A Constructive Introduction to Quantum Field Theory

Φ^4_3 on the torus II: a priori estimates and the limit

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SISSA lectures, Lecture 7 — notes: Lecture 7, §§13–18

Notation from Lecture 6

- **Lattice**: $\Lambda_N = (N^{-1}\mathbb{Z}/\mathbb{Z})^3$, $\int f = N^{-3}\sum_x f(x)$, $\|\cdot\|_p$ the L^p norms, $\Delta = \Delta_N$.
- **Free operators**: resolution $q \in [q_0, \infty)$, $K_q = q - \Delta$ (precision), $C_q = K_q^{-1}$ (unresolved covariance), $\|C_q\|_{p\to p} = q^{-1}$; $C = C_{q_0}$ the Gaussian reference, $K_{q_0} = q_0 - \Delta$ the free field operator, $q_0 > 0$ the reference mass.
- **Couplings**: $U = \frac{\lambda}{12}\int \varphi^4$, $g_q = \lambda q^{-1/2}$, $r_q = (q/\lambda)^{1/2}$ (cube $\asymp K_q$), auxiliary scale q_*.
- **Processes**: Polchinski process Φ_q ($\Phi_{q_0} = 0$), innovation W, bridge noise B, forecast $\psi_q = \mathbb{E}_q \Phi_\infty$, Cov_q its conditional covariance, filtration $\mathcal{F}_q = \sigma(\Phi_s : s \leqslant q)$; $\psi_q = m_q(\Phi_q)$, $Y_q = y_q(\Phi_q)$.
- **Forces**: exact $\mathscr{F}_q$ ($K_q\Phi_q = K_q\psi_q + \mathscr{F}_q(\psi_q)$), explicit $\bar{\mathscr{F}}_q(y) = \frac{\lambda}{3}y^3 + M_q y + \tilde{\ell}_q(y)$; response $H_q = (K_q + D\bar{\mathscr{F}}_q)^{-1}$, defect drift b_q, $\Xi_q = \mathsf{Cov}_q - H_q(Y_q)$, coordinate correction ϱ_q.

Where we are

At each lattice N, exactly (Lecture 6):

$$\mathrm{d}Y_q = [b_q(Y_q) + H_q(Y_q)\, Z_q]\mathrm{d}q + H_q(Y_q)\,\mathrm{d}W_q,$$
$$\mathrm{d}Z_q = -[b_q(Y_q) + H_q(Y_q)\, Z_q]\mathrm{d}q + \Xi_q\,\mathrm{d}W_q,$$

$$\psi = Y + Z = \int \mathrm{Cov}\,\mathrm{d}W.$$

A) Coercivity: $H_q(y) \asymp (K_q + \lambda y^2)^{-1}$, $\|H_q\|_{p\to p} \leqslant (1+\varepsilon_q)/q$, $\varepsilon_q = C(g_q + g_q^2)$.

B) Source: $\|b_q^{\mathsf{b}}\|_\infty \leqslant C\lambda^{7/2} q^{-5/2}$, $\|b_q^{\mathsf{I}}(y)\|_2^2 \leqslant C\lambda^4 q^{-5}\|t_q(y)\|_6^6$.

C) Coordinates: $K_q(Y_q + \varrho_q(Y_q)) + \frac{\lambda}{3} Y_q^3 = K_q \Phi_q$, $\|D\,\varrho_q\|_{p\to p} \leqslant \varepsilon_q$.

Goal

Z small and Y rarely at large fields, **uniformly in N**. Neither equation can be estimated alone.

How FBSDE estimates usually work, and why not here

Usual (Ma–Yong): Lipschitz coefficients, Itô on $\|Z\|^2$ and Gronwall; close on a short interval or under monotonicity.

Here: infinite interval, coefficients not uniformly Lipschitz, the coupling is not small. Instead we use **structure**:

- the backward equation is **linear in Z** given Y: a conditional representation with an explicit propagator;
- the forward equation is, by the field equation of Lecture 5, a cubic **elliptic** equation at each resolution: coercive, estimated **pathwise**;
- the two estimates are linked by two weighted energies with **matched weights**.

At fixed N, T all moments are finite (finite dimensions): used only to justify absorptions. Ma–Yong 1999

The energies

Lecture 7

The noise of the field equation

Lecture 6: $\Phi_q = (q - q_0) C_q \Phi_\infty + C_q B_{q-q_0}$, a noisy measurement. Since $\psi_q = \mathbb{E}_q \Phi_\infty$,

$$\omega_q := K_q \Phi_q - (q - q_0)\, \psi_q = \mathbb{E}_q[B_{q-q_0}]:$$

the noise observable of Lecture 5 is the **filtered measurement noise**.

Lemma. *For convex Ψ, $\mathbb{E}\, \Psi(\omega_q) \leqslant \mathbb{E}\, \Psi(B_{q-q_0})$, whatever the interaction. So $\zeta_q = C_q\, \omega_q$ has*

$$\mathbb{E}\, \mathrm{e}^{u \zeta_q(x)} \leqslant \exp\left(\frac{u^2}{2} (q - q_0)\, a^2(q) \right), \qquad (q - q_0)\, a^2(q) \lesssim \sqrt{q}.$$

- Proof: conditional Jensen. One line.
- ζ is not Gaussian, but it has the **tails of the free field**, uniformly in N and in the interaction.

