Supplement to Lectures 6–7: detailed proofs

Φ^4_3 on the torus: coefficients, consistency, and older routes

M. GUBINELLI
Mathematical Institute, University of Oxford

November 2026

Table of contents

19 Elementary kernels . . . 2
Forces and their primitives . . . 2
20 Renormalisation with two correction steps . . . 4
20.1 Tensors and the iteration . . . 4
20.2 Two elementary receiving estimates . . . 5
20.3 The finite derivative budgets . . . 7
20.4 The coefficients, with a boundary value . . . 9
21 Global coercivity and the complete large-field error . . . 13
21.1 Clipping . . . 13
21.2 Global coercivity . . . 13
21.3 The complete defect . . . 14
22 The bare mass and its limit . . . 16
23 Identifying the deterministic coefficients: conventions . . . 18
24 The topology of the centred quadratic kernel . . . 18
25 Physical vertex measures and their derivatives . . . 19
26 Compact resolvents receive weak Hessians . . . 20
27 Convergence of the seed and the two trace steps . . . 22
28 The scalar fixed point converges . . . 23
29 Clipping and the complete drift . . . 26
30 Random inputs . . . 29
31 The coordinate, high modes and stochastic integrals . . . 30
32 Complements not used in the lectures . . . 34
32.1 Regularity in space and scale . . . 34
32.2 The limit by compactness: the noise . . . 34
32.3 The limit by compactness: the result . . . 36
32.4 Density at a fixed scale . . . 37
32.5 Feynman–Kac comparison along the explicit flow . . . 37
32.6 Uniqueness through the comparison . . . 40
32.7 Symmetrised Girsanov comparison . . . 42
References . . . 44

This supplement contains the detailed proofs which Lectures 6–7 cite or condense. It is reference material: the lectures can be read without it. The results are recent, joint work with S.-J. Meyer + AI. Unless stated otherwise, notation is that of the lectures.

Where each result is proved.

- Free resolvents, Lemma 10.1: Lemma 19.1.
- The coefficient construction, Proposition 11.2: Section 20, in particular Proposition 20.6, with the rooted calculus and the trace comparison. The dependence on the boundary value is Proposition 20.8.
- The explicit counterterm, Theorem 11.3: Section 22.
- Global coercivity and the source bounds, Lemma 12.1 and Proposition 12.2, with the term-by-term ledger: Section 21.
- The facts of (12.5), (12.8)–(12.9) and the bound on $\mathfrak{D}$ used in Lemma 18.1: Lemma 21.2.
- Consistency of the coefficients, Section 18.1 of Lecture 7, and Lemma 18.3: Sections 23–31.

- Results removed from the lectures, which are not needed there: regularity at finite resolution, the proof of the limit FBSDE by compactness and the Riccati equation, the density at a fixed scale, and two earlier proofs of uniqueness: Section 32.

The auxiliary scale q_* is as in the lectures. In Sections 19–31 it is the lowest scale of the action; below it the action is frozen (Section 12.4 of Lecture 6).

19 Elementary kernels

We use the notation of Lecture 6: $K_q = q - \Delta_N$, $C_q = K_q^{-1}$, $G_q(y) = (K_q + \lambda y^2)^{-1}$, H for the inverses of $K_q + D^2\Gamma$, $M_q = \mu_q I + \Theta_q$ for the quadratic path, with $\Theta_q = \Theta_q^{[3]}$ for the final one and $M^{[j]}$ for the level masses of Section 20.4, κ_q for the clip, ϱ_q for the coordinate correction, and $\mathrm{Ctr}_E T$ for contraction. Recall $g_q = \lambda q^{-1/2}$ and $r_q = (q/\lambda)^{1/2}$, so that $\lambda r_q^2 = q$. Generic convolution kernels are denoted E, E', and the letters L, L' denote arbitrary kernels. Distance d is torus distance.

Forces and their primitives. Lecture 6 is written in terms of forces. The proofs below expand the trace, so they are written for the primitives of those forces: $h^{[j]}$ with $Dh^{[j]} = \ell^{[j]}$ for the corrections (11.8), $h^{[1]} = \mathcal{L}$ with $D\mathcal{L} = \ell^{[1]}$ for the seed (11.6), $\bar\Gamma$ with $D\bar\Gamma = \bar{\mathcal{F}}$, e with $De = \mathcal{D}$, and the trace map $\frac{1}{2}\mathrm{Tr}[(K+M+D^2(U+h))^{-1} - (K+M)^{-1}]$, whose derivative is the loop force $\mathcal{T}$ of Lecture 6. A vertex with k legs is D^k of the primitive and D^{k-1} of the force, so the norms S_k and $\mathcal{N}_k$ agree. No additive constant of a primitive enters any estimate.

There are two scales, which must not be confused. The **reference mass** $q_0 > 0$ has arbitrary size; it fixes the covariance $C = C_{q_0}$ of the measure, and it is the lower end of the range of Lemma 19.1. The **auxiliary scale** $q_* \geqslant \max(1, q_0)$ is large, depending on λ and on the bound B_0 below; all the operators of Sections 19–21 after Lemma 19.1 are used at scales $q \geqslant q_*$ and do not involve q_0. So from Lemma 19.3 on we assume $q \geqslant 1$. Constants may depend on the coupling and on finitely many derivative orders, but not on N or on the scale interval. For a kernel, $\|C\|_{\mathrm{row}} = \sup_x \int |C(x,z)| \mathrm{d}z$ is the row norm (Section 10.1 of Lecture 6); it bounds $\|C\|_{p\to p}$ for every p when C is symmetric.

Lemma 19.1. (Free and killed resolvents) *This is Lemma 10.1 of Lecture 6, with further items. For $q \geqslant q_0$, uniformly in N, with constants that depend on q_0 only through scales $q < 1$: the kernel of C_q is nonnegative, $C_q 1 = q^{-1}1$, $\|C_q\|_{p\to p} = q^{-1}$ for $1 \leqslant p \leqslant \infty$, and*

$$a_2(q) := \mathrm{Tr}\, C_q^2 \lesssim q^{-1/2} \wedge N^3 q^{-2}, \quad \mathrm{Tr}\, C_q \lesssim N \wedge N^3 q^{-1}, \quad \int d(0,z) C_q(0,z)^2 \,\mathrm{d}z \leqslant C q^{-1}. \tag{19.1}$$

For $y \in H_N$ and $\lambda > 0$,

$$0 \leqslant G_q(y)(x,z) \leqslant C_q(x,z), \quad G_q(y)(q + \lambda y^2) = 1. \tag{19.2}$$

Proof. The heat integral $C_q = \int_0^\infty \mathrm{e}^{-qt} \mathrm{e}^{t\Delta_N} \mathrm{d}t$ proves positivity, the row mass and the operator norms. Killing the lattice random walk at rate λy^2 proves the first part of (19.2). The second part is $(K_q + \lambda y^2)1 = q + \lambda y^2$.

The symbol of $-\Delta_N$ is $4N^2 \sum_i \sin^2(\pi n_i / N)$, with $k = 2\pi n$, comparable to $|k|^2$ on the centred frequency cube. Thus $a_2(q) = \sum_k (q + c_k)^{-2}$ and $\mathrm{Tr}\, C_q = \sum_k (q + c_k)^{-1}$, with c_k this symbol, $c_0 = 0$. Counting the N^3 modes gives the bounds $N^3 q^{-2}$ and $N^3 q^{-1}$, with universal constants. Nonzero modes have $|k| \geqslant 2\pi$. Splitting at $|k| = \sqrt{q}$ bounds $\sum_{k\neq 0} (q + c|k|^2)^{-2}$ by $C q^{-1/2}$ for all $q > 0$, and $\sum_{k\neq 0} (q + c|k|^2)^{-1} \leqslant CN$. The zero mode contributes q^{-2} to a_2 and q^{-1} to the trace: for $q \geqslant 1$ these are at most $q^{-1/2}$ and 1, and for $q_0 \leqslant q < 1$ they are at most $q_0^{-3/2} q^{-1/2}$ and q_0^{-1}.

Let $q \geqslant 1$. A dyadic shell is a smooth Fourier cutoff supported where the frequency magnitude is comparable to L, with L ranging over powers of two; together with a low-frequency cutoff these form a partition of unity. Such a shell of radius $L \leqslant CN$ has kernel bounded by $C_m L^3 (q + L^2)^{-1} (1 + L\,d)^{-m}$. This follows by summation by parts; on the final shell use the periodic sine symbol. Summing gives $C_q(0,z) \leqslant C\min(d^{-1}, q^{-1}d^{-3})$ off the diagonal. For its weighted square split the radial integral at $q^{-1/2}$: $\int_0^{q^{-1/2}} r\,\mathrm{d}r$ and $q^{-2}\int_{q^{-1/2}}^{O(1)} r^{-3}\,\mathrm{d}r$ are both $O(q^{-1})$. Discrete annuli give the same bound above the lattice kinetic scale; the diagonal has zero distance weight.

Let $q_0 \leqslant q < 1$. Separate the zero mode, whose kernel is the constant q^{-1}: the kernel of $C_q - q^{-1}$ has multiplier $1_{k\neq 0}(q + c_k)^{-1}$, and that of $C_q - q^{-1} - (C_1 - 1)$ has multiplier bounded by $c_k^{-2} 1_{k\neq 0}$, which is summable uniformly in N. Hence $C_q(0,z) \leqslant q^{-1} + C + C_1(0,z)$. Inserting this and the case $q = 1$ gives $\int d(0,z) C_q(0,z)^2\,\mathrm{d}z \leqslant C(q^{-2} + 1) \leqslant C(1 + q_0^{-1})q^{-1}$. □

Remark 19.2. (**The zero mode**) The dependence on q_0 in Lemma 19.1 is genuine and comes only from the constant mode, whose eigenvalue of C_q is q^{-1}. On the one-site lattice $a_2(q) = q^{-2}$ and $\operatorname{Tr} C_q = q^{-1}$, so that the bounds $a_2 \lesssim q^{-1/2}$ and $\operatorname{Tr} C_q \lesssim N$ cannot be uniform as $q \to 0$. The constants are universal for $q \geqslant 1$. Since $q_* \geqslant 1$, no constant below depends on q_0 through this lemma.

