_q)\|_6^6\, \mathrm{d}q$ (Lecture 7).

The coordinate equation

With $U' = \frac{\lambda}{3} y^3$, **up to a small Lipschitz perturbation, the forecast solves the cubic elliptic equation driven by the field at resolution q**:

$$K_q \Phi_q = K_q \left(Y_q + \varrho_q(Y_q) \right) + \frac{\lambda}{3} Y_q^3,$$

$$\varrho_q(y) = C_q \left(M_q y + \tilde{\ell}_q(y) \right), \quad \| D\, \varrho_q \|_{p \to p} \leqslant C \left(g_q + g_q^2 \right).$$

The force below the auxiliary scale

The expansion needs g_q small, so $\bar{\mathscr{F}}_q$ is built for $q \geqslant q_*$. Below q_* nothing needs constructing: freeze it,

$$\bar{\mathscr{F}}_q(y) = \bar{\mathscr{F}}_{q_*}(y) + (q_* - q)y.$$

Then $K_q + D\bar{\mathscr{F}}_q = K_{q_*} + D\bar{\mathscr{F}}_{q_*}$: coercivity is automatic, $H_q = H_{q_*}$, and $y_q(x) = y_{q_*}(C_{q_*}K_q x)$.

The defect is explicit: $\mathscr{D}_s(y) = -y + \frac{1}{2}\mathsf{C}\mathrm{tr}_{H^2_{q_*}} D^2\bar{\mathscr{F}}_{q_*}(y)$, with $\|\mathscr{D}_s(y)\|_p \leqslant C\|y\|_p$.

So the system holds on $[q_0, \infty)$ with $\Upsilon_{q_0} = 0$: the initial value is deterministic, with no large mass. q_* stays only in the proofs, where the expansion and the closure of the loop need g_q small.

Summary of today

At every lattice:

- μ_N, M_q and a globally coercive $\bar{\mathscr{F}}_q$, all deterministic;
- the exact FBSDE for (Y, Z);
- a source whose bounded-field part is integrable (two steps), and whose large-field part is controlled by sixth powers of Y on large fields.

Next: estimates on (Y, Z) uniform in N.