

A Constructive Introduction to Quantum Field Theory

Lecture 1 — Relativistic quantum fields and imaginary time

M. Gubinelli

Mathematical Institute, University of Oxford

Notes: §1–§2

What is constructive QFT?

- **Goal:** construct, with full rigour, *interacting* relativistic quantum fields, i.e. models satisfying axioms encoding quantum mechanics, special relativity and causality.
- **Status:**
 - $d = 2$: $P(\varphi)_2$, Yukawa₂, sine-Gordon, ...
 - $d = 3$: Φ_3^4 , ...
 - $d = 4$: no interacting example; Φ_4^4 is trivial (Aizenman–Duminil-Copin 2021).
- **Lesson of the 1970s:** work in *imaginary time*. A QFT becomes a probability measure on $\mathcal{S}'(\mathbb{R}^d)$, which looks like the continuum limit of a classical spin system at a critical point.
- **Modern tools** (this course): stochastic quantisation, singular SPDEs, variational methods; RG and cluster expansions as complements.

Plan of the lecture

Part I. Minkowski picture

1. Wightman axioms
2. Why operator-valued distributions
3. Källén–Lehmann representation
4. The quantum wave equation

Part II. Imaginary time

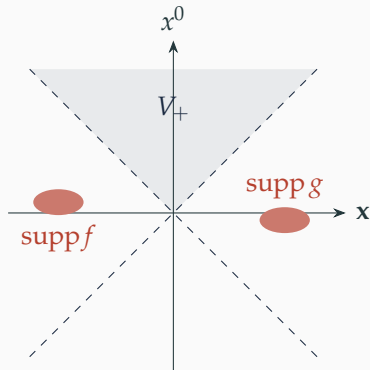
5. A lattice Hamiltonian on a circle
6. Positivity of energy $\Rightarrow$ Wick rotation
7. Feynman–Kac: quantum $\rightarrow$ classical
8. Statistical mechanics and criticality
9. Osterwalder–Schrader axioms

Conventions: $\hbar = c = 1$, spacetime dimension $d = 1 + s$, $\langle f, g \rangle = \int \bar{f} g$,
 $\hat{f}(k) = \int e^{-ik \cdot x} f(x) dx$.

Relativistic quantum fields

Kinematics

- Minkowski space $\mathbb{R}^{1,s}$, $x = (x^0, \mathbf{x})$,
 $x \cdot y = x^0 y^0 - \mathbf{x} \cdot \mathbf{y}$.
- x, y **spacelike separated** if $(x - y)^2 < 0$.
- Forward cone $V_+ = \{p : p^0 > |\mathbf{p}|\}$, closure $\overline{V}_+$.
- Restricted Lorentz group $\mathcal{L}_+^\uparrow$; Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$ acting by $x \mapsto \Lambda x + a$.
- Wigner: a relativistic quantum system carries a (projective) unitary representation of $\mathcal{P}_+^\uparrow$.



Wightman axioms (I)

(W0) Relativistic quantum theory

Separable Hilbert space $\mathcal{H}$; strongly continuous unitary representation U of $\mathcal{P}_+^\uparrow$;
 $U(a, \mathbb{1}) = e^{iP \cdot a}$, $H = P^0$.

Spectrum condition: $\text{spec}(P) \subset \overline{V}_+$, so $H \geq |\mathbf{P}| \geq 0$.

Vacuum: unique (up to phase) unit vector Ω with $U(a, \mathbb{1})\Omega = \Omega$.

(W1) Fields

Dense $\mathcal{D} \ni \Omega$; for $f \in \mathcal{S}(\mathbb{R}^d)$ an operator $\varphi(f)$ on $\mathcal{D}$, $\varphi(f)\mathcal{D} \subset \mathcal{D}$, $U\mathcal{D} \subset \mathcal{D}$;
 $f \mapsto \langle \chi, \varphi(f)\psi \rangle$ tempered; $\langle \chi, \varphi(f)\psi \rangle = \langle \varphi(\bar{f})\chi, \psi \rangle$.