For a symmetric centred convolution kernel Θ, that is, with $\Theta 1 = 0$, put

$$v_1(\Theta) = \int d(0,z)|\Theta(0,z)|\,\mathrm{d}z, \quad \eta_q(\Theta) = q^{-1/2} v_1(\Theta).$$

The diagonal is determined by centring. We never need its absolute row norm. Define the convolution algebra and module norms

$$\|E\|_{\mathcal{A}_q} = \int (1 + \sqrt{q}d)|E|, \quad \|E\|_{\mathcal{H}_q}^2 = \int (1 + \sqrt{q}d)|E|^2.$$

Here a translation-invariant kernel is identified with its displacement function $E(z) = E(0, z)$, all integrals are in that displacement, and $(E * E')(z) = \int E(z-u)E'(u)\,\mathrm{d}u$ is convolution. Products EE' inside a single spatial integral instead mean pointwise products. The triangle inequality for distance, Minkowski, and Cauchy–Schwarz give

$$\|E * E'\|_{\mathcal{A}_q} \leqslant \|E\|_{\mathcal{A}_q}\|E'\|_{\mathcal{A}_q}, \quad \|E * E'\|_{\mathcal{H}_q} \leqslant \|E\|_{\mathcal{H}_q}\|E'\|_{\mathcal{A}_q}, \quad \int d|EE'| \leqslant q^{-1/2}\|E\|_{\mathcal{H}_q}\|E'\|_{\mathcal{H}_q}. \tag{19.3}$$

Also $\|C_q^j\|_{\mathcal{H}_q} \leqslant C_j q^{-(j-1)-1/4}$; for $j > 1$ use the heat integral bound $C_q^j \leqslant C_j q^{-(j-1)} C_{q/2}$. We also use $\|C_q\|_{\mathcal{A}_q} \leqslant C/q$. Indeed the unwrapped lattice walk has second moment $6t$, so its expected torus distance at time t is at most $\sqrt{6t}$. Integrating against $\mathrm{e}^{-qt}\,\mathrm{d}t$ bounds the first moment of C_q by $C q^{-3/2}$.

Lemma 19.3. (**Centre before taking absolute values**) *Define spatial translation by* $(\sigma_z f)(x) = f(x+z)$ *and its symmetric second difference by* $\delta_z^2 = \sigma_z + \sigma_{-z} - 2I$, *where* I *denotes the identity operator. Then*

$$\|\delta_z^2 C_q\|_{\mathcal{A}_q} \leqslant C q^{-1/2} d(0,z), \quad \|C_q\Theta\|_{\mathcal{A}_q} \leqslant C\eta_q(\Theta). \tag{19.4}$$

For $M = \mu I + \Theta$ *this implies*

$$\|G_q(y)M\|_{\infty\to\infty} + \|C_q^{1/2} M C_q^{1/2}\|_{2\to 2} \leqslant C(|\mu|/q + \eta_q(\Theta)). \tag{19.5}$$

Proof. A shell of radius L and multiplier size a_L contributes at most $C\,a_L \min((L\,d)^2, 1)(1 + \sqrt{q}/L + \sqrt{q}d)$ to the first norm. Use two derivatives of its kernel when $L\,d < 1$, and translates otherwise. Put $\Lambda = \sqrt{q}$ and $t = \Lambda d$. Let ϑ_Λ be a smooth even Fourier multiplier equal to one below frequency Λ and zero above 2Λ, with the usual scaled derivative bounds. Decompose

$$C_q = q^{-1}\vartheta_\Lambda - q^{-1}(-\Delta_N)C_q\vartheta_\Lambda + C_q(I - \vartheta_\Lambda).$$

The last two terms have shell sizes $q^{-2}L^2$ below Λ and L^{-2} above it. After dividing by q^{-1} the sums are

$$\sum_{x\leqslant 1} x^2\min((xt)^2,1)(1+x^{-1}+t), \quad \sum_{x\geqslant 1} x^{-2}\min((xt)^2,1)(1+x^{-1}+t).$$

Splitting at $x=t^{-1}$ bounds each by Ct; for $t\leqslant 1$ the only borderline high-shell term is $t^2(1+|\log t|)\leqslant Ct$. The first term in the decomposition is bounded by $Cq^{-1}t$ too. Above the lattice frequency take $\vartheta_\Lambda = I$; nonzero lattice distances have $t\geqslant 1$, and the delta kernel has the same bound.

Now use the signed identity $C_q\Theta = \frac{1}{2}\int \Theta(z)\delta_z^2 C_q\,\mathrm{d}z$. The receiving identity $G\Theta = (I - G\lambda y^2)C_q\Theta$ and $\|I - G\lambda y^2\|_{\infty\to\infty}\leqslant 2$ give the row bound. The form bound follows from the commuting Fourier multipliers of C_q and M and the row bound on C_qM. □

Remark 19.4. (A necessary endpoint modification) The estimate $\|\delta_z^2 C_q\|_1 \lesssim d(0,z)^2$ generally has a logarithmic loss at kinetic order two. We use one distance power and pay $q^{-1/2}$ instead. The recursion below produces exactly the compensating first-moment bound. This choice also rules out an unrecorded Laplacian in weak kernel limits: on $d<\delta$ a smooth second difference is bounded by $C\delta d$.

20 Renormalisation with two correction steps

This deterministic section assumes no interacting field estimates. It contains the proof of Proposition 11.2 of Lecture 6.

20.1 Tensors and the iteration

The kernels of field derivatives will be called *vertices*. A physical k-tensor T represents a multilinear form by

$$T[u_1,\dots,u_k] = \int T(x_1,\dots,x_k)\prod_{i=1}^{k} u_i(x_i)\prod_{i=1}^{k}\mathrm{d}x_i.$$

A slot means one argument x_i. Fixing a slot as a *root*, and integrating all others absolutely, defines the rooted norm

$$s(T) = \max_i \sup_{x_i}\int |T(x_1,\dots,x_k)|\prod_{j\neq i}\mathrm{d}x_j.$$

For a translation-invariant tensor define its first spatial moment by

$$v_1(T) = \max_{i,a<b}\sup_{x_i}\int d(x_a,x_b)|T(x_1,\dots,x_k)|\prod_{j\neq i}\mathrm{d}x_j.$$

All derivative tensors are symmetric in their slots. A local derivative is allowed to contain physical deltas: $\delta_x(z) = N^3 1_{x=z}$ has integral one. This convention prevents hidden lattice-volume factors when slots coincide. On the field ball $\mathcal{B}_q$, the set of $y\in H_N$ with $\|y\|_\infty < 2r_q$, put

$$\mathsf{S}_k(f) = \sup_{y\in\mathcal{B}_q} s(D^k f(y)), \quad \mathcal{N}_{k,q}(f) = \mathsf{S}_k(f) + \sqrt{q}\,v_1(D^k f(0)).$$

Thus S_k controls the derivative everywhere in the field ball, whereas the spatial moment is required only at the zero field. The scalar bound $\mathsf{S}_k(f)$ is distinct from the posterior covariance S_q of (10.2). For a symmetric k-tensor T, contraction of two slots against a kernel E means the physical integral

$$(\mathrm{Ctr}_E T)(x_1,\dots,x_{k-2}) = \int E(u,v)T(x_1,\dots,x_{k-2},u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

For tensor measures the same formula is interpreted after testing the remaining slots. Slots joined to inverse kernels in such a product are called operator slots; the remaining ones are external slots. The seed is the mass-subtracted logarithmic determinant, (11.6) of Lecture 6,

$$\mathcal{L}_q(y) = \frac{1}{2}\mathrm{Tr}[\log(I + C_q^{1/2}\lambda y^2 C_q^{1/2}) - C_q^{1/2}\lambda y^2 C_q^{1/2}]. \tag{20.1}$$

The square root and logarithm are defined by diagonalising the positive finite-dimensional operators; in particular this is a finite matrix logarithm. The subtraction removes its quadratic field term. It is nonpositive with zero constant and quadratic jets, and

$$\partial_q \mathcal{L}_q(y) = \frac{1}{2}\mathrm{Tr}[G_q(y) - C_q] + \frac{\lambda}{2}a_2(q)\|y\|_2^2. \tag{20.2}$$

For an even action define $(\Pi f)(y) = f(y) - f(0) - \frac{1}{2}D^2 f(0)[y,y]$; this removes the constant and quadratic Taylor coefficients, also called its low jets. For a quadratic path $M_q = \mu_q I + \Theta_q$ (a scalar and an even, translation-invariant, centred kernel), whenever the displayed inverses exist, define the primitive of the trace map (11.7) of Lecture 6 (its field gradient is that map with $\ell = Dh$)

$$\mathcal{J}_q[M,h](y) = \frac{1}{2}\mathrm{Tr}[(K_q + M_q + D^2(U+h)(y))^{-1} - (K_q + M_q)^{-1}] \tag{20.3}$$

Its quadratic jet at zero is, by evenness, the operator $D^2\mathcal{J}_q[M,h](0) = \mathfrak{S}_q[M,h]$ of (20.8) below. For a translation-invariant operator L write $L = \langle L\rangle I + L^\circ$, with $\langle L\rangle$ its constant-mode eigenvalue and L° centred. Given a scalar path μ, the recursion has three nonlinear functions and two centred kernels, as in (11.8): with the level masses $M^{[j]} = \mu I + \Theta^{[j]}$ and $\Theta^{[1]} = 0$,

$$h^{[1]} = \mathcal{L}, \quad h_q^{[j+1]} = \mathcal{L}_q - \int_q^\infty \Pi(\mathcal{J}_s[M^{[j]},h^{[j]}] - \partial_s\mathcal{L}_s)\,\mathrm{d}s, \quad \Theta_q^{[j+1]} = -\int_q^\infty \mathfrak{S}_s[M^{[j]},h^{[j]}]^\circ \mathrm{d}s, \quad j = 1,2. \tag{20.4}$$

Integrals are at fixed field, and the field balls increase with scale. The integrals are improper at infinite scale; their convergence is proved in Proposition 20.6. No derivative-losing infinite iteration is used.

20.2 Two elementary receiving estimates

All kernels in this subsection are physical kernels, with normalised spatial measure. A closed cycle means the product of alternating vertex and inverse kernels, integrated over all joined operator slots. An external slot is left unintegrated until the rooted norm is taken. A marked factor is simply a selected factor that carries a difference estimate or a spatial-distance weight. Define

$$m(L) = \max\Big(\sup_x \int |L(x,u)|\,\mathrm{d}u, \sup_u \int |L(x,u)|\,\mathrm{d}x\Big),$$

$$\mathfrak{l}(L) = \max(\sup_x \|L(x,\cdot)\|_2, \sup_u \|L(\cdot,u)\|_2).$$

Schur's inequality gives $\|L\|_{p\to p}\leqslant m(L)$ for all $1\leqslant p\leqslant\infty$. Also $\mathfrak{l}(LL')$, $\mathfrak{l}(L'L)\leqslant m(L')\mathfrak{l}(L)$: on one side use the L^2 operator bound for L', and on the other use Minkowski and its absolute row or column mass. These statements do not require positive kernels.