The field equation

Lecture 5, forecast form, in this profile: $K_{q_0}\psi_q + \mathscr{F}_q(\psi_q) = \omega_q$, with the exact (unknown) force. Explicit force $\bar{\mathscr{F}}_q = K_q \varrho_q + \frac{\lambda}{3} y^3$, $\psi = \Upsilon + Z$:

$$\boxed{K_{q_0}\Upsilon_q + \bar{\mathscr{F}}_q(\Upsilon_q) = \omega_q + (q - q_0)\, Z_q}$$

- Elliptic at each q, **no memory**, **pathwise**; coercivity from the explicit **cube**.
- The backward equation enters only through $(q - q_0)\, Z_q$.
- Pointwise bounds need a smooth noise: $\zeta = C_q \omega$, $V = \Upsilon + \varrho(\Upsilon) - \zeta$,

$$K_{q_0} V_q + \frac{\lambda}{3} \Upsilon_q^3 = (q - q_0)\, (\zeta_q - \varrho_q(\Upsilon_q) + Z_q).$$

Mass only q_0: all the coercivity comes from the cube.

Thresholds and energies

Thresholds: $w_q = \operatorname{sgn}(V_q)\left(|V_q| - \frac{3}{4} r_q\right)_+$, $\quad t_q = Y_q 1_{|Y_q| > r_q}$, $\quad \xi_q = \zeta_q 1_{|\zeta_q| > r_q/32}$.

$$E_T = \frac{1}{\lambda} \int q^{1/2} \|Z_q\|_2^2 \, \mathrm{d}q \qquad \text{backward energy}$$

$$J_T = \int q^{-3/2} \left[q \, \|w_q\|_4^4 + \lambda \|w_q\|_6^6 + \lambda \|t_q\|_6^6 \right] \mathrm{d}q \qquad \text{large-field action}$$

$$G_T = \int \left[q^{-1/2} \|\xi_q\|_4^4 + \lambda \, q^{-3/2} \|\xi_q\|_6^6 \right] \mathrm{d}q \qquad \text{noise tails}$$

quantity	weight	reason
$\|Z_q\|^2$	$q^{1/2}$	forcing: $q^{-3/2} \cdot \lambda^{-1} q^2 \|Z\|_2^2$
$\lambda \|t_q\|_6^6$	$q^{-3/2}$	$q^{7/2} \|b^{\mathsf{I}}\|^2 \leqslant C \lambda^3 \cdot \lambda \, q^{-3/2} \|t\|_6^6$
$\|b_q^{\mathsf{I}}\|^2$	$q^{7/2}$	propagator s/q, $\int_q^\infty s^{-3/2} = 2 q^{-1/2}$

Only **thresholded** sixth powers: $\mathbb{E}\|\zeta_q\|_6^6 \sim q^{3/2}$ would give $G_T \sim \lambda N^2$.

Forward coercivity

Lecture 7

The scalar model

Lemma. (scalar forward coercivity) *If* $a \geqslant 0$, $0 \leqslant \omega \leqslant q$ *and* $av + \frac{\lambda}{3}(v+g)^3 = \omega(g+z)$, *then, with* $r = (q/\lambda)^{1/2}$,

$$\frac{q}{8} w^4 + \frac{\lambda}{16} w^6 \leqslant 54 q \xi^4 + \frac{4225}{6} \lambda \xi^6 + \frac{12}{\lambda} q^2 z^2,$$

where $w = \operatorname{sgn}(v)\,(|v| - \frac{3}{4} r)_+$, $\xi = g\, 1_{|g|>r/32}$; *and* $|t| \leqslant \frac{32}{7}(|w| + |\xi|)$.

- The excess is controlled **at each resolution separately** by the noise tails and by the error Z.
- No smallness of the coupling, and no integration in q: multiply by $q^{-3/2}$ only to match the energies.

Proof: test with w^3

Where $w \neq 0$: $|v| = |w| + \frac{3}{4} r$, same sign, so $a\, v\, w^3 \geqslant 0$.

- **Cube.** $\operatorname{sgn}(w)\,(v+g) \geqslant |w| + \frac{11}{16} r - |\xi|$, and $\lambda r^2 = q$:

$$\frac{\lambda}{3} w^3 (v+g)^3 \geqslant \frac{\lambda}{12} w^6 + \frac{q}{4} w^4 + \frac{q\, r}{13} |w|^3 - \frac{4225}{6} \lambda\, \xi^6.$$

- **Noise.** $\varpi\, g\, w^3 \leqslant \frac{q\, r}{32} |w|^3 + q|\xi||w|^3$: the first term is absorbed by the cube, and $q|\xi||w|^3 \leqslant \frac{q}{8} w^4 + 54\, q\, \xi^4$.
- **Error.** $q|z||w|^3 \leqslant \frac{\lambda}{48} w^6 + \frac{12}{\lambda} q^2 z^2$.

Above the threshold the cube provides a sixth power, a mass q and a cubic term which dominates the noise below the threshold.

Two elementary inequalities

For $x, \rho, u, r \geqslant 0$:

$$(x-\rho)^3 \geqslant \frac{3}{4}x^3 - 65\,\rho^3,$$

$$\frac{1}{3}u^3\left(u+\frac{11}{16}r-\rho\right)^3 \geqslant \frac{1}{12}u^6 + \frac{1}{4}r^2u^4 + \frac{1}{13}r^3u^3 - \frac{4225}{6}\rho^6.$$

Proof. $\frac{1}{4}x^3 + 64\,\rho^3 - 3\,x^2\rho = \frac{1}{4}(x-8\,\rho)^2(x+4\,\rho) \geqslant 0$. Then $x = u+b$, $b = \frac{11}{16}r$: $\frac{3}{4}b^2 > \frac{1}{4}r^2$, $\frac{1}{4}b^3 > \frac{1}{13}r^3$, and $\frac{65}{3}u^3\rho^3 \leqslant \frac{1}{6}u^6 + \frac{4225}{6}\rho^6$.