Wightman axioms (II)

(W2) Covariance, (W3) Locality, (W4) Cyclicity

$U(a, \Lambda)\varphi(f)U(a, \Lambda)^{-1} = \varphi(f \circ (a, \Lambda)^{-1})$; $[\varphi(f), \varphi(g)] = 0$ if supports are spacelike separated; polynomials in φ applied to Ω are dense.

Formally $\varphi(f) = \int \varphi(x)f(x)dx$, $\varphi(x^0, \mathbf{x}) = e^{iHx^0}\varphi(0, \mathbf{x})e^{-iHx^0}$.

Wightman functions $W_n(f_1, \dots, f_n) = \langle \Omega, \varphi(f_1) \cdots \varphi(f_n)\Omega \rangle \in \mathcal{S}'(\mathbb{R}^{dn})$:

- positivity $\sum_{j,k} W_{j+k}(g_j^* \otimes g_k) \geq 0$; Poincaré invariance;
- spectral condition: Fourier support in $\overline{V}_+^{n-1}$ (difference variables);
- locality; cluster property $\Leftrightarrow$ unique vacuum.

Theorem (Wightman reconstruction; Streater–Wightman 1964)

A sequence (W_n) of tempered distributions with $W_0 = 1$ satisfying positivity, invariance, spectral condition and locality is the sequence of Wightman functions of a Wightman theory, unique up to unitary equivalence.

- To construct a QFT it suffices to construct its Wightman functions.
- **Difficulty:** positivity is a nonlinear, nonlocal constraint, hard to check and unstable under approximations.
- The Euclidean approach replaces it by **reflection positivity**, which is stable under limits.
- The axioms say nothing about dynamics: no Lagrangian; the interaction is hidden in the higher W_n .

The free field

$\omega(\mathbf{p}) = (|\mathbf{p}|^2 + m^2)^{1/2}$, $d\Omega_m(\mathbf{p}) = \frac{d\mathbf{p}}{(2\pi)^s 2\omega(\mathbf{p})}$ (invariant measure on $H_m^+ = \{p^2 = m^2, p^0 > 0\}$), Fock space over $L^2(d\Omega_m)$:

$$\varphi(x) = \int d\Omega_m(\mathbf{p}) [a(\mathbf{p})e^{-ip \cdot x} + a^*(\mathbf{p})e^{ip \cdot x}]_{p^0=\omega(\mathbf{p})},$$

with $[a(\mathbf{p}), a^*(\mathbf{q})] = (2\pi)^s 2\omega(\mathbf{p}) \delta(\mathbf{p} - \mathbf{q})$, $a(\mathbf{p})\Omega = 0$. Then

$$W_2(x - y) = \int d\Omega_m(\mathbf{p}) e^{-ip \cdot (x - y)}, \quad \|\varphi(f)\Omega\|^2 = \int d\Omega_m |\tilde{f}|^2 < \infty,$$

but

$$\|\varphi(x)\Omega\|^2 = W_2(0) = \int d\Omega_m = +\infty \quad (s \geq 1).$$

UV divergence: logarithmic for $s = 1$, power-like for $s \geq 2$. **Not an artefact of the free field.**

No fields at a point

Proposition (Wightman 1964)

Assume (W0)–(W2) and $\varphi(f)\Omega = \int f(x)U(x, \mathbb{1})\psi \, dx$ for some $\psi \in \mathcal{H}$ (“ $\varphi(0)\Omega = \psi$ exists”). Then $\psi \in \mathbb{C}\Omega$, hence $\varphi(f)\Omega = c \left(\int f \right) \Omega$.

Lemma

For $d \geq 2$, a finite $\mathcal{L}_+^\uparrow$ -invariant measure on $\overline{V}_+$ is $c \delta_0$.

- Combined with locality and Reeh–Schlieder: φ is a multiple of $\mathbb{1}$.
- Quantum fields are **operator-valued distributions**: nonlinear expressions in $\varphi(x)$ need a definition $\Rightarrow$ renormalisation.