Lemma 20.1. (A rooted closed cycle) *This is (11.5) of Lecture 6 with its proof. Let $T_1,\dots,T_p$ be symmetric physical vertices, each with at least one external slot. Close their two operator slots into a cycle with $p+1$ symmetric inverse factors H; two consecutive inverse factors may be combined into H^2. Denote the resulting external tensor by $\mathcal{Z}$. Then the absolute rooted norm of $\mathcal{Z}$ is at most*

$$\mathfrak{l}(H)^2 m(H)^{p-1}\prod_{i=1}^{p} s(T_i). \tag{20.5}$$

If all factors are translation invariant and every inverse factor has $\mathcal{A}_q$ norm at most C/q and $\mathcal{H}_q$ norm at most $Cq^{-1/4}$, then its first-moment version is

$$s(\mathcal{Z})+\sqrt{q}v_1(\mathcal{Z})\leqslant C_p q^{-1/2}q^{-(p-1)}\prod_i\,(s(T_i)+\sqrt{q}v_1(T_i)). \tag{20.6}$$

The same conclusions hold for mixed inverse factors with common bounds. A marked factor with both norms smaller by ρ gives the additional factor ρ. For cycles with p inverse factors and p vertices, $p\geqslant 2$, replace $p-1$ by $p-2$ in both bounds.

Proof. Choose the external root in T_1. Integrate all external slots of the other vertices in absolute value. Their two-slot marginals $\bar{T}_i$ have both Schur masses at most $s(T_i)$, by symmetry. The open chain between the two operator slots of T_1 starts and ends with an inverse. Its kernel is bounded by

$$\|H(u,\cdot)\|_2\|\text{middle chain}\|_{2\to 2}\|H(\cdot,v)\|_2\leqslant\mathfrak{l}(H)^2m(H)^{p-1}\prod_{i>1}s(T_i).$$

Integrating the rooted T_1 gives (20.5). Absolute kernels were used throughout; no cancellation or positivity of H was assumed.

For the spatial assertion, translation invariance identifies every rooted integral with its full spatial average. If the marked pair is in one vertex, charge its distance to that vertex's v_1. Otherwise follow either arc of the closed cycle between the two marked external slots. The triangle inequality charges each piece to a within-vertex displacement or an inverse edge. A vertex charge gives a two-slot marginal with weighted row and column masses at most $s(T_i)+\sqrt{q}v_1(T_i)$. For edge charges the two arcs contain an inverse each. After the intervening vertices have been collapsed, write their convolution chains as E_1,E_2. The closing integrals obey

$$\int(1+\sqrt{q}d)|E_1(z)E_2(-z)|\,\mathrm{d}z\leqslant\|E_1\|_{\mathcal{H}_q}\|E_2\|_{\mathcal{H}_q}.$$

Use one inverse in each $\mathcal{H}_q$ norm and every other inverse and collapsed vertex in $\mathcal{A}_q$, using (19.3). There are $p-1$ remaining inverses. This gives (20.6); there are at most Cp distance charges. The identical proof treats marked factors and the version with one fewer inverse. The weighted estimate uses $q^{-1/2}$; the sharper unweighted factor $\mathfrak{l}(H)^2\leqslant Ca_2(q)$ is retained separately when needed. □

Remark 20.2. (The vertex must pay for its own spatial extent) Take the nonlocal fourth vertex $T(x_1,x_2,x_3,x_4)=1$. Since $C_q^2 1=q^{-2}1$, its contraction is the constant two-slot kernel $\mathrm{Ctr}_{C_q^2}T=q^{-2}$. Consequently

$$\frac{v_1(\mathrm{Ctr}_{C_q^2}T)}{s(\mathrm{Ctr}_{C_q^2}T)}=\int_{\mathbb{T}^3}d(0,z)\,\mathrm{d}z>0.$$

On a lattice replace the integral by its normalised sum. Thus closing two free inverses cannot by itself give a $q^{-1/2}$ spatial gain for an arbitrary nonlocal vertex. This is why (20.6) retains every within-vertex moment.

Lemma 20.3. (An inserted multiplication and a nonlocal vertex) *For arbitrary kernels L, L', a symmetric third tensor T, and a multiplication input ϕ, the two linear maps*

$$\phi \mapsto \operatorname{diag}(L M_\phi L'), \quad \phi \mapsto \operatorname{Ctr}_{L M_\phi L'} T$$

have L^p operator norms at most $\mathfrak{l}(L)\mathfrak{l}(L')$ and $s(T)\mathfrak{l}(L)\mathfrak{l}(L')$, respectively, for every $1 \leqslant p \leqslant \infty$. If a bounded covariance kernel E satisfies $\|E\|_\infty := \sup_{x,z} |E(x,z)| \leqslant c$, inserting a multiplication input in any one slot of $\operatorname{Ctr}_E T$ costs at most $c\, s(T) \|\phi\|_p$.

Proof. For the diagonal map the positive majorant kernel is $|L(x,a)L'(a,x)|$. Its row and column masses are bounded by $\mathfrak{l}(L)\mathfrak{l}(L')$, using Cauchy–Schwarz in a and x, respectively.

For the tensor contraction its majorant is

$$\mathcal{J}(x,a) = \int |T(x,u,v)||L(u,a)L'(a,v)| \, \mathrm{d}u \, \mathrm{d}v.$$

For its row mass first integrate a, obtaining at most $\mathfrak{l}(L)\mathfrak{l}(L')s(T)$. For its column mass put $\bar{T}(u, v) = \int |T(x,u,v)| \, \mathrm{d}x$. Symmetry makes $\|\bar{T}\|_{2\to 2} \leqslant s(T)$, and therefore

$$\int \mathcal{J}(x,a) \, \mathrm{d}x \leqslant s(T) \|L(\cdot,a)\|_2 \|L'(a,\cdot)\|_2.$$

Schur's inequality proves the assertion, including both endpoints $p = 1, \infty$. For an input in a contracted slot, bound E by c and collapse the unused slot of T; symmetry again supplies both Schur masses. An input in the open slot multiplies a function bounded by $c\, s(T)$. These proofs also apply to diagonal vertex measures: the physical delta integrates to one, so it introduces no extra lattice factor. □

20.3 The finite derivative budgets

Lemma 20.4. (Finite trace calculus) *This is step (a) of the proof of Proposition 11.2 of Lecture 6. For $4 \leqslant k \leqslant 10$, $\mathcal{N}_{k,q}(\mathcal{L}) \leqslant C_k g_q q\, r_q^{2-k}$. For the trace comparison fix $k \geqslant 4$. Assume that $h, \hat{h}$ are translation invariant and even, have zero constant and quadratic jets, and obey the* individual *bounds*

$$\mathcal{N}_{j,q}(h) + \mathcal{N}_{j,q}(\hat{h}) \leqslant C_j g_q q\, r_q^{2-j}, \quad 4 \leqslant j \leqslant k+2.$$

Assume also that M lies in the coefficient ball $|\mu|/q + \eta_q(\Theta) \leqslant C_0(g_q + g_q^2)$ and that q is large enough for all the corresponding row Neumann series to converge. If the difference obeys $\mathcal{N}_{j,q}(h - \hat{h}) \leqslant \varepsilon_q q\, r_q^{2-j}$, $4 \leqslant j \leqslant k+2$, then

$$\mathcal{N}_{k,q}(\mathcal{T}_q[M,h] - \mathcal{T}_q[M,\hat{h}]) \leqslant C_k \varepsilon_q q^{1/2} r_q^{-k}. \tag{20.7}$$

Changing M between two paths in the same ball gives the bound with $|\delta\mu|/q + q^{-1/2} v_1(\delta\Theta)$ replacing ε_q. The initial comparison $\Pi(\mathcal{T}_q[M,\mathcal{L}] - \partial_q \mathcal{L})$ has $\varepsilon_q = C(g_q + |\mu|/q + \eta_q(\Theta))$.

Proof. Let $H = (K_q + M_q + D^2(U+h)(y))^{-1}$, and $\hat{H}$ the same with $\hat{h}$. Taylor's formula, evenness, and the zero quadratic jet give $\mathsf{S}_2(h) \leqslant C g_q q$ and $\mathsf{S}_3(h) \leqslant C g_q q / r_q$; the same holds for $\hat{h}$, and for their difference with g_q replaced by ε_q. Thus the hypotheses control every Hessian vertex used in differentiating the inverse, including those with zero or one field derivative. Together with the local quartic they give $s(D^{j+2}(U+h)) \leqslant C_j q\, r_q^{-j}$ for $0 \leqslant j \leqslant k$.

More explicitly, $D^k H[u_1,\dots,u_k]$ is the sum over ordered partitions $(P_1,\dots,P_p)$ of the index set $1,\dots,k$ of

$$(-1)^p H D^{|P_1|+2}(U+h)[u_{P_1}]H\cdots D^{|P_p|+2}(U+h)[u_{P_p}]H.$$

Here two slots of each vertex are operator slots, and u_P fills its remaining slots. Taking the trace closes this chain. Lemma 20.1 gives its rooted absolute bound

$$C_k a_2(q) q^{-(p-1)} \prod_i s(T_i) \leqslant C_k q^{1/2} r_q^{-k}.$$

The factor $a_2(q)$ comes from the two L^2 inverse rows, and each of the other $p-1$ inverses costs C/q. Symmetry gives both Schur bounds for every collapsed vertex, also at nonconstant fields. The row Neumann series preserves the row and row-L^2 bounds: use (19.5) for M, and the just proved S_2 bound for the nonlinear Hessian. All constants depend on the finite list of individual derivative bounds above.

Telescope the two partition sums. A changed vertex contributes ε_q by the difference budgets. A changed inverse equals $-H(D^2h - D^2\hat{h})\hat{H}$ and contributes the same factor, since its extra inverse costs q^{-1} and its Hessian costs $\varepsilon_q q$. More precisely, group that product as one marked inverse; its m and ℓ bounds are $C\varepsilon_q/q$ and $C\varepsilon_q\sqrt{a_2(q)}$. This proves the unweighted part of (20.7). The worst vertex uses $k+2$ derivatives.

For a quadratic inverse difference put $\rho = |\delta\mu|/q + q^{-1/2} v_1(\delta\Theta)$. At a nonconstant field, (19.5) and the row Neumann series give $\|H\delta M\|_{\infty\to\infty} \leqslant C\rho$. Thus $\delta H = -H\delta M\hat{H}$ has absolute row mass $C\rho/q$ and row L^2 bound $C\rho\sqrt{a_2(q)}$. Transposing and exchanging the two backgrounds supplies the column bounds. Apply the same cycle estimate with this marked inverse. One must not replace this argument by an absolute row bound on $\delta\Theta$ itself.

At zero field all inverses are the convolution $R_0 = (K+M)^{-1}$. The centred cancellation and the convolution Neumann series give

$$\|R_0\|_{\mathcal{A}_q} \leqslant C/q, \quad \|R_0\|_{\mathcal{H}_q} \leqslant C q^{-1/4}.$$

The identity $\delta R_0 = -R_0 \delta M \hat{R}_0$, with $\|C_q \delta M\|_{\mathcal{A}_q} \leqslant C\rho$, gives the same two bounds multiplied by ρ for δR_0. Equation (20.6) now proves the first-moment part of the trace comparison, including changes of M. Within-vertex distances use exactly the individual moment budgets in the statement.