The lattice version

Proposition. (forward coercivity) *For q_* large, at every N, pathwise and for every q,*

$$q\|w_q\|_4^4+\lambda\|w_q\|_6^6+\lambda\|t_q\|_6^6\leqslant C\,(q\|\xi_q\|_4^4+\lambda\|\xi_q\|_6^6+\lambda^{-1}q^2\|Z_q\|_2^2).$$

So $J_T\leqslant C\,(G_T+E_T)$, and $\|G_T\|_{L^p}\leqslant C\lambda^3 p^5$.

- **Elliptic term:** $w^3=\phi'(V)$, ϕ convex, so $\langle w^3,K_{q_0}V\rangle\geqslant 0$.
- **Coordinate correction:** $\varrho=\varrho^0+\varrho^1$, $\|\varrho^0\|_\infty\leqslant\varepsilon r$ shifts the threshold, $\|\varrho^1\|_j\leqslant C\,\varepsilon\,(\|w\|_j+\|\xi\|_j)$ is absorbed.
- **Noise tails:** $\mathbb{P}\,(|\zeta_q(x)|>r_q/32)\leqslant 2\,\mathrm{e}^{-c\sqrt{q}/\lambda}$, by the lemma.

L^∞ coercivity

Proposition. *For q_* large, at every N, pathwise and for every q,*

$$\|Y_q\|_\infty \leqslant C\left[\left(\frac{q}{\lambda}\right)^{1/3}\|\zeta_q\|_\infty^{1/3} + \|\zeta_q\|_\infty + q^{1/4} + q\|Z_q\|_2\right].$$

So $t_q = 0$ whenever $\|\zeta_q\|_\infty + q\|Z_q\|_2 \leqslant c\, r_q$.

- Remove the error: $\mathscr{H} = (q - q_0)\, C_{q_0} Z_q$, with $\|\mathscr{H}\|_\infty \lesssim q\|Z_q\|_2$.
- At a maximum of $W = V - \mathscr{H}$, the elliptic term has a sign: $\frac{\lambda}{3} Y^3 \leqslant q\|d\|_\infty$. **The cube alone gives the bound.**
- **Large fields occur only when the noise or the error is large.**
- With $\|\zeta_q\|_\infty \lesssim q^{1/4}$ and $\|Z_q\|_2 \lesssim q^{-5/4}$ (Doob on $\mathbb{E}_q A^{\mathsf{I}}$): $\|Y_q\|_\infty \lesssim q^{5/12} \ll r_q$, up to logarithms.

Backward feedback and closure

Lecture 7

The backward representation

Propagator: $\partial_s \mathcal{P}_{q,s} = \mathcal{P}_{q,s} H_s(Y_s)$, $\mathcal{P}_{q,q} = I$, and by (A)

$$\|\mathcal{P}_{q,s}\| \leqslant L_0 \frac{s}{q}, \qquad L_0 = \exp\left(\int_{q_*}^{\infty} \varepsilon_u \frac{\mathrm{d}u}{u}\right) \leqslant \mathrm{e}^{C(g_{q_*} + g_{q_*}^2)}.$$

Lemma. (conditional representation)

$$Z_q = \mathbb{E}_q\left[\mathcal{P}_{q,T} Z_T + \int_q^T \mathcal{P}_{q,s}\, b_s(Y_s)\, \mathrm{d}s\right].$$

Proof: $\mathrm{d}_s(\mathcal{P}_{q,s} Z_s) = -\mathcal{P}_{q,s} b_s \mathrm{d}s + \mathcal{P}_{q,s} \Xi_s \mathrm{d}W_s$, a true martingale at fixed N.

Z = expected source on the unresolved scales, transported back, plus the boundary error. Compare $F_t = \mathbb{E}_t[F_T]$ in Lecture 4: here the source is the **defect**.

Three contributions

Bounded fields (two correction steps!):

$$\|Z_q^{\mathrm{b}}\| \leqslant \int_q^\infty L_0 \frac{s}{q}\, C\lambda^{7/2} s^{-5/2} \mathrm{d}s = 2C L_0 \lambda^{7/2} q^{-3/2}, \qquad \frac{1}{\lambda}\int q^{1/2}\|Z^{\mathrm{b}}\|^2 \leqslant C\lambda^6 q_*^{-3/2}.$$

Boundary: $q\|Z_q^{\mathrm{t}}\| \leqslant L_0\, \mathbb{E}_q \|T Z_T\|$, contributing $\frac{2L_0^2}{\lambda q_*^{1/2}}\delta_{N,T,p}$, $\delta_{N,T,p} = \|T Z_T\|^2_{L^{2p}(\mathbb{P};L^2)}$.

Large fields: $A^{\mathrm{l}} = \int s^{7/2}\|b_s^{\mathrm{l}}\|^2 \leqslant C\lambda^3 J_T$; Cauchy–Schwarz and Jensen,

$$\|Z_q^{\mathrm{l}}\|^2 \leqslant 2L_0^2 q^{-5/2}\, \mathbb{E}_q A^{\mathrm{l}}, \qquad \left|\frac{1}{\lambda}\int q^{1/2}\|Z^{\mathrm{l}}\|^2\right|_{L^p} \leqslant \frac{C\lambda^2}{q_*}\|J_T\|_{L^p}.$$

This is how the clipping error is controlled

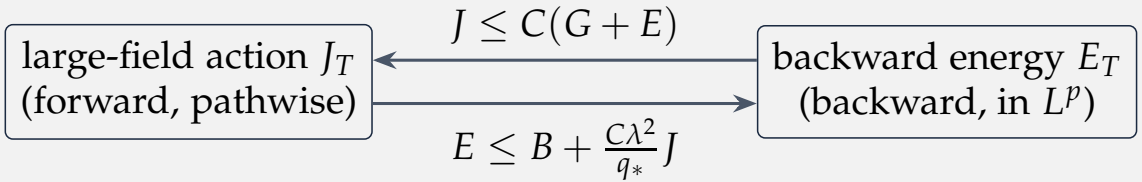
b^{l} is not small, but it enters the backward energy only through the large-field action J_T, with the small factor λ^2/q_*.