Lemma. A boost of rapidity θ acts as $p^0 \pm p^1 \mapsto e^{\pm\theta}(p^0 \pm p^1)$. The sets

$$S_j^\pm = \{p \in \overline{V}_+ : 2^j \leq p^0 \pm p^1 < 2^{j+1}\}, \quad j \in \mathbb{Z},$$

are mapped to each other by the boost with $e^\theta = 2$; they are disjoint and cover $\overline{V}_+ \setminus \{0\}$. Invariance and finiteness force $\mu(S_j^\pm) = 0$.

Proposition.

1. Covariance gives $\int f(x)U(\Lambda x)(U(0, \Lambda)\psi - \psi)dx = 0$ for all f , so $U(0, \Lambda)\psi = \psi$.
2. The spectral measure μ of ψ for translations is finite, supported in $\overline{V}_+$ and Lorentz invariant, so $\mu = \|\psi\|^2\delta_0$.
3. Hence $U(a)\psi = \psi$ for all a , and $\psi \in \mathbb{C}\Omega$ by uniqueness of the vacuum.

The Källén–Lehmann representation

Let $W_2^{(M)}$ be the free two-point function of mass $M \geq 0$ and $\langle \Omega, \varphi(f)\Omega \rangle = v \int f$.

Theorem (Källén 1952, Lehmann 1954)

*Under (W0)–(W2) there is a positive, polynomially bounded measure ρ on $[0, \infty)$ (the **spectral measure** of the field) such that*

$$W_2(x - y) = v^2 + \int_{[0, \infty)} \rho(dM^2) W_2^{(M)}(x - y).$$

$(\rho(\{0\}) = 0 \text{ if } d = 2.)$

Sketch. $\psi_f = \varphi(f)\Omega - v(\int f)\Omega$ gives $W_2 - v^2$ of positive type $\Rightarrow$ (Bochner–Schwartz) Fourier transform of a positive measure μ , supported in $\bar{V}_+$, $\mu(\{0\}) = 0$, Lorentz invariant. Orbits are the hyperboloids H_M^+ (and the cone for $M = 0$), each with a unique invariant measure $d\Omega_M$. Disintegrate along M .

Reading the spectrum off ρ

- **Free field:** $\rho = \delta_{m^2}$. Conversely $\rho = \delta_{m^2}$, $m > 0$, forces a free field (Jost–Schroer).
- **Generalised free fields** $\varphi = \sum_i c_i \varphi_{M_i}$ have $\rho = \sum_i c_i^2 \delta_{M_i^2}$ and no interaction: $\rho \neq \delta_{m^2}$ is necessary, not sufficient.
- **Particles:** an isolated atom $Z\delta_{m^2}$ is a particle of mass m ; multi-particle continuum from $(2m)^2$ (or $(3m)^2$ with $\varphi \mapsto -\varphi$ symmetry). Weakly coupled $P(\varphi)_2$: Glimm–Jaffe–Spencer 1974.
- **No point fields again:** $W_2^T(0) = \int \rho(dM^2) \int d\Omega_M = \infty$.
- **Canonical fields:** CCR at sharp times $\Rightarrow \int \rho = 1$, so $Z \leq 1$. In $d = 4$ perturbatively $\int \rho = \infty$ (wave-function renormalisation).

The quantum wave equation

The free field solves, as an operator-valued distribution,

$$(\square + m^2)\varphi = 0, \quad \square = \partial_t^2 - \Delta_{\mathbf{x}},$$

and its sharp-time fields $\varphi(t, g)$, $\pi(t, g)$, $\pi = \partial_t \varphi$, satisfy the CCR

$$[\varphi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\varphi, \varphi] = [\pi, \pi] = 0.$$

Picture

A linear wave equation with quantum initial data $(\varphi(0, \cdot), \pi(0, \cdot))$, “distributed” according to the vacuum.

Interacting version: $(\square + m^2)\varphi + \lambda\varphi^3 = 0$ together with the CCR.