For the seed, a derivative of order $k \geqslant 4$ of the log determinant has $p \geqslant 2$ inverse factors and p vertices: the local potential λy^2 has only first and second derivatives. The same cycle cut has $p-2$ intermediate inverses and gives $C_k q^2 a_2(q) r_q^{-k} \leqslant C_k g_q q\, r_q^{2-k}$. At zero its degree $2j$ term is $(-1)^{j+1}\mathrm{Tr}(C_q \lambda y^2)^j/(2j)$; the same two-chain argument proves its spatial bound. Taylor gives $\mathsf{S}_2 \leqslant C g_q q$, $\mathsf{S}_3 \leqslant C g_q q/r_q$. Finally (20.2) says that $\mathcal{T}_q[0,0] - \partial_q \mathcal{L}_q$ is purely quadratic. Consequently

$$\Pi(\mathcal{T}_q[M,\mathcal{L}] - \partial_q \mathcal{L}) = \Pi(\mathcal{T}_q[M,\mathcal{L}] - \mathcal{T}_q[M,0] + \mathcal{T}_q[M,0] - \mathcal{T}_q[0,0]).$$

The first difference has a marked seed vertex and factor g_q; the second has a marked quadratic inverse and factor $|\mu|/q + \eta_q(\Theta)$. This proves the stated initial comparison. □

Remark 20.5. (Small Hessians alone do not suffice) The higher individual bounds in this lemma are necessary (compare Exercise 11.3 of Lecture 6). On the one-site lattice take $q = \lambda = 1$, $M = 0$, and fixed small $\alpha, \delta > 0$. Put

$$\hat{h}_n(y) = \frac{\alpha}{n^2}\left(\cos(ny) - 1 + \frac{n^2y^2}{2}\right), \quad h_n(y) = \hat{h}_n(y) + \frac{\delta}{12}y^4.$$

Both corrections have zero quadratic jet and uniformly small Hessians on $|y|<2$, while all derivatives of their difference through order six are bounded independently of n. Nevertheless, expanding $(1+y^2+h_n''(y))^{-1}$ to order four gives

$$D^4(\mathcal{T}[h_n]-\mathcal{T}[\hat{h}_n])(0)=12(2\delta+\alpha\delta n^2+\delta^2).$$

Thus a trace Lipschitz estimate cannot follow from Hessian smallness and difference bounds alone. The example rescales to any q,λ: put $y=r_q s$ and replace each correction $h(s)$ by $q r_q^2 h(y/r_q)$. The displayed trace derivative is then multiplied by $q^{-1}r_q^{-4}$ and still diverges with n at fixed q. In our recursion the needed higher individual bounds follow at each finite step from the seed and the previous increment; they are not extra assumptions on the field.

20.4 The coefficients, with a boundary value

Retain the quadratic jet and set, with $R_{0,q}=(K_q+M_q)^{-1}$, as in (11.9),

$$\mathfrak{S}_q[M,h]=-\lambda\mathrm{Tr}(R_{0,q}^2)I-\frac{1}{2}\mathrm{C}\mathrm{tr}_{R_{0,q}^2}D^4h(0),\quad \mathfrak{S}_q 1=\beta_q 1,\quad \mathfrak{S}_q:=\mathfrak{S}_q[M^{[3]},h^{[3]}]. \tag{20.8}$$

Let $\mathcal{B}=[-B_0\lambda q_*^{1/2},B_0\lambda q_*^{1/2}]$, an interval of boundary values, not to be confused with the field ball $\mathcal{B}_q$. For $b_*\in\mathcal{B}$ the scalar mass solves

$$\mu_q=b_*+\int_{q_*}^{q}\beta_s\,\mathrm{d}s,\quad \beta_s=\langle\mathfrak{S}_s[M^{[3]},h^{[3]}]\rangle. \tag{20.9}$$

Here $\mathfrak{S}_q[M,h]$ is a deterministic operator-valued source, the linear coefficient of the loop force at zero field, and β_q is the constant-mode eigenvalue of the final one. The identity operator is written I when its space is clear. Translation invariance makes the constant mode a scalar. The centred parts of the sources at the first two levels define $\Theta^{[2]}$, $\Theta^{[3]}$ in (20.4), integrated from infinity; only the scalar of the final level is imposed, from q_*. The linear force is $M_q:=M_q^{[3]}=\mu_q I+\Theta_q^{[3]}$, so that $\partial_q M=\beta I+\mathfrak{S}_q[M^{[2]},h^{[2]}]^\circ$, $\mu_{q_*}=b_*$ and $v_1(\Theta_q)\to 0$ as $q\to\infty$. The bare mass is $\mu_N=\lim_{q\to\infty}\mu_q$, as in (11.11), whose existence is a fixed-lattice assertion proved below. Note that q_0 does not occur in (20.4), (20.8) or (20.9): these operators are built from K_q and the level masses alone.

Give scalar paths on $[q_*,\infty)$ the norm

$$\|\mu\|_*=\sup_q\frac{|\mu_q|}{\lambda\sqrt{q}}.$$

Proposition 20.6. (The coefficient construction closes) *This is Proposition 11.2 of Lecture 6, with the dependence on b_* made explicit; the bare mass is treated in Lemma 20.7. There is a universal constant $C_*\geqslant 1$ and, for each $B_0<\infty$, a scale $\hat{q}=\hat{q}(\lambda,B_0)$ such that the following holds for $q_*\geqslant\hat{q}$, $q_*\geqslant\max(1,q_0)$, and $b_*\in\mathcal{B}$. With $R=B_0+C_*$, the recursion (20.4) and the scalar problem (20.8)–(20.9) on $[q_*,\infty)$ have a unique solution with $\|\mu\|_*\leqslant R$, and, as in (11.12),*

$$|\mu_q|\leqslant C\lambda\sqrt{q},\quad v_1(\Theta_q^{[j]})\leqslant C\lambda^2 q^{-1/2},\quad \|\partial_q M_q\|_{\mathrm{row}}\leqslant C g_q,\quad \eta_q(\Theta^{[3]}-\Theta^{[2]})\leqslant C g_q^3. \tag{20.10}$$

Starting with ten derivatives leaves six at the final depth, and, as in (11.13),

$$\mathcal{N}_{k,q}(h^{[j]}-h^{[j-1]})\leqslant C_k g_q^j q r_q^{2-k},\quad 4\leqslant k\leqslant 12-2j,\quad j=2,3. \tag{20.11}$$

The complete defect of $\Gamma_q^{\mathrm{o}}=U+\frac{1}{2}\langle y,M_q y\rangle+h_q^{[3]}$ on $\mathcal{B}_q$ is, as in (11.14), $e_q^{\mathrm{o}}=\Pi(\mathcal{J}_q[M^{[2]},h^{[2]}]-\mathcal{J}_q[M^{[3]},h^{[3]}])+\frac{1}{2}\langle y,\mathcal{E}_q^c y\rangle$, with the centred kernel $\mathcal{E}_q^c=\mathfrak{S}_q[M^{[2]},h^{[2]}]^\circ-\mathfrak{S}_q[M^{[3]},h^{[3]}]^\circ$, and

$$\mathsf{S}_4(e_q^{\mathrm{o}})\leqslant C\lambda^5 q^{-3},\quad \|\mathcal{E}_q^c\|_{\mathrm{row}}+q^{1/2}v_1(\mathcal{E}_q^c)\leqslant C g_q^4. \tag{20.12}$$

The constants in (20.10)–(20.12) may depend on B_0, but not on N or on $b_* \in \mathcal{B}$.

Proof. Here every Θ is even and centred. At each fixed N the norm $\|\cdot\|_*$ defines a complete weighted supremum space of continuous scalar paths; restrict to the closed subset $\mu_{q_*} = b_*$. It meets the ball of radius R, since $|b_*|/(\lambda\sqrt{q}) \leqslant B_0(q_*/q)^{1/2} \leqslant B_0$ for $q \geqslant q_*$. Work provisionally with radius $R \geqslant 1$ and $R g_{q_*}$ smaller than a fixed constant.

The recursion on a given scalar path. Apply the trace lemma by a finite induction, with the level masses. The seed supplies its individual bounds through order ten, and $\Theta^{[1]} = 0$. The first comparison, $\Pi(\mathcal{T}[M^{[1]}, \mathcal{L}] - \partial\mathcal{L})$, with $M^{[1]} = \mu I$, is the initial comparison of Lemma 20.4 with $\varepsilon_q = C(g_q + |\mu_q|/q) \leqslant C R g_q$; it constructs $h^{[2]} - h^{[1]}$ through order eight, and adding the seed gives the individual bounds for $h^{[2]}$ at these orders. The centred source of the first level, $\mathfrak{S}_q[M^{[1]}, \mathcal{L}]^\circ$, contracts the fourth seed vertex with $R^2_{0,q}$, whose supremum is $C q^{-1/2}$; the distance stays in the vertex, so its first moment is at most $C\lambda^2 q^{-3/2}$ and $v_1(\Theta^{[2]}_q) \leqslant 2C\lambda^2 q^{-1/2}$, that is $\eta_q(\Theta^{[2]}) \leqslant C g_q^2$. The local quartic contributes only to the scalar. The second comparison, $\mathcal{T}[M^{[2]}, h^{[2]}] - \mathcal{T}[M^{[1]}, h^{[1]}]$, changes the vertex by $h^{[2]} - h^{[1]}$ and the mass by the centred kernel $\Theta^{[2]}$: write it as a vertex comparison at the mass $M^{[2]}$ plus a mass comparison at the primitive $h^{[1]}$, both covered by Lemma 20.4, the second with $\eta_q(\Theta^{[2]})$ replacing ε_q. Both have relative size g_q^2, so it constructs $h^{[3]} - h^{[2]}$ through order six and hence the individual bounds for $h^{[3]}$. The centred source changes by the same two insertions: contracting the fourth vertex of $h^{[2]} - h^{[1]}$ with R^2_0, and the fourth vertex of $h^{[1]}$ with the marked difference of R^2_0, whose supremum is $C\eta_q(\Theta^{[2]})q^{-1/2}$ by the marked inverse bounds of Lemma 20.4, gives a first moment $C q^{-1/2} g_q^3$, and integrating from infinity $\eta_q(\Theta^{[3]} - \Theta^{[2]}) \leqslant C g_q^3$. These in turn permit the final fourth-order defect comparison. At each step the Hessian Schur norm is $C g_q q$, and every level mass lies in the coefficient ball, $|\mu_q|/q + \eta_q(\Theta^{[j]}) \leqslant C R(g_q + g_q^2)$, so the inverses exist before the next trace is used. The centred kernels enter only through their first moments, absorbed in the resolvents by (19.5); they are never cycle vertices. In this induction the trace lemma and

$$\int_q^\infty s^{1-k/2-(a+1)/2}\,\mathrm{d}s = \frac{q^{2-k/2-(a+1)/2}}{k/2-2+(a+1)/2}, \qquad k \geqslant 4, \quad a \geqslant 1,$$

prove the two increment bounds. Here are the details at the improper endpoint. Put $F_s^{[j]} = \Pi(\mathcal{T}_s[M^{[j]}, h^{[j]}] - \partial_s\mathcal{L}_s)$. At the first step the initial comparison gives, for $4 \leqslant k \leqslant 8$,

$$\mathcal{N}_{k,s}(F^{[1]}) \leqslant C_k(1+R) g_s s^{1/2} r_s^{-k} = C_k(1+R)\lambda^{1+k/2} s^{-k/2}.$$

For $F^{[2]}$ through order six, telescope it as $F^{[1]} + \Pi(\mathcal{T}[M^{[2]}, h^{[2]}] - \mathcal{T}[M^{[1]}, h^{[1]}])$. The additional term has the smaller factor g_s^2 instead of g_s. All these powers are integrable. In the moment part of the integral, $\sqrt{q} \leqslant \sqrt{s}$ permits the same domination for $\mathcal{N}_{k,q}$. The projected integrands have zero constant and quadratic jets and are even. Taylor's formula therefore bounds their derivatives of order zero through three, on each fixed bounded field set, by the integrable fourth-order bound $C_R\lambda^3 s^{-2}$ times the corresponding field powers. Thus differentiation under the improper integral is justified through order eight at depth two and six at depth three. No unknown terminal jets or interacting moments are being imposed at infinity.