Closing the loop

Theorem. (cutoff-uniform energy estimate) *One $q_* = q_*(\lambda)$: for all N, T, $p \geqslant 1$,*

$$\|E_T\|_{L^p} + \|J_T\|_{L^p} \leqslant C_\lambda\,(1 + p^5 + \delta_{N,T,p}).$$

$x = \|E_T\|_p$, $y = \|J_T\|_p$: $\quad y \leqslant A_0\,(C\,\lambda^3\,p^5 + x), \quad x \leqslant B_0 + B_1\,\delta + \theta\,y, \quad \theta = C\,\lambda^2\,/\,q_* = C\,g_{q_*}^2$. Choose $A_0\,\theta \leqslant \frac{1}{2}$; $y < \infty$ at fixed N, T, absorb.



The smallness is in the **feedback**, independent of p: one q_* for all p; the growth p^5 is inherited from G_T.

Consequences

Once the boundary term is removed (Section 16):

$$\sup_N \|E_\infty^N\|_{L^p} + \sup_N \|J_\infty^N\|_{L^p} \leqslant C_\lambda\,(1+p^5), \qquad \sup_N \mathbb{E}\|Z_q^N\|_2^2 \leqslant C_\lambda\, q^{-5/2},$$

and $\mathbb{E}\exp\{c\,(E_\infty + J_\infty)^{1/5}\} < \infty$.

- The approximate forecast differs from the exact one by $O(q^{-5/4})$ in $L^2(\mathbb{P}; L^2)$ (root mean square), uniformly in the lattice.
- $H_q Z_q = O(q^{-9/4})$ in $L^2(\mathbb{P}; L^2)$: up to an integrable drift, the forward equation is the explicit SDE $\mathrm{d}Y = b(Y)\mathrm{d}q + H(Y)\mathrm{d}W$.

Comparison with the variational method

Comparison with Boué–Dupuis (Lecture 4)

There: drift $u = -\frac{\lambda}{3}\dot{Q}^{1/2}{:}X^3{:} + r$ ($\frac{\lambda}{12}\int {:}\varphi^4{:}$ in Lecture 4). Here: $\bar{\mathscr{F}}$ does all such changes of variables at once in the forecast coordinate; Z plays the role of r. Coercivity of the quartic: **pathwise** in the forward equation; the remainder: **in expectation** through the backward one.

Barashkov–G. 2020

Removing the cutoffs

So far, at every lattice N and upper scale T:

$$\|E_T\|_{L^p} + \|J_T\|_{L^p} \leqslant C_\lambda \, (1 + p^5 + \delta_{N,T,p}), \qquad \delta_{N,T,p} = \|T Z_T\|^2_{L^{2p}(\mathbb{P};L^2)}.$$

Three things remain:

1. the boundary term vanishes: $T Z_T \to 0$ at fixed N;
2. uniform energy bounds $\Rightarrow$ tightness of the measures;
3. identify the limit: coefficients, the explicit density, then the equations.

The subtle point

Convergence of the fields does not pass an FBSDE to the limit: the backward martingale could acquire **extra noise**. Here the limit is a density over an explicit flow, and the equations are read off from it.

The boundary at infinite resolution

Lecture 7

The boundary lemma

$\bar{\mathscr{F}}_q$ was built to converge to the bare force: $\mu_q \to \mu_N$, $\Theta_q \to 0$, corrections integrated from $q = \infty$.

Lemma. (fixed N, constants depend on N)

$$\|Y_q - \Phi_\infty\|_{L^p} + \|q\, Z_q\|_{L^p} \leqslant C_{N,p}\, q^{-1/2}, \qquad \textit{hence} \qquad \delta_{N,T,p} \leqslant C_{N,p}\, T^{-1}.$$

- The field at resolution q: $\Phi_q = (q - q_0)\, C_q\, \Phi_\infty + C_q\, B_{q-q_0} \to \Phi_\infty$.
- $\Phi_q = Y_q + C_q\, \bar{\mathscr{F}}_q(Y_q)$, so $Y_q - \Phi_q = O(q^{-1})$ and $\bar{\mathscr{F}}_q(Y_q) \to D\, V_N(\Phi_\infty)$.
- Exact minus approximate coordinate identity:

$$q\, Z_q = q\, C_q\, [\bar{\mathscr{F}}_q(Y_q) - \mathbb{E}_q\, D\, V_N(\Phi_\infty)];$$

compare $\mathbb{E}_q\, G$ with the **observed** $G_q^0 = D\, V_N(\Phi_q)$: $\|\mathbb{E}_q\, G - G\| \leqslant 2\, \|G - G_q^0\|$.

No uniform estimate, no posterior concentration.

UV stability

Let $T \to \infty$ at fixed N (paths and ν_N do not depend on T; monotone convergence; $\delta_{N,T,p} \to 0$). **Only then** take $\sup_N$.

Theorem. (UV stability) *For $q_* = q_*(\lambda)$ large,*

$$\sup_N \left(\|E_\infty^N\|_{L^p} + \|J_\infty^N\|_{L^p}\right) \leqslant C_\lambda\,(1+p^5), \qquad \sup_N \mathbb{E}\|Z_q^N\|_2^2 \leqslant C_\lambda\, q^{-5/2},$$

and $\sup_N \mathbb{E}\exp\{c\,(E_\infty^N + J_\infty^N)^{1/5}\} < \infty$.