The free field on a circle

$s = 1$, space $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, momenta $k \in \frac{2\pi}{L}\mathbb{Z}$, $\omega_k = (k^2 + m^2)^{1/2}$:

$$\varphi(0, x) = \sum_k \frac{a_k e^{ikx} + a_k^* e^{-ikx}}{\sqrt{2L\omega_k}}, \quad [a_k, a_{k'}^*] = \delta_{kk'}, \quad H_0 = \sum_k \omega_k a_k^* a_k.$$

- An infinite family of independent harmonic oscillators, one per mode.
- With a UV cutoff $|k| \leq R$:

$$\langle \Omega, \varphi_R(0, x)^2 \Omega \rangle = \sum_{|k| \leq R} \frac{1}{2L\omega_k} = \frac{1}{2\pi} \log \frac{R}{m} + O(1) \rightarrow \infty.$$

- Wick ordering $:\varphi_R^2: = \varphi_R^2 - \langle \Omega, \varphi_R^2 \Omega \rangle$ removes exactly this divergence (Exercise 1.7).

Interactions: three problems

1. **Products of distributions.** $\varphi(x)^3$ is meaningless and must be renormalised: Wick ordering suffices in $d = 2$; more counterterms in $d = 3$; triviality in $d = 4$.
2. **Haag's theorem.** In infinite volume the interacting field at fixed time is not unitarily equivalent to the free one: there is no interaction picture.
3. **Semiboundedness.** Even on $\mathbb{T}_L$, $H = H_0 + \lambda \int : \varphi(0, x)^4 : dx$ is positive plus unbounded below. Nelson 1966: bounded below (hypercontractivity). Volume cutoff removed by Glimm–Jaffe; Wightman axioms for weak coupling $P(\varphi)_2$ by Glimm–Jaffe–Spencer 1974.

The Euclidean approach (Symanzik, Nelson, Osterwalder–Schrader, Guerra–Rosen–Simon) simplifies this enormously.

From real to imaginary time

A lattice Hamiltonian on the circle

Lattice $\Lambda_\varepsilon = \varepsilon\mathbb{Z}/L\mathbb{Z}$, $N = L/\varepsilon$ sites, $\mathcal{H}_\varepsilon = L^2(\mathbb{R}^{\Lambda_\varepsilon}, d\varphi)$, φ_x multiplication, $\pi_x = -i\varepsilon^{-1}\partial_{\varphi_x}$, so $[\varphi_x, \pi_y] = i\varepsilon^{-1}\mathbb{1}_{x=y}$.

$$H_\varepsilon = \sum_x \varepsilon \left[\frac{1}{2} \pi_x^2 + \frac{1}{2} (\nabla_\varepsilon \varphi)_x^2 + \frac{1}{2} m^2 \varphi_x^2 + V(\varphi_x) \right] = -\frac{1}{2\varepsilon} \Delta_{\mathbb{R}^N} + U_\varepsilon(\varphi).$$

- Heisenberg fields solve the **lattice wave equation**

$$\partial_t^2 \varphi_x = (\Delta_\varepsilon \varphi)_x - m^2 \varphi_x - V'(\varphi_x)$$

— the quantum wave equation, now rigorous.

- Schrödinger operator with confining potential: self-adjoint, compact resolvent, e^{-tH_ε} trace class with positive kernel.
- Perron–Frobenius: simple ground state $\Omega_\varepsilon > 0$, energy E_0 . $\bar{H} := H_\varepsilon - E_0 \geq 0$.

Positivity of the energy $\Rightarrow$ Wick rotation

For bounded F_j and $F_j(t) = e^{itH}F_je^{-itH}$:

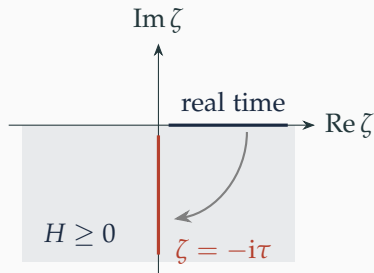
$$W(t_1, \dots, t_n) = \langle \Omega_\epsilon, F_1 e^{-i(t_1 - t_2)\bar{H}} F_2 \cdots F_n \Omega_\epsilon \rangle.$$

Proposition

W extends analytically to $\text{Im}(t_j - t_{j+1}) < 0$. At $t_j = -i\tau_j$, $\tau_1 \geq \cdots \geq \tau_n$, it gives the *Schwinger functions*

$$S(\tau_1, \dots, \tau_n) = \langle \Omega_\epsilon, F_1 e^{-(\tau_1 - \tau_2)\bar{H}} F_2 \cdots F_n \Omega_\epsilon \rangle,$$

which determine W .