For fixed q and any compact subset of $\mathcal{B}_q$, the same bounds hold in a common neighbourhood in field and scale, since $\mathcal{B}_s$ increases with s. The lower-endpoint derivative is therefore legitimate, and $\Pi\partial_q\mathcal{L}_q = \partial_q\mathcal{L}_q$ gives

$$\partial_q h^{[j+1]} = \Pi\mathcal{T}_q[M^{[j]}, h^{[j]}].$$

Moreover $\mathsf{S}_4(h^{[j]}) \leqslant C\lambda^2 q^{-1/2}$ and the zero low jets show that each $h_q^{[j]}$ and its first three field derivatives vanish as $q \to \infty$ on any fixed bounded field set.

The scalar fixed point. Contracting fourth tensors with the kernel R_0^2, whose supremum is $Cq^{-1/2}$, gives for the source $\mathfrak{S}_q = \mathfrak{S}_q[M^{[j]}, h^{[j]}]$ at every level j

$$|\beta_q| + \|\mathfrak{S}_q\|_{\mathrm{row}} \leqslant C(g_q + g_q^2), \quad v_1(\mathfrak{S}_q^\circ) \leqslant C\lambda^2 q^{-3/2}.$$

In the centred part the distance stays in the fourth tensor. These bounds give $\|\partial_q M\|_{\mathrm{row}} \leqslant C g_q$ and, for the scalar map $\mathcal{A}$ defined by the right side of (20.9),

$$\frac{|\mathcal{A}[\mu]_q|}{\lambda\sqrt{q}} \leqslant \frac{|b_*|}{\lambda\sqrt{q}} + C(1 + g_{q_*}) \leqslant B_0 + C(1 + g_{q_*}).$$

Here is the order of the constants. With $R \geqslant 1$ and $R\, g_{q_*}$ smaller than a fixed constant, the inverse Neumann sums are bounded, for example, by two. The first increment has size $C(1 + R)g_q^2$, the next has one further g_q; after adding the seed, the individual derivative budgets have constants independent of R in this regime. Thus the source bounds above have a fixed constant, and the output bound reads $B_0 + C_*$, with $C_* \geqslant 1$ independent of R and of B_0. Enlarging C_* if necessary, the centred outputs obey $v_1(\Theta_q^{[j]}) \leqslant C_* \lambda^2 q^{-1/2}$ with the same constant, so every level mass lies in the ball $\|M^{[j]}\|_* \leqslant R$ of (23.1). Now put $R = B_0 + C_*$ and then choose q_* to satisfy the provisional smallness requirement.

For two scalar paths $\mu, \hat{\mu}$ in the ball with the same boundary value b_*, put $d = \|\mu - \hat{\mu}\|_*$. The difference is a scalar insertion of relative size $|\delta\mu_q| / q \leqslant g_q d$ at every level. Through the finite recursion, the trace comparison with mass changes gives $\mathcal{N}_k(\delta h^{[j]}) \leqslant C\, g_q^2 q\, r_q^{2-k} d$ and, integrating the centred sources from infinity, $\eta_q(\delta\Theta^{[j]}) \leqslant C g_q^3 d$. Contracting with R_0^2 as above, the scalar source changes by $|\delta\beta_q| \leqslant C g_q^2 d$. The scalar integral in (20.9) starts from the same value, so

$$|\delta\mathcal{A}[\mu]_q| \leqslant C\lambda^2 \log(q/q_*) d.$$

After normalisation this is at most $C g_{q_*} d \log x / \sqrt{x}$ with $x = q/q_* \geqslant 1$; the supremum of $\log x / \sqrt{x}$ is $2/\mathrm{e}$. This is a contraction estimate on the whole half-line, with no prescribed bound on the bare mass. Increasing q_* once more makes it less than $d/2$. The closed ball is complete, so Banach's theorem gives the solution. Every choice here depends on the coupling, on B_0 and on the finite derivative budget, not on N or on a probability moment order.

The defect. Evenness gives $D^2\big(\frac{1}{2}\mathrm{Tr}\, H\big)(0) = -\frac{1}{2}\mathrm{C}\mathrm{tr}_{R_0^2} D^4\Gamma^{\mathrm{o}}(0)$. The scalar mass equation supplies exactly the constant mode of this quadratic jet, while $\partial_q h^{[3]} = \Pi\widehat{\mathcal{J}}_q[M^{[2]}, h^{[2]}]$ and $\partial_q\Theta^{[3]} = \mathfrak{S}_q[M^{[2]}, h^{[2]}]^\circ$. This proves the defect identity. Apply the trace bound to the last increment to obtain the fourth-derivative estimate, with $\varepsilon_q \leqslant C g_q^3$ from (20.11) and $\eta_q(\Theta^{[3]} - \Theta^{[2]}) \leqslant C g_q^3$. The centred kernel $\mathcal{E}_q^c$ has no tadpole contribution, since the local term is a multiple of the identity; it is the centred part of $\mathrm{Ctr}_{\delta(R_0^2)} D^4 h^{[3]}(0) + \mathrm{Ctr}_{R_0^2} D^4(h^{[3]} - h^{[2]})(0)$, up to the factor $-\frac{1}{2}$. The row norm of a contraction is at most the supremum of the kernel times the rooted norm of the tensor, and that of a centred part at most twice that of the operator. With $\sup|\delta(R_0^2)| \leqslant C g_q^3 q^{-1/2}$ and $\sup|R_0^2| \leqslant C q^{-1/2}$, this gives $\|\mathcal{E}_q^c\|_{\mathrm{row}} \leqslant C q^{-1/2}(g_q^3 \lambda g_q + \lambda g_q^3) \leqslant C g_q^4$; the first moment follows in the same way, the distance staying in the vertex. □

From now on $q \geqslant q_*$ whenever $\bar{\Gamma}_q$ occurs, and all constants may depend on λ, B_0 but not on N or on $b_* \in \mathcal{B}$.

Lemma 20.7. (The bare mass exists at each lattice) *This is the last assertion of Proposition 11.2 of Lecture 6. For each fixed N, the preceding infinite-scale construction satisfies*

$$\|\mathfrak{S}_q\|_{\mathrm{row}} + |\beta_q| \leqslant C\lambda(1 + g_q) a_2(q), \quad |\mu_N - b| \leqslant C\lambda \mathrm{Tr}\, C_{q_*} \leqslant C\lambda N, \tag{20.13}$$

$$|\mu_N - \mu_q| \leqslant C\lambda N^3 / q, \quad \|\Theta_q\|_{\mathrm{row}} \leqslant C N \lambda^2 q^{-1/2} \longrightarrow 0.$$

The last two limits concern $q\to\infty$ at fixed N. In particular $|\mu_N|\leqslant|b_*|+C\lambda N$, and $\operatorname{Tr}C_{q_*}\leqslant\operatorname{Tr}C$.

Proof. Keep the sharp finite-lattice bubble in the unweighted trace bounds: $\|R^2_{0,q}\|_\infty\leqslant Ca_2(q)$ and $\mathsf{S}_4(h_q^{[3]})\leqslant C\lambda^2q^{-1/2}=C\lambda g_q$. Contracting the fourth tensor in (20.8) therefore gives the first inequality, including both Schur masses. Since $\int_{q_*}^\infty a_2(s)\,\mathrm{d}s=\operatorname{Tr}C_{q_*}$, its scalar integral is absolutely convergent, and (20.9) gives the bound on μ_N-b_*. The Fourier symbol bound gives $\operatorname{Tr}C_{q_*}\leqslant C\sum_{|k|\leqslant CN}(1+|k|^2)^{-1}\leqslant CN$. For the upper tail use $a_2(s)\leqslant N^3s^{-2}$. Finally every nonzero lattice displacement has distance at least N^{-1}. The absolute off-diagonal row mass of Θ is at most $Nv_1(\Theta)$; centring bounds the diagonal by that same mass. This proves the last estimate. For the one-site lattice $\Theta=0$. □

The leading N-dependent part of μ_N is $-\lambda\operatorname{Tr}C_{q_*}$, and it differs from $-\lambda\operatorname{Tr}C$ by a quantity bounded in N. The next quadratic contraction contains the logarithmic two-loop mass divergence. Keeping its full running kernel avoids an incorrect local replacement. There is no quartic or kinetic counterterm in the bare potential. The number μ_N generally diverges as $N\to\infty$; only the running mass at each finite scale has a continuum limit. We never interchange these two limits of the mass. The explicit form of μ_N is Theorem 22.2, and the limits of the coefficients are in Proposition 22.1.