- $C_{N,p}$ diverges with N: never take $\sup_N$ of $C_{N,p}/T$.
- The force: $F_q = \mathbb{E}_q[-D\,V_N(\Phi_\infty)] = K_q Z_q - \bar{\mathscr{F}}_q(Y_q)$, the explicit force at the approximate forecast, corrected by $K_q Z_q$.

Restarted laws

The estimates use the start of the flow only through the noise. Let $\mathbb{P}^{q,x}$ be the exact flow conditioned on $\Phi_q = x$: $\Phi_t = C_t(K_q x + (t-q)\hat{\Phi} + B'_{t-q})$, with prior $\propto \mathrm{e}^{\langle z, K_q x\rangle - \frac{1}{2}\langle z, K_q z\rangle - V_N(z)}\,\mathrm{d}z$.

The same posterior, the same system, and $\zeta_t = C_t K_q x + \zeta^0_t$ with $\|C_t K_q x\|_\infty \leqslant 2\|x\|_\infty$. The field equation is $K_q V_t + \frac{\lambda}{3} Y_t^3 = (t-q)(\zeta_t - \varrho_t(Y_t) + Z_t)$.

Restarted UV stability. Uniformly in N and $\|x\|_\infty \leqslant R$, the bounds on E, J and $\mathbb{E}\|Z_s\|_2^2 \leqslant C_R s^{-5/2}$ hold on $[q \vee q_*, \infty)$.

Below q_*: deduced. ψ is a martingale, so $\mathbb{E}\|\psi_s\|_2^{2p} \leqslant \mathbb{E}\|\psi_{q_*}\|_2^{2p}$. The coordinate equation at q_* bounds Y_s, and $Z_s = \psi_s - Y_s$.

The role of the estimates

- **Noise bound:** free-field tails for ζ, whatever the interaction.
- **Forward coercivity:** the cube bounds large fields by noise tails and Z.
- **Backward representation:** Z is the transported defect; $C\lambda^2/q_*$ closes the loop.
- **Boundary:** at fixed N. Why the upper scale goes first.
- **UV stability:** the control from the explicit to the exact flow has finite entropy.
- **Restarted laws:** the same from any field at any scale; gives the kernels.
- **Defect, Lipschitz coordinate:** $\int |e|$ integrable; the explicit flow is a strong SDE.
- **Terminal tail:** tightness in H^{-s}. **Sources:** tails, end results only.

The formula for ν needs the first five at the start of the flow, with first moments. The FBSDE follows by Girsanov.

Tightness

Lecture 7

Suprema by a maximum principle

Embed by cells $\square_x$: $\iota_N : H_N \to H = L^2(\mathbb{T}^3)$ isometric, $j_N = \iota_N^*$.

Lemma. (supremum after closure) *Let* $\mathcal{Y} = \sup_q q^{-5/12} (\log(2+q))^{-1/6} \|Y_q\|_\infty$, $\mathcal{Z} = \sup_q q^{5/4} \|Z_q\|_2$. *Uniformly in* N,

$$\mathcal{Y} \leqslant C_\lambda (1 + \mathcal{N} + \mathcal{Z}), \qquad \|\mathcal{N}\|_{L^p} \leqslant C\sqrt{p}, \qquad \|\mathcal{Z}^2\|_{L^p} \leqslant C_\lambda (1 + p^5).$$

- From the L^∞ coercivity above: **lattice maximum principle** in the field equation, no memory.
- $\mathcal{N}$: supremum of the noise (convex domination, Dudley, Doob). $\mathcal{Z}$: Doob on $\mathbb{E}_q A^{\mathrm{I}}$, after the boundary term is removed.
- The exponent $5/12 < \frac{1}{2}$ keeps Y below the threshold.

Finite-resolution suprema have all moments, uniformly in N. At the terminal resolution there is no such bound: Φ_∞ is a distribution.

The terminal field

Embed by cells: $\iota_N : H_N \to H = L^2(\mathbb{T}^3)$ isometric, $j_N = \iota_N^*$.

$H_q^2 \leqslant C\, q\, C_q$ gives, for $\frac{1}{2} < s < \frac{3}{2}$,

$$\int_Q^\infty \|(1-\Delta)^{-s/2} H_q\|_{\mathrm{HS}}^2 \,\mathrm{d}q \leqslant C \int_Q^\infty \frac{1}{q} \sum_k \frac{(1+|k|^2)^{-s}}{q + c|k|^2} \,\mathrm{d}q \leqslant C_s Q^{-\left(s-\frac{1}{2}\right)},$$

and the drifts have squared tails $O(Q^{-3})$, $O(Q^{-5/2})$, $O(Q^{-3/2})$. Doob:

$$\sup_N \mathbb{E} \sup_{t \geqslant Q} \|Y_t - Y_Q\|_{H^{-s}}^2 \leqslant C_s Q^{-\left(s-\frac{1}{2}\right)}.$$

ν_N is tight in $H^{-s}(\mathbb{T}^3)$ for every $s > \frac{1}{2}$, with moments uniform in N. (Threshold of the free field in $d = 3$.)

Identifying the limit

Lecture 7

The deterministic coefficients

Uniform bounds do not identify a limit: a weakly convergent vertex cannot in general be contracted against a varying covariance. Three tools:

- **Weak vertex measures**: bounded rooted norms $\Rightarrow$ signed measures, weakly compact; the centred kernel in a first-moment topology. The limits solve the same fixed point, unique in its ball: the whole sequence converges.
- **Compact sandwich**: $G_N \to G$ in HS, $\mathsf{B}_N \to \mathsf{B}$ weakly, symmetric, uniformly bounded, $\sup_N \|G_N \mathsf{B}_N\| \leqslant \theta < 1$, $\|G\mathsf{B}\| \leqslant \theta \Rightarrow (I + G_N \mathsf{B}_N)^{-1} G_N \to (I + G\mathsf{B})^{-1} G$ in HS.
- **Contraction against a moving covariance**: rows of the inverses are Hölder + HS convergence $\Rightarrow (H^N)^2 \to H^2$ uniformly; weak vertices can be contracted.