Where positivity of the energy is used

1. **Analyticity.** $e^{-i\zeta\bar{H}} = \int e^{-i\zeta\lambda} dE(\lambda)$ is a contraction for $\text{Im } \zeta \leq 0$, since $\lambda \geq 0$: analytic in the interior, continuous up to the boundary.
2. **Uniqueness.** An analytic function on the half plane vanishing on $-i\mathbb{R}_+$ vanishes; iterate variable by variable.
3. **Semigroup.** $e^{-\tau\bar{H}}$ is a contraction semigroup, which can be represented as a path integral (Feynman–Kac).
4. **Projection onto the vacuum.** $e^{-T\bar{H}} \rightarrow |\Omega\rangle\langle\Omega|$ as $T \rightarrow \infty$.

Relativistic version: Wightman functions are boundary values of functions analytic in the forward tube; Lorentz covariance extends them to all non-coincident Euclidean points (Schwinger functions are *symmetric* and *Euclidean invariant*).

Discretising imaginary time

Kinetic semigroup: $e^{\frac{\delta}{2\varepsilon}\Delta}(\varphi, \varphi') = (2\pi\delta/\varepsilon)^{-N/2} \exp(-\frac{\varepsilon}{2\delta}|\varphi - \varphi'|^2)$.

Transfer matrix (symmetric Trotter step):

$$T_\delta = e^{-\delta U/2} e^{\frac{\delta}{2\varepsilon}\Delta} e^{-\delta U/2}, \quad T_\delta(\varphi, \varphi') \propto \exp\left(-\frac{\varepsilon}{2\delta} \sum_x (\varphi_x - \varphi'_x)^2 - \frac{\delta}{2}(U(\varphi) + U(\varphi'))\right),$$

and $T_\delta^{\lfloor \tau/\delta \rfloor} \rightarrow e^{-\tau H_\varepsilon}$ (Trotter).

Space-time lattice $\Lambda = (\delta\mathbb{Z}/T\mathbb{Z}) \times (\varepsilon\mathbb{Z}/L\mathbb{Z})$, $T = J\delta$:

$$\mathrm{Tr}(T_\delta^J) \propto \int e^{-S_{\delta,\varepsilon}(\varphi)} \prod_{u \in \Lambda} d\varphi_u,$$

$$S_{\delta,\varepsilon}(\varphi) = \sum_{(\tau,x)} \varepsilon\delta \left[\frac{1}{2} (\partial_\tau^\delta \varphi)^2 + \frac{1}{2} (\nabla_\varepsilon \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + V(\varphi) \right].$$

The Minkowski metric has become Euclidean.

Feynman–Kac–Nelson

Lattice Gibbs measure $\mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi) = Z^{-1} \mathrm{e}^{-S_{\delta,\varepsilon}(\varphi)} \prod_u \mathrm{d}\varphi_u$:

$$\frac{\mathrm{Tr}(F_1 T_\delta^{k_1} F_2 \cdots F_n T_\delta^{k_n})}{\mathrm{Tr}(T_\delta^J)} = \int F_1(\varphi_{\tau_1}) \cdots F_n(\varphi_{\tau_n}) \mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi).$$

Proposition

$$\langle \Omega_\varepsilon, F_1 \mathrm{e}^{-(\tau_1 - \tau_2) \tilde{H}} F_2 \cdots F_n \Omega_\varepsilon \rangle = \lim_{T \rightarrow \infty} \lim_{\delta \rightarrow 0} \int F_1(\varphi_{\tau_1}) \cdots F_n(\varphi_{\tau_n}) \mathrm{d}\mu_{\delta,\varepsilon}^{T,L}.$$

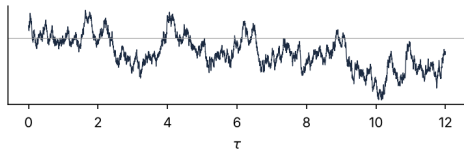
Sketch. Gaussian factor = Brownian loops with diffusion ε^{-1} ; $\delta \sum_\tau U(\varphi_\tau)$ is a Riemann sum for $\int_0^T U(\omega_\tau) \mathrm{d}\tau$; then $T \rightarrow \infty$ uses the spectral gap.