Proposition 20.8. (Dependence on the boundary value) *This is the step Response of the proof of Theorem 11.3 of Lecture 6. For q_* large depending on λ,B_0, let $b_{*,1},b_{*,2}\in\mathcal{B}$ with scalar paths $\mu^{(1)}$, $\mu^{(2)}$ and level masses $M^{[j],(1)},M^{[j],(2)}$, and $\delta b=b_{*,1}-b_{*,2}$. Then at every fixed N,*

$$\sup_{q\geqslant q_*}|\delta\mu_q|\leqslant 2|\delta b_*|,\quad \sup_{q\geqslant q_*}q^{3/2}v_1(\delta\Theta_q^{[j]})\leqslant C\lambda^2|\delta b_*|,\quad |\delta\mu_N-\delta b_*|\leqslant C\lambda q_*^{-1/2}|\delta b_*|. \tag{20.14}$$

Proof. Let $D=\sup_q|\delta\mu_q|$ and $X=\sup_{q,j}q^{3/2}v_1(\delta\Theta_q^{[j]})$. Both are finite at fixed N: Lemma 20.7 gives $a_2(q)=O_N(q^{-2})$, so the centred source is $O_N(q^{-3})$ and $v_1(\Theta_q)=O_N(q^{-2})$. Put $\rho_q=D/q+Xq^{-2}$, which bounds $|\delta\mu_q|/q+q^{-1/2}v_1(\delta\Theta_q)$. The trace comparison gives $\mathcal{N}_{k,s}(\mathcal{J}_s[M^{[j],(1)},h]-\mathcal{J}_s[M^{[j],(2)},h])\leqslant C\rho_s s^{1/2}r_s^{-k}$ at each level. Since the seed does not depend on the mass, integrating from q to ∞ with $k=4$ and using the finite recursion,

$$s(D^4\delta h_q^{[3]}(0))\leqslant C\lambda^2(Dq^{-3/2}+Xq^{-5/2}),\quad v_1(D^4\delta h_q^{[3]}(0))\leqslant C\lambda^2(Dq^{-2}+Xq^{-3}).$$

Let $R_0,\hat R_0$ be the two zero-field inverses. These are commuting Fourier multipliers, each with symbol bounded by $C(q+c|k|^2)^{-1}$ by (19.5), and $\delta(R_0^2)=-(R_0^2\hat R_0+R_0\hat R_0^2)\delta M$. So the local term of (20.8) varies by $\lambda\operatorname{Tr}((R_0^2\hat R_0+R_0\hat R_0^2)\delta M)$, to leading order $2\lambda\operatorname{Tr}(R_0^3\delta M)$. This is at most $C\lambda(Dq^{-3/2}+Xq^{-5/2})$, as $|\widehat{\delta\Theta}(k)|\leqslant 2|k|v_1(\delta\Theta)$ and $\sum_k|k|(q+c|k|^2)^{-3}\leqslant Cq^{-1}$. Contracting the fourth vertex with R_0^2, whose supremum is $Cq^{-1/2}$, and the vertex $D^4h^{[3]}(0)$ with $\delta(R_0^2)$, whose supremum is at most $C\rho_qq^{-1/2}$ by the marked inverse bounds of Lemma 20.4, gives

$$|\delta\beta_q|\leqslant C\lambda(Dq^{-3/2}+Xq^{-5/2}),\quad v_1(\delta\mathfrak{S}_q-\delta\beta_qI)\leqslant C\lambda^2(Dq^{-5/2}+Xq^{-7/2}).$$

The same bounds hold for the centred sources of the iteration at each level, whose kernels are integrated from infinity; as the kernels are outputs of the finite recursion, this gives directly $X\leqslant C\lambda^2D$, the bound at the first level feeding the second. Integrating the scalar part from q_* to q,

$$D\leqslant|\delta b_*|+C\lambda q_*^{-1/2}D+C\lambda q_*^{-3/2}X.$$

For q_* large this gives $D \leqslant 2|\delta b_*|$, and integrating $\delta\beta$ over $[q_*, \infty)$ gives the last bound. □

21 Global coercivity and the complete large-field error

This section proves (12.2), Lemma 12.1 and Proposition 12.2 of Lecture 6.

21.1 Clipping

Choose a smooth odd τ, equal to the identity on $[-1,1]$, constant outside $[-2,2]$, with $0 \leqslant \tau' \leqslant 1$ and $|\tau| < 2$. Define pointwise, as in (12.1),

$$\kappa_q(y)(x) = r_q \tau(y(x)/r_q), \quad \tilde{h}_q = h_q^{[3]} \circ \kappa_q, \quad \bar{\Gamma}_q = U + \frac{1}{2}\langle y, M_q y\rangle + \tilde{h}_q. \tag{21.1}$$

The quartic and quadratic terms are not clipped. The finite chain rule gives globally, as in (12.2),

$$s(D^j \tilde{h}_q) \leqslant C g_q q r_q^{2-j} \quad (j = 2,3,4), \quad s(D^2 \partial_q \tilde{h}_q) \leqslant C g_q. \tag{21.2}$$

For the scale term use $\partial_q \kappa_q y = (\kappa_q y - y\tau'(y/r_q))/(2q)$, of size $C r_q/q$, and differentiate the composition. These terms remain in the defect.

Proof. We verify the clipping budgets (21.2). In this proof $h = h_q^{[3]}$ is the unclipped correction and $J = D\kappa_q(y)$ is the diagonal operator multiplying by $\tau'(y/r_q)$. Taylor on the unclipped ball gives $\|D h\|_\infty \leqslant C g_q q r_q$ as well as the stated Hessian and higher budgets. In $D^j(h \circ \kappa_q)$, each partition with m blocks has one $D^m h$ and clipping factors whose product is bounded by $C r_q^{m-j}$. Indeed $|\kappa_q^{(a)}| \leqslant C_a r_q^{1-a}$ for $a \geqslant 1$. Coincident slots give physical deltas integrating to one. Root any external slot and use symmetry of the h vertex to obtain the asserted Schur bound for $2 \leqslant j \leqslant 4$.

For scale differentiation first use $\partial_q h^{[3]} = \Pi \widetilde{J_q}[M^{[2]}, h^{[2]}]$. With $\Gamma^{[2]} = U + h^{[2]}$ and $H^{[2]} = (K_q + M_q^{[2]} + D^2\Gamma^{[2]})^{-1}$, differentiating this trace twice gives the two types

$$\mathrm{Tr}(H^{[2]} D^3\Gamma^{[2]}[u] H^{[2]} D^3\Gamma^{[2]}[v] H^{[2]}), \quad -\frac{1}{2}\mathrm{Tr}(H^{[2]} D^4\Gamma^{[2]}[u,v] H^{[2]}).$$

Lemma 20.1 bounds both rooted kernels by $C a_2(q) q r_q^{-2} \leqslant C g_q$. Subtracting the quadratic jet at zero preserves this bound. Hence $\mathsf{S}_2(\partial_q h^{[3]}) \leqslant C g_q$ and $\|D\partial_q h^{[3]}\|_\infty \leqslant C g_q r_q$ on the ball. Write $z = \kappa_q y$ and $w = \partial_q \kappa_q y$. Globally,

$$|w| \leqslant C r_q/q, \quad |Dw| \leqslant C/q, \quad |D^2 w| \leqslant C/(q r_q), \quad \partial_q \tilde{h} = (\partial_q h)(z) + \langle Dh(z), w\rangle.$$

Two derivatives of the first term cost $C g_q$ by the preceding two bounds. The second gives respectively $D^3 h[J,J,w]$, $D^2 h[\kappa_q'', w]$, the two terms $D^2 h[J, Dw]$, and $Dh D^2 w$. Each costs $C g_q$ in rooted Schur norm, using the displayed budgets. This proves the scale assertion in (21.2), including motion of the clipping scale. □

21.2 Global coercivity

Write $H_q(y) = (K_q + D^2\bar{\Gamma}_q(y))^{-1}$, as in (10.11).

Lemma 21.1. (Uniform resolvent bounds) *This is Lemma 12.1 of Lecture 6, with one further bound. For large q_* the inverse exists globally, and as in (12.3), for all $y\in H_N$ and $q\geqslant q_*$,*

$$c(K_q+\lambda y^2)\leqslant H_q(y)^{-1}\leqslant C(K_q+\lambda y^2),\quad \|H_q(y)\|_{p\to p}\leqslant\frac{1+C(g_q+g_q^2)}{q}\quad (1\leqslant p\leqslant\infty),\tag{21.3}$$

$$\operatorname{diag}H_q(y)^2\leqslant Ca_2(q),\quad M_yH_q(y)^2M_y\leqslant C(\lambda q)^{-1}I.$$

Proof. Use (19.5) and $\|D^2\tilde h\|_{2\to2}\leqslant Cg_qq$ for the forms. The row Neumann series in $H=(I+G(M+D^2\tilde h))^{-1}G$ has first-factor norm $1+C(g_q+g_q^2)$. Symmetry gives columns and interpolation gives p. Every row of G has L^2 norm at most $\sqrt{a_2(q)}$; Minkowski in the same series proves the covariance estimate. Finally use $H^2\leqslant(C/q)H$ and $M_yHM_y\leqslant C\lambda^{-1}I$. One must not square a Loewner inequality to obtain this assertion. □

21.3 The complete defect

The defect is $e_q(y)=\partial_q\bar\Gamma_q(y)-\frac12\operatorname{Tr}[H_q(y)-H_q(0)]$ and the drift is $b_q(y)=-H_q(y)De_q(y)$, as in (10.11). Put $\chi_q(y)=1_{|y|>r_q}$, pointwise, and split the drift as

$$b_q^{\mathrm b}(y)=-H_q(y)D\kappa_q(y)De_q^{\mathrm o}(\kappa_qy),\quad b_q^{\mathrm l}=b_q-b_q^{\mathrm b}.$$

Lemma 21.2. (The complete source bounds) *This is Proposition 12.2 of Lecture 6, with the correction bound of (12.5). For $t=t_q(y)=y\chi_q(y)$, uniformly in $y\in H_N$, N, $q\geqslant q_*$ and $p\in[1,\infty]$,*

$$\|b_q^{\mathrm b}(y)\|_p\leqslant C\lambda^{7/2}q^{-5/2},\tag{21.4}$$

$$\|b_q^{\mathrm l}(y)\|_p\leqslant C\lambda q^{-3/2}(\|t\|_p+r_q^{-1}\|y^2\chi_q(y)\|_p),$$

$$\|b_q^{\mathrm l}(y)\|_2^2\leqslant C\lambda^4q^{-5}\|t\|_6^6.$$

Moreover $\|b_q(y)\|_p\leqslant C\lambda q^{-3/2}\|y\|_p$, $\|De_q(y)\|_p\leqslant Cg_q\|y\|_p$; the trace part (the loop part $\mathcal U$ of (12.8) of Lecture 6) $D\mathrm Y_q(y)=\lambda y\operatorname{diag}H_q(y)^2+\frac12C\operatorname{tr}_{H_q(y)^2}D^3\tilde h_q(y)$ obeys $\|D\mathrm Y_q(y)\|_p\leqslant Cg_q\|y\|_p$; and

$$\varrho_q(y):=C_q(M_qy+D\tilde h_q(y)),\quad \|D\varrho_q(y)\|_{p\to p}\leqslant\varepsilon_q:=C(g_q+g_q^2).\tag{21.5}$$

Proof. The exact gradient is

$$De=\partial_qM_qy+D\partial_q\tilde h+\lambda y\operatorname{diag}H^2+\frac12C\operatorname{tr}_{H^2}D^3\tilde h.\tag{21.6}$$