Hence $H_q^N(Y_q^N) \to H_q(Y_q)$ in $L^2(I;\mathsf{HS})$ and $b_q^N(Y_q^N) \to b_q(Y_q)$ in $L^2(I;H)$.

The exact flow as a controlled explicit flow

$K_q y + \bar{\mathscr{F}}_q(y) = K_q x$, $\bar{f}_q(x) = K_q(y - x)$: the explicit force solves Burgers with a source $-K_q H_q \mathscr{D}_q \circ y$.

The first scalar. $\mathscr{D}$ and $\bar{f}$ are gradients: $e_q = \int_0^1 \langle \mathscr{D}_q(ty), y \rangle \, \mathrm{d}t$, $D\bar{V} = -\bar{f}$, and

$$\partial_q \bar{V} + \frac{1}{2}\mathrm{Tr}(C_q^2 D^2 \bar{V}) - \frac{1}{2}\|C_q D\bar{V}\|_2^2 = e_q \circ y + c(q).$$

Explicit flow $\mathbb{R}^{q,x}$: $\mathrm{d}x = -C_s(x - y_s(x))\,\mathrm{d}s + C_s\,\mathrm{d}W$, whose coordinate solves $\mathrm{d}Y = b(Y)\,\mathrm{d}s + H(Y)\,\mathrm{d}W$. It contains no Z.

y_s is Lipschitz for the uniform norm (maximum principle), uniformly in N: a strong continuum SDE.

The exact flow is a controlled explicit flow. $\Delta = V - \bar{V}$ solves a Hamilton–Jacobi–Bellman equation with control u, drift $C_s u_s$, and running cost $e + c + \frac{1}{2}\|u\|^2$. Its optimal feedback is $-C_s D\Delta_s = Z_s$.

Density and sandwich

With $A_N(q,x) = V_q(x) - \bar{V}_q(x) - \int_q^\infty c$, at fixed N:

$$\frac{\mathrm{d}\mathbb{P}^{q,x}}{\mathrm{d}\mathbb{R}^{q,x}} = \mathrm{e}^{A_N} \exp(-\int_q^\infty e_s(Y_s)\,\mathrm{d}s), \quad \mathsf{Ent}(\mathbb{P}^{q,x}|\mathbb{R}^{q,x}) = \frac{1}{2}\int_q^\infty \mathbb{E}\|Z_s\|_2^2\,\mathrm{d}s,$$

$$A_N = \mathsf{Ent} + \mathbb{E}^{\mathbb{P}}\int e, \quad \text{so} \quad \mathbb{E}^{\mathbb{P}}\int e \leqslant A_N \leqslant \mathbb{E}^{\mathbb{R}}\int e.$$

The defect on all fields: $|e_s(y)| \leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4 + C\lambda^4 s^{-2}\|\pi_s y\|_2^2 + C\lambda^3 s^{-5/2}\|t_s(y)\|_6^6$. So both sides are bounded uniformly in N, by the energies under $\mathbb{P}$ and by Itô under $\mathbb{R}$.

No penalty, no sub- or supersolutions, no exponential moment of the defect: only expectations of e, and the entropy of the control.

The limit law

Let $\iota_N x_N \to x$ uniformly. Couple the flows by the same W: paths converge uniformly, $\int e^N \to \int e$ in probability. **The entropy bound makes the densities uniformly integrable**, so they converge in L^1:

$$\Pi_q O(x) = \mathrm{e}^{A(q,x)} \mathbb{E}^{\mathbb{R}^{q,x}}\Big[O(Y_\infty)\mathrm{e}^{-\int_q^\infty e}\Big], \quad \mathrm{e}^{-A(q,x)} = \mathbb{E}^{\mathbb{R}^{q,x}}\mathrm{e}^{-\int_q^\infty e},$$

jointly continuous in (q,x).

The measure is the kernel at $(q_0,0)$. With $\mathbb{R}$ the flow from $Y_{q_0}=0$, along the whole sequence,

$$\int O\,\mathrm{d}\nu = \frac{\mathbb{E}^{\mathbb{R}}\Big[O(Y_\infty)\mathrm{e}^{-\int_{q_0}^\infty e}\Big]}{\mathbb{E}^{\mathbb{R}}\Big[\mathrm{e}^{-\int_{q_0}^\infty e}\Big]}, \quad -\log\int \mathrm{e}^{-V_N}\,\mathrm{d}\mu_C - \mathscr{C}_N(q_0) \to A(q_0,0).$$

As in Gubinelli–Meyer for sine–Gordon, but the defect is unbounded: the entropy replaces the bound.

The filtering model and the FBSDE

Under $\mathbb{P} = f\mathbb{R}$, with $f = \mathrm{e}^{A(q_0,0)}\mathrm{e}^{-\int e}$, $\Phi = Y_\infty$, $\mathscr{F}_q = \sigma(x_s : s \leqslant q)$:

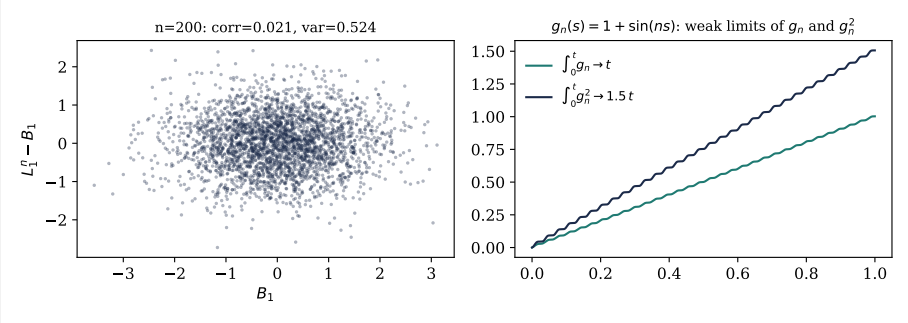
- $\Phi \sim \nu$, $x_q = (q - q_0)C_q\Phi + C_q B_{q-q_0}$ with B independent of Φ;
- $\psi_q = \mathbb{E}[\Phi|\mathscr{F}_q] = m_q(\Phi_q)$, a continuous function of the field;
- $W = P - \int \psi$ is a Brownian motion, and with $Y = y(x)$, $Z = \psi - Y$, the FBSDE holds from $Y_{q_0} = Z_{q_0} = 0$;
- $\psi = \int \mathrm{Cov}\,\mathrm{d}W$, no orthogonal part, Cov the conditional covariance;
- the lattice processes converge in law.

Proof. $\mathscr{F}$ is Brownian under $\mathbb{R}$. So $\mathrm{d}D = D\langle Z', \mathrm{d}W^{\mathbb{R}}\rangle$, and Girsanov gives $\mathrm{d}P = (Y + Z')\,\mathrm{d}s + \mathrm{d}W$. By independence of B the drift is ψ. For $M = N/D$, Itô gives $\mathrm{d}M = \langle n/D - MZ, \mathrm{d}W\rangle$.

Hidden noise

$L^n_t = \int_0^t g_n(s)\,\mathrm{d}B_s$, $g_n = 1 + \sin(ns)$: integrands $\to 1$ weakly, squares $\to \frac{3}{2}$.

$$(L^n, B) \Longrightarrow (B + 2^{-1/2}\tilde{B},\ B), \qquad \tilde{B} \perp B.$$



Left: $(B_1, L^n_1 - B_1)$, $n = 200$: correlation ≈ 0, variance of the gap $\approx \frac{1}{2}$ — the residual does not vanish and is orthogonal to B. Right: $\int_0^t g_n \to t$ but $\int_0^t g_n^2 \to \frac{3}{2}t$.

A limit by compactness could carry an extra orthogonal martingale. Here the density excludes it: the filtration is Brownian under the reference.

The renormalised field equation

Theorem. (integration by parts) $G(\varphi)=g(\langle\varphi,h_1\rangle,\dots,\langle\varphi,h_k\rangle)$, $g\in C_b^\infty$, h,h_i *smooth:*

$$\lim_N \mathbb{E}\left[G\left(\iota_N\Phi_\infty^N\right)\langle\mu_N\Phi_\infty^N+\frac{\lambda}{3}(\Phi_\infty^N)^3,j_Nh\rangle\right]$$

$$=\mathbb{E}_\nu\left[D_hG(\Phi)-G(\Phi)\langle\Phi,(q_0-\Delta)\,h\rangle\right].$$

Proof: Gaussian integration by parts in $\mathrm{e}^{-V_N-\frac{1}{2}\langle\varphi,K_{q_0}\varphi\rangle}$; tightness and uniform moments.

- The terminal case of the DS equations of Lecture 5.
- The renormalised cube $\mu_N\Phi+\frac{\lambda}{3}\Phi^3$ is identified only **tested in expectation**, not as a random distribution.
- Limiting force without terminal representative: **not proved**.
- At every finite resolution the force is explicit: $F_q=K_qZ_q-\bar{\mathcal{F}}_q(Y_q)$.

The field equation at resolution q

In the limit: $\omega_q = K_q\Phi_q - (q-q_0)\,\psi_q$, $F_q = K_q(\psi_q - \Phi_q)$, $J_q = (q-q_0)\,C_q \to I$.

Theorem. *For every q, with no renormalisation constant, pathwise as distributions,*

$$K_{q_0}\Phi_q = J_q F_q + \omega_q, \qquad \mathbb{E}\, D_{J_q h} G(\Phi_q) = \mathbb{E}[G(\Phi_q)\langle \omega_q, h\rangle].$$

$\langle F, h\rangle$, $\langle \omega, h\rangle$ are martingales, and $\|\langle F_q + \bar{\mathscr{F}}_q(Y_q), h\rangle\|_{L^2} \lesssim \|h\|_{H^2} q^{-1/4}$.

- **Proof of DS:** $\omega_q = \mathbb{E}_q B_{q-q_0}$ and $\Phi_q = J_q\Phi_\infty + C_q B_{q-q_0}$: Gaussian integration by parts in B.
- **The renormalised cube is an observable:** the unique martingale in the class of the explicit $-\bar{\mathscr{F}}_q(Y_q)$.
- $q \to \infty$ recovers the integration by parts formula. Not proved: that F is closed.

Quartic tails: the tilt

Hairer–Steele proved quartic tails by tilting the dynamics. Here: the filter is available for every prior, and the noise bound does not depend on the prior.

Let $\hat{\nu} \propto \mathrm{e}^{h\langle \varphi, f\rangle} \nu_N$. This is the flow restarted at $(q_0, hC_{q_0} f)$. **Shift the coordinate:**

$$K_q Y_q + \bar{\mathcal{F}}_q(Y_q) = K_q \Phi_q + hf, \quad Z_q = \hat{\psi}_q - Y_q.$$

The system is the same, $T Z_T \to 0$ at fixed N, and the field equation gains hf: the noise gains $d_q = hC_q f$, $\|d_q\|_\infty \leqslant |h| \|f\|_\infty / q$.

So UV stability holds on $[q_h, \infty)$ when $q_h^{3/2} \geqslant 70 \lambda^{1/2} |h| \|f\|_\infty$: where the cube at its threshold, $q\, r_q$, dominates the source.