Vacuum expectations in imaginary time are moments of a classical probability measure.

Example: one oscillator and the Ornstein–Uhlenbeck process

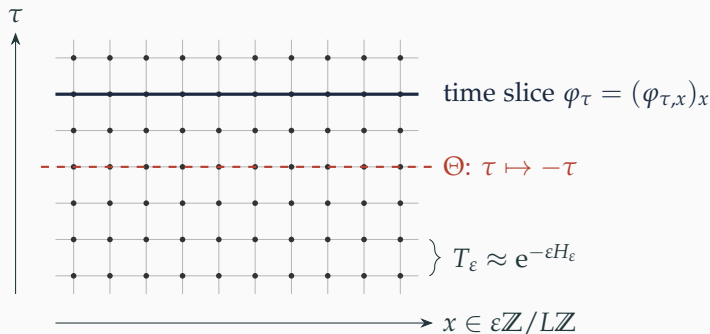
$$H = -\frac{1}{2}\partial_q^2 + \frac{1}{2}m^2q^2, \quad \Omega(q) \propto e^{-mq^2/2},$$
$$E_0 = \frac{m}{2}.$$

- Ground-state transform:
 $\Omega^{-1}\bar{H}\Omega = -\frac{1}{2}\partial_q^2 + mq\partial_q$, minus the generator of $dX = -mXdt + dB$.
- Schwinger function
 $\langle \Omega, q e^{-\tau\bar{H}} q \Omega \rangle = \frac{e^{-m\tau}}{2m}.$
- This is $(-\partial_\tau^2 + m^2)^{-1}$: the **GFF** in $d = 1$.



A sample of the stationary OU process, $m = 1$.

The space-time lattice



Rows are quantum configurations at fixed imaginary time; the transfer matrix connects consecutive rows. As $T \rightarrow \infty$, the rows form a stationary Markov process whose one-time marginal is $\Omega_\varepsilon(\varphi)^2 d\varphi$.

Isotropic lattice $\delta = \varepsilon$:

$$S_\varepsilon(\varphi) = \sum_{\langle u,v \rangle} \frac{1}{2}(\varphi_u - \varphi_v)^2 + \varepsilon^2 \sum_u \left[\frac{1}{2}m^2 \varphi_u^2 + V(\varphi_u) \right].$$

- $\mu_\varepsilon \propto \prod_{\langle u,v \rangle} e^{-\frac{1}{2}(\varphi_u - \varphi_v)^2} \prod_u \nu_\varepsilon(d\varphi_u), \quad \nu_\varepsilon(da) = e^{-\varepsilon^2(\frac{1}{2}m^2 a^2 + V(a))} da.$
- A **ferromagnetic nearest-neighbour spin system** with unbounded spins: short-range classical statistical mechanics.
- In dimension d : $\varepsilon^{d-2} \sum_{\langle u,v \rangle}$ and $\varepsilon^d \sum_u$.
- Nearest-neighbour $\Rightarrow$ **reflection positive**: the Euclidean shadow of $e^{-\varepsilon H} \geq 0$. Conversely, RP $\Rightarrow$ Hilbert space and $H \geq 0$ (Exercises 2.5, 2.7).

A dictionary

Quantum (real time)	Euclidean field theory	Statistical mechanics
Hilbert space positivity, $H \geq 0$	reflection positivity	RP of n.n. interactions
$e^{-\varepsilon H}$	time translation by ε	transfer matrix
vacuum Ω	infinite-volume measure	infinite-volume Gibbs state
inverse temperature β	length of the time circle	—
mass gap $E_1 - E_0$	exponential decay	inverse correlation length
continuum limit	UV cutoff $\varepsilon \rightarrow 0$	critical point, scaling limit
renormalisation	counterterms	tuning to criticality

Continuum limit = critical point

- In lattice units the mass is $\varepsilon m \rightarrow 0$: correlation length $\xi \approx (\varepsilon m)^{-1} \rightarrow \infty$ lattice spacings.
- Continuum QFT = **scaling limit of short-range spin systems approaching criticality** (Wilson). Conversely, RP critical models with nontrivial scaling limits give candidate QFTs.
- Griffiths–Simon: φ^4 spins as block spins of Ising models; critical 4D Ising and φ_4^4 have Gaussian scaling limits (ADC 2021).