For this calculation let $h=h_q^{[3]}$, $z=\kappa_qy$, and $J=D\kappa_q(y)$, the diagonal derivative of the clipping map, and write $M'=\partial_qM_q$. Evenness, the zero scalar quadratic jet of $e^{\mathrm o}$ and its centred quadratic jet $\mathcal E_q^c$ give $\|De^{\mathrm o}(z)\|_\infty\leqslant\mathsf S_4(e^{\mathrm o})(2r_q)^3/6+\|\mathcal E_q^c\|_{\mathrm{row}}2r_q\leqslant C\lambda^{7/2}q^{-3/2}$, by (20.12). Multiplication by J and the C/q inverse bound therefore give the bounded-field estimate in (21.4). Here are the identities for the large-field calculation. Put $w=\partial_q\kappa_qy$, $f_1=Dh(z)$, $f_2=D^2h(z)$, and $f_3=D^3h(z)$. Here w is only the local abbreviation for the scale derivative of the clip. The unclipped and clipped inverses are

$$H^{\mathrm o}=(K_q+M_q+\lambda z^2+D^2h(z))^{-1},\quad H=H_q(y).$$

Their inverse difference, denoted $\mathsf D$ to distinguish it from field differentiation, obeys

$$\mathsf D:=H^{-1}-(H^{\mathrm o})^{-1}=\lambda M_{y^2-z^2}+(J-I)f_2J+f_2(J-I)+M_{f_1\kappa_q''},$$

$$H^2-(H^{\mathrm o})^2=-H^2\mathsf DH^{\mathrm o}-H\mathsf D(H^{\mathrm o})^2.$$

The full gradient difference is

$$De - JDe^{\mathrm{o}}(z) = M'y - JM'z + Jf_2 w + (Dw)^* f_1$$
$$+\lambda(y - Jz)\mathrm{diag}H^2 + \lambda Jz\,\mathrm{diag}(H^2 - (H^{\mathrm{o}})^2)$$
$$+\frac{1}{2}\mathrm{Ctr}_{H^2}[D^3(h\circ\kappa_q) - Jf_3] + \frac{1}{2}J\mathrm{Ctr}_{H^2-(H^{\mathrm{o}})^2} f_3.$$

These are the groups of Section 12.3 of Lecture 6. Here J in the last line acts on the open slot, κ_q'' denotes the pointwise second derivative of the scalar clipping function, and products such as $\kappa_q'' u\,v$ are pointwise. In the next identity the symbol $\sum_{\mathrm{cyc}}$ denotes the three cyclic permutations of the directions u, v, v'. Retain the entire chain rule

$$D^3(h\circ\kappa)[u,v,v'] = D^3h[Ju, Jv, Jv'] + \sum_{\mathrm{cyc}} D^2h[\kappa'' u\,v, Jv'] + Dh[\kappa''' u\,v\,v'].$$

Suppress q and put $r = r_q$, $g = g_q$, $a_2 = a_2(q)$, $\chi = \chi_q(y)$ and $t = y\chi$. Let

$$\sigma_1 = \|f_1\|_\infty \leqslant C g q r, \quad \sigma_2 = m(f_2) \leqslant C g q, \quad \sigma_3 = s(f_3) \leqslant C g q / r.$$

Clipping obeys the following pointwise bounds, all uniform in y:

$$|z| \leqslant \min(|y|, 2r), \quad |y - z| + |y - Jz| \leqslant C|t|, \quad |1 - J| \leqslant C\chi, \quad |\kappa''| \leqslant C\chi / r, \tag{21.7}$$
$$|\kappa'''| \leqslant C\chi / r^2, \quad 0 \leqslant y^2 - z^2 \leqslant y^2\chi, \quad |w| \leqslant C r\chi / q, \quad |Dw| \leqslant C\chi / q, \quad \chi \leqslant |t| / r \leqslant y^2\chi / r^2.$$

The estimates for H hold also for H^{o} on the unclipped ball, by the same Neumann-series proof. In particular, for H^a either of the inverses H, H^{o},

$$m(H^a) \leqslant C / q, \quad \mathfrak{l}(H^a) \leqslant C\sqrt{a_2}, \quad \mathfrak{l}((H^a)^2) \leqslant C q^{-1}\sqrt{a_2}.$$

Receive every inverse insertion before estimating the field. Write $\mathcal{D}_2 = H^2 - (H^{\mathrm{o}})^2$. In each term of its exact identity the two outer inverse chains have product of ℓ norms at most $C a_2 / q$. For a local insertion use Lemma 20.3. In an insertion $(J - I) f_2 J$ or $f_2(J - I)$, absorb f_2, J into the adjacent chain; this costs at most σ_2 in its ℓ norm. Finally $|f_1\kappa''| \leqslant C\sigma_1\chi / r$. The lemma therefore gives, for all p,

$$\|\mathrm{diag}\mathcal{D}_2\|_p \leqslant C\frac{a_2}{q}[\lambda\|y^2\chi\|_p + (\sigma_2 + \sigma_1 / r)\|\chi\|_p] \leqslant C\frac{\lambda a_2}{q}\|y^2\chi\|_p, \tag{21.8}$$
$$\|\mathrm{Ctr}_{\mathcal{D}_2} f_3\|_p \leqslant C\sigma_3\frac{\lambda a_2}{q}\|y^2\chi\|_p.$$

In the last inequality use $\sigma_2 + \sigma_1 / r \leqslant C g q$, $\chi \leqslant \lambda y^2\chi / q$, and small g. This argument controls the column mass of the nonlocal contraction as well as its row mass.

Receive the third-order chain rule. In its first term, after removing the open-slot J, the difference of the two contracted-slot factors is bounded by $\chi(u) + \chi(v)$. The last assertion of Lemma 20.3, with $\sup_{x,u} |H^2(x,u)| \leqslant C a_2$, gives $C a_2\sigma_3\|\chi\|_p$. Of the three second-derivative terms, one identifies the two contracted slots and the other two identify the open slot with a contracted slot. Their majorant kernels have Schur masses at most $C a_2\sigma_2 / r$. The last term identifies all three slots and costs $C a_2\sigma_1 / r^2$. Write $\mathcal{E}_3 = D^3(h\circ\kappa_q) - Jf_3$. Consequently

$$\|\mathrm{Ctr}_{H^2}\mathcal{E}_3\|_p \leqslant C a_2(\sigma_3 + \sigma_2 / r + \sigma_1 / r^2)\|\chi\|_p \leqslant C g^2\|t\|_p. \tag{21.9}$$

Here $q / r^2 = \lambda$ and $\lambda a_2 \leqslant C g$. This is an operator estimate; the marked input sites can produce output at other spatial sites.

Bound the six groups in the complete gradient difference. The following ledger follows directly from (21.7)–(21.9):

$$\|M'y - JM'z\|_p \leqslant Cg\|t\|_p, \tag{21.10}$$

$$\|Jf_2 w + (Dw)^* f_1\|_p \leqslant Cg\|t\|_p,$$

$$\|\lambda(y - Jz)\mathrm{diag} H^2\|_p \leqslant Cg\|t\|_p,$$

$$\|\lambda Jz\mathrm{diag}\mathcal{D}_2\|_p \leqslant Cgr^{-1}\|y^2\chi\|_p,$$

$$\frac{1}{2}\|C\mathrm{tr}_{H^2}\mathcal{E}_3\|_p \leqslant Cg^2\|t\|_p,$$

$$\frac{1}{2}\|JC\mathrm{tr}_{\mathcal{D}_2} f_3\|_p \leqslant Cg^2 r^{-1}\|y^2\chi\|_p.$$

For the first row split the expression exactly as $M'(y-z) + (I-J)M'z$. The first part uses $m(M') \leqslant Cg$; the second uses $\|M'z\|_\infty \leqslant Cgr$ and $\chi \leqslant |t|/r$. The second row uses the w, Dw bounds and σ_1, σ_2. For the fourth row note $\lambda^2 r a_2/q = \lambda a_2/r \leqslant Cg/r$; for the sixth use $\sigma_3 \lambda a_2/q \leqslant Cg^2/r$.

Put $\mathfrak{D}_q(y) = De_q(y) - D\kappa_q(y) De_q^{\mathrm{o}}(\kappa_q y)$. Summing the ledger gives, before multiplication by H, for all y and p,

$$\|\mathfrak{D}_q(y)\|_p \leqslant Cg_q(\|t_q(y)\|_p + r_q^{-1}\|y^2\chi_q(y)\|_p). \tag{21.11}$$

This is the bound used in Lemma 18.1 of Lecture 7; rows one, two, three–four and five–six are the four groups of Section 12.3 of Lecture 6. Since $b^1 = -H\mathfrak{D}$, using $\|H\|_{p\to p} \leqslant C/q$ proves

$$\|b^1\|_p \leqslant C\frac{g}{q}(\|t\|_p + r^{-1}\|y^2\chi\|_p).$$

On the tail $|t| + r^{-1}y^2\chi \leqslant 2r^{-2}|t|^3$; squaring the $p=2$ inequality gives $\|b^1\|_2^2 \leqslant C\lambda^4 q^{-5}\|t\|_6^6$. This retains the same coefficients needed for the backward energy.

For the global linear estimate, freeze $H(y)^2$ in (21.6) and write $D^3\tilde{h}(y) = \int_0^1 D^4\tilde{h}(uy)[y]\,\mathrm{d}u$. Collapsing the covariance slots gives a two-slot kernel with both Schur masses at most $Ca_2 s(D^4\tilde{h}) \leqslant Ca_2 g q/r^2 \leqslant Cg^2$. The other three terms in that gradient have L^p bounds $Cg\|y\|_p$, using $m(M') \leqslant Cg$, $s(D^2\partial_q\tilde{h}) \leqslant Cg$, and $\lambda\|\mathrm{diag} H^2\|_\infty \leqslant Cg$. Multiplication by H proves $\|b(y)\|_p \leqslant Cgq^{-1}\|y\|_p$. In particular, at $q = q_*$, the last two terms of (21.6), which form DY in (12.8) of Section 12.4 of Lecture 6, are bounded by $Cg_{q_*}\|y\|_p$. Finally $D\varrho = C_q M + C_q D^2\tilde{h}$. The centred convolution estimate bounds the first summand by $C(|\mu|/q + \eta_q(\Theta))$ on every L^p; the Schur bounds for C_q and the clipped Hessian bound the second by Cg. This proves (21.5). As $\varrho_q(0) = 0$ by evenness and $DU = \frac{\lambda}{3}y^3$, the equation $K_q y + D\bar{\Gamma}_q(y) = K_q x$ is (12.5) of Lecture 6. □

22 The bare mass and its limit

This section gives the consistency of the mass path with a moving boundary value and the proof of the explicit counterterm, Theorem 11.3 of Lecture 6. All constants may depend on λ, q_0, B_0, but not on N or on the boundary value.