An exact identity, with no cube and no derivative: $\hat{\Phi} = Y_q + Z_q + (\omega_q - B_{q-q_0}) / (q - q_0)$.

Quartic tails: closing

Test the identity with f, $\|f\|_* = \|f\|_1 + \|f\|_2 + \|f\|_\infty = 1$, and average over $[q_h, 2q_h]$, $X = \langle \hat{\Phi}, f \rangle$:

$$\hat{\mathbb{E}}|X| \leqslant C r_{q_h} + C \leqslant C(1+|h|)^{1/3}.$$

As $L(h) = \log \mathbb{E} e^{h\langle \Phi, f \rangle}$ has $L'(h) = \hat{\mathbb{E}} X$, we get $L(h) \leqslant C(|h| + |h|^{4/3})$. Chernoff:

$$\mathbb{E}\exp(\beta \langle \Phi, f \rangle^4 / \|f\|_*^4) \leqslant C, \quad \sup_N \mathbb{E}\exp(\beta_\rho \|\iota_N \Phi\|_{H^{-\rho}}^4) < \infty, \quad \rho > \frac{3}{2}.$$

Gaussian tails at the free threshold. A quadratic tilt $W = \frac{\beta}{2}\|\iota_N \varphi\|_{H^{-s}}^2$, with boundary term DW, gives $\sup_N \mathbb{E}\exp(\beta_s \|\iota_N \Phi\|_{H^{-s}}^2) < \infty$, $s > \frac{1}{2}$.

The quartic moment at s near $\frac{1}{2}$ would need spatially resolved sources.

Dyson–Schwinger along the scales

DS along the scales. For $G = g(\langle \Phi_{q_1}, h_1 \rangle, \dots, \langle \Phi_{q_k}, h_k \rangle)$, $q_i \leqslant q$:

$$\mathbb{E} \sum_i \partial_i g \langle J_{q_i} h, h_i \rangle = \mathbb{E}[G \langle \omega_q, h \rangle] = \mathbb{E}[G(\langle \Phi_q, K_{q_0} h \rangle + \langle \bar{\mathscr{F}}_q(Y_q), J_q h \rangle)] + O(q^{-1/4}).$$

So the explicit force $\bar{\mathscr{F}}_q(Y_q)$ closes the hierarchy at large q, up to an error which vanishes as $q \to \infty$.

Uniqueness in law and the variational formula

Scalar form. $A_t = A(t, x_t)$ solves

$$A_t = \int_t^\infty \left(e_s(Y_s) + \frac{1}{2}\|Z_s\|_2^2 \right) \mathrm{d}s + \int_t^\infty \langle Z_s, \mathrm{d}W_s \rangle.$$

With the forward equation $\mathrm{d}P = (Y + Z)\,\mathrm{d}s + \mathrm{d}W$ this has **uniqueness in law**: Girsanov removes Z, pathwise uniqueness gives $\mathbb{R}$, and the scalar equation gives the density. Z is the feedback $m(x) - y(x)$.

Renormalised variational formula. The Gibbs principle over $\mathbb{R}$:

$$-\log \int \mathrm{e}^{-O}\,\mathrm{d}\nu = \mathscr{G}(O) - \mathscr{G}(0), \quad \mathscr{G}(O) = \inf_u \mathbb{E}\left[O(Y_\infty^u) + \int_{q_0}^\infty \left(e_s(Y_s^u) + \frac{1}{2}\|u_s\|_2^2 \right) \mathrm{d}s \right],$$

$\mathrm{d}Y^u = (b(Y^u) + H(Y^u)u)\,\mathrm{d}s + H(Y^u)\,\mathrm{d}W$, $Y_{q_0}^u = 0$. Minimiser for $O = 0$: $u = Z$. **No infinite constant, no initial law to optimise.**

What is not proved

- **The vector FBSDE alone:** Z need not be a gradient. It would suffice that any solution has a Lipschitz decoupling field: a pointwise bound on $\mathrm{Cov}_q(x)$, or high-scale contraction with non-degeneracy. Only $L^2(\mathrm{HS})$ control is available.
- **Strong solutions:** is the Polchinski process adapted to W? The same bound. Allinger–Mitter does not apply, as $\|\Phi\|_2 = \infty$.
- **Tails:** the quartic moment at s near $\frac{1}{2}$.
- **Axioms:** reflection positivity holds on the lattice and passes to ν for every coordinate reflection (standard; exercise); rotation invariance and clustering need infinite volume.
- **Infinite volume:** the force and its Burgers flow are intensive, but the entropy and the defect are extensive, so estimates must be made local. The noise from infinity in the nonlocal Ξ must also be controlled.

Summary of the course

Wightman → Euclidean (L1) → free field (L2) → stochastic quantisation, scale decomposition (L3) → Polchinski, FBSDE, variational problem (L4) → field equation at every resolution, sub-Gaussian noise (L5) → Φ^4_3 through the FBSDE in forecast coordinates (L6–7).

Lecture 4 (formal)	Lectures 6–7	in the limit
forecast $\psi_t = \mathbb{E}_t \Phi_T$	$\psi = Y + Z$	closed martingale
response $H_t = \mathrm{Cov}_t$	$\mathrm{Cov} = H(Y) + \Xi$	conditional covariance
$\mathscr{F}_t$, force equation	$\bar{\mathscr{F}}_q$ explicit, defect $\mathscr{D}$	deterministic coefficients
force F_t	$K_q Z_q - \bar{\mathscr{F}}_q(Y_q)$	observable at finite q
DS equation	integration by parts	tested in expectation

UV stability = an explicit force predicts the conditional mean up to a small error. So the measure is an explicit density over an explicit flow, with that error as the optimal control.