Renormalisation already appears

For $V = \lambda a^4$ and fixed (m, λ) the limit is trivial:

$c_\varepsilon = \mathbb{E}[\varphi_u^2] = \frac{1}{2\pi} \log \frac{1}{\varepsilon m} + O(1) \rightarrow \infty$. Take $V_\varepsilon(a) = \lambda(a^4 - 6c_\varepsilon a^2)$, i.e. $\lambda:a^4$: up to a constant. Enough in $d = 2$; not in $d = 3$.

Formally, as $\varepsilon \rightarrow 0$ and $L, T \rightarrow \infty$:

$$\mu(d\varphi) \stackrel{?}{=} Z^{-1} \exp\left(-\int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right] dx\right) \mathcal{D}\varphi.$$

- Neither $\mathcal{D}\varphi$ nor the action make sense: typical φ are distributions.
- The Gaussian part defines the **Gaussian free field** μ_m (next lecture).
- Then $\mu \propto e^{-\int V_{\text{ren}}(\varphi)} \mu_m$: meaningful in finite volume for $d = 2$ (and, with more work, $d = 3$).

Question. Which measures on $\mathcal{S}'(\mathbb{R}^d)$ correspond to relativistic QFTs?

Osterwalder–Schrader axioms

$$\hat{\mu}(f) = \int e^{i\varphi(f)} \mu(d\varphi), \quad \Theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x}).$$

OS0 *Analyticity:* $z \mapsto \hat{\mu}(\sum_j z_j f_j)$ entire.

OS1 *Regularity:* $|\hat{\mu}(f)| \leq \exp(c(\|f\|_{L^1} + \|f\|_{L^p}^p)), p \in [1, 2].$

OS2 *Euclidean invariance.*

OS3 *Reflection positivity:* for real f_i supported in $\{\tau > 0\}$,
 $(\hat{\mu}(f_i - \Theta f_j))_{ij} \geq 0.$

OS4 *Ergodicity* under time translations.

Theorem (Osterwalder–Schrader 1973/75)

OS0–OS3 $\Rightarrow$ the moments of μ are the Schwinger functions of a unique Wightman theory; OS4 $\Leftrightarrow$ unique vacuum.

Reconstruction: the idea

- $\mathcal{E}_+$ = functions of $\varphi|_{\{\tau>0\}}$, with $(F, G) = \mathbb{E}_\mu[\overline{\Theta F} G] \geq 0$ by OS3.
- Quotient by null vectors and complete: physical Hilbert space $\mathcal{H}$.
- Time translations by $t \geq 0$ preserve $\mathcal{E}_+$ and descend to a self-adjoint contraction semigroup e^{-tH} : **positivity of the energy is recovered from reflection positivity**.
- Vacuum = class of 1; analytic continuation $\tau \rightarrow it$ gives the Wightman functions.

The Euclidean strategy

(1) regularise (lattice, torus); (2) renormalise; (3) uniform estimates, remove cutoffs; (4) verify OS axioms (RP survives limits, Euclidean invariance recovered, regularity, clustering); (5) prove non-Gaussianity.

Exercises and reading

Exercises (numbering of the notes)

- 1.3 Which restrictions of the free field make sense?
- 1.5–1.6 Canonical sum rule; generalised free fields.
- 2.2 Harmonic oscillator and Ornstein–Uhlenbeck.
- 2.4 Correlation length and criticality.
- 2.5, 2.7 Reflection positivity on the lattice; toy OS reconstruction.

Next lecture: the Gaussian free field.

Reading

- Streater–Wightman, *PCT, Spin and Statistics, and All That*.
- Glimm–Jaffe, *Quantum Physics*, Euclidean chapters.
- Simon, *Functional Integration and Quantum Physics*.
- Osterwalder–Schrader, CMP 31 (1973), 42 (1975).