Proposition 22.1. (Limits with a moving boundary) *If $b_{*,N} \to b_*$ in $\mathcal{B}$, the scalar paths μ^N with boundary values $b_{*,N}$ converge to the unique continuum scalar fixed point with boundary value b_*, and the centred kernels $\Theta^{[j],N}$ converge, in the weak path topology of Section 24, to its outputs; the nonlinear coefficients and β_q^N converge accordingly at each fixed q, and Propositions 27.2, 29.1 and Theorem 30.1 hold along $b_{*,N}$. In particular $\hat{\Theta}_q^N(k) \to \hat{\Theta}_q(k)$ for each fixed k.*

Proof. This is Theorem 28.3, whose proof does not use this section. □

Theorem 22.2. (The mass and its explicit counterterm) *Write $a_2 = \mathrm{Tr}\, C_q^2$, $C_q^{(2)}$ for the kernel of C_q^2, and $f_N(b_*) = \mu_N(b_*) + \lambda \mathrm{Tr}\, C - \frac{2}{3}\lambda^2 \int C^3$. Then $\beta_q = -\lambda a_2 - \lambda \mu_q a_2' + \lambda^2 a_2^2 + 2\lambda^2 \int C_q^2 C_q^{(2)} + \mathcal{R}_q$ with $|\mathcal{R}_q| \leqslant C\lambda^3 q^{-3/2}$, and*

$$f_N(b_*) = b_*(1 + \lambda a_2(q_*)) + \lambda(\mathrm{Tr}\ C - \mathrm{Tr}\ C_{q_*}) - \frac{2}{3}\lambda^2 \int (C^3 - C_{q_*}^3) + \int_{q_*}^{\infty} (\mathcal{R}_q + \lambda(\beta_q + \lambda a_2) a_2)\, \mathrm{d}q. \tag{22.1}$$

As $N \to \infty$, f_N converges to a limit f pointwise on $\mathcal{B}$, and for B_0 larger than a universal constant, each $r \in \mathbb{R}$ and q_* large depending on λ, q_0, B_0, r, there is a unique $b_{*,N} \in \mathcal{B}$ with $f_N(b_{*,N}) = r$, that is,

$$\mu_N = r - \lambda \mathrm{Tr}\, C + \frac{2}{3}\lambda^2 \int C^3, \tag{22.2}$$

and $b_{*,N} \to f^{-1}(r)$ along the whole sequence. Moreover $|f_N(b_{*,1}) - f_N(b_{*,2}) - (b_{*,1} - b_{*,2})| \leqslant C g_{q_*} |b_{*,1} - b_{*,2}|$.

Proof. Expansion. Expand $R_0 = C - CMC + CMCMR_0$ with $C = C_q$. Then $-\lambda \mathrm{Tr}\, R_0^2 = -\lambda a_2 + 2\lambda \mathrm{Tr}(C^3 M) + O(\lambda a_2 \rho^2)$, where $\rho = |\mu|/q + \eta_q \leqslant C g_q$. As $a_2' = -2\mathrm{Tr}\, C^3$, this is $-\lambda a_2 - \lambda \mu a_2' + 2\lambda \mathrm{Tr}(C^3 \Theta)$ up to the error, and $|\mathrm{Tr}(C^3\Theta)| \leqslant C q^{-1} v_1(\Theta) \leqslant C\lambda^2 q^{-3/2}$. For the vertex, write $h^{[3]} = \mathcal{L} + (h^{[2]} - \mathcal{L}) + (h^{[3]} - h^{[2]})$. The increments have fourth rooted norm at most $C g_q^2 \lambda$, and contracted with R_0^2, whose supremum is $C q^{-1/2}$, they give $O(\lambda^3 q^{-3/2})$. The quartic part of the seed is $L(y) = -\frac{\lambda^2}{4} \int\int C(x,z)^2 y(x)^2 y(z)^2$. So

$$-\frac{1}{2}\mathrm{Tr}(R_0^2 D^2 L(y)) = \frac{1}{2}\langle y, (\lambda^2 a_2 \mathrm{diag}(R_0^2) + 2\lambda^2 C^2 \cdot R_0^2) y \rangle,$$

where $\cdot$ denotes the pointwise product of kernels. Its constant-mode eigenvalue is $\lambda^2 a_2 \mathrm{diag}(R_0^2) + 2\lambda^2 \int C^2 R_0^2$, and replacing R_0^2 by $C^{(2)}$ costs a relative factor $C g_q$, that is, $O(\lambda^3 q^{-3/2})$. This proves the expansion.

Integration. Three identities: $\partial_q \mathrm{Tr}\, C_q = -a_2$; $\partial_q C_q(x) = -C_q^{(2)}(x)$, so $\partial_q \int C_q^3 = -3 \int C_q^2 C_q^{(2)}$; and $\partial_q(\mu a_2) = \beta a_2 + \mu a_2'$, so $-\lambda \mu a_2' + \lambda^2 a_2^2 = -\lambda \partial_q(\mu a_2) + \lambda(\beta + \lambda a_2) a_2$. At fixed N, as $q \to \infty$, $\mathrm{Tr}\, C_q$, $\int C_q^3$ and $\mu_q a_2(q)$ tend to 0. Since $\lambda^2 \cdot 2 \cdot \frac{1}{3} = \frac{2}{3}\lambda^2$, integrating β over $[q_*, \infty)$ gives $\mu_N = b_*(1 + \lambda a_2(q_*)) - \lambda \mathrm{Tr}\, C_{q_*} + \frac{2}{3}\lambda^2 \int C_{q_*}^3 + \int (\mathcal{R} + \lambda(\beta + \lambda a_2) a_2)$, which is (22.1). The last integrand is $O(\lambda^3 q^{-3/2})$, because $\beta + \lambda a_2 = O(\lambda^2/q)$: indeed $\mu a_2' = O(\lambda q^{1/2} q^{-3/2})$.

Limit. The multiplier of $\mathrm{Tr}\, C - \mathrm{Tr}\, C_{q_*}$ is $(q_* - q_0)((q_0 + c_k)(q_* + c_k))^{-1} \lesssim (1 + |k|)^{-4}$, summable, with $c_k \geqslant c|k|^2$, so this sum converges. Next $C^3 - C_{q_*}^3 = (C - C_{q_*})(C^2 + C C_{q_*} + C_{q_*}^2)$, where $C - C_{q_*}$ converges uniformly, by the same summability, and the second factor converges in L^1. At fixed q, β_q^N, μ_q^N and the explicit terms converge by Proposition 22.1 and the convergence of the free kernels, hence so does $\mathcal{R}_q^N$. The integrand is dominated by $C\lambda^3 q^{-3/2}$, so $f_N \to f$ pointwise.

Inversion. By Proposition 20.8 and $\lambda a_2(q_*) \leqslant C g_{q_*}$, f_N is increasing with $|f_N(b_{*,1}) - f_N(b_{*,2}) - \delta b_*| \leqslant C g_{q_*} |\delta b_*|$, and the same holds for f. By (22.1), $f_N(0) = \lambda(\mathrm{Tr}\, C - \mathrm{Tr}\, C_{q_*}) + O(\log q_*)$, and $\mathrm{Tr}\, C - \mathrm{Tr}\, C_{q_*} \leqslant C(q_0) + q_*^{1/2}/10$ uniformly in N (the continuum value of the coefficient is $\frac{1}{4\pi} < \frac{1}{10}$, and the lattice sum is bounded by $\frac{1}{10}$ uniformly in N). So for B_0 above a universal constant and then q_* large depending on λ, q_0, B_0, r, $f_N(\mathcal{B})$ contains r for all N. Monotonicity and pointwise convergence give $b_{*,N} \to f^{-1}(r)$. □

The measure depends only on μ_N, so the choice of q_* is immaterial for (22.2): we may take any large q_*.

23 Identifying the deterministic coefficients: conventions

This part proves the statements of Section 18.1 of Lecture 7 and Lemma 18.3: the convergence of the coefficients along fields, which enter Theorems 18.4 and 18.6. The estimates of Proposition 11.2 do not by themselves identify the coefficients of a limit: a weakly converging vertex cannot automatically be contracted against a varying covariance. That additional assertion is proved here.

Notation. We use that of Lectures 6–7: $K_q = q - \Delta_N$, $C_q = K_q^{-1}$, $g_q = \lambda q^{-1/2}$, $r_q = (q/\lambda)^{1/2}$, $a_2(q) = \operatorname{Tr} C_q^2$, the quadratic path $M_q = \mu_q I + \Theta_q$, $R_{0,q} = (K_q + M_q)^{-1}$, $G_q(y) = (K_q + \lambda y^2)^{-1}$, the seed $\mathcal{L}_q$, the corrections $h_q^{[j]}$ of (11.8), the sources $\mathfrak{S}_q[M,h]$ of (20.8), with $\mathfrak{S}_q 1 = \beta_q 1$ at the final level and centred parts $\mathfrak{S}_q^\circ$, the centred kernels $\Theta_q^{[j]}$ with $\Theta_q = \Theta_q^{[3]}$, the clip κ_q, $\tilde h_q = h_q^{[3]} \circ \kappa_q$, the action $\bar\Gamma_q$, the response $H_q(y) = (K_q + D^2\bar\Gamma_q(y))^{-1}$, the defect e_q, the drift $b_q = -H_q D\, e_q$, and the coordinate correction $\varrho_q(y) = C_q(M_q y + D\tilde h_q(y))$ of (12.5). $\operatorname{Ctr}_R T$ contracts two slots of the tensor T with the kernel R. The letter H without a scale index is the space $L^2(\mathbb{T}^3)$; the response always carries a scale index or a field. HS is the Hilbert–Schmidt class on H. The boundary value (below) is b_*; $b_q(y)$ is the drift.

Superscripts or subscripts N distinguish the cutoffs: $\mu^N = (\mu_q^N)_{q \geqslant q_*}$ is a running mass path, while μ_N is reserved for the single bare mass. $k \in \mathbb{Z}^3$ and $e_k(x) = \mathrm{e}^{2\pi i k\cdot x}$ (the k of Lecture 6 is 2π times this one); e_k^N is the restriction of e_k to Λ_N, and Λ_N^* is the centred cube of N^3 frequencies. On Λ_N^* the symbol $c_k = 4N^2 \sum_i \sin^2(\pi k_i/N)$ of $-\Delta_N$ satisfies $c|k|^2 \leqslant c_k \leqslant 4\pi^2|k|^2$.

Identifications. Fields are embedded by the cell function ι_N, operators by $\iota_N \cdot j_N$, with $j_N = \iota_N^*$, $j_N\iota_N = I$ and $\mathsf{P}_N = \iota_N j_N \to I$ strongly. All embedded inverses are $\iota_N R j_N$ on H, with zero action on the orthogonal complement of the cell functions. Their resolvent identities are valid in this convention; an inverse on the complementary subspace is never required. For a smooth ϕ we put $\phi_N = j_N\phi$, so $\|\phi_N\|_\infty \leqslant \|\phi\|_\infty$ and $\|\iota_N\phi_N - \phi\|_\infty \leqslant C\|\phi\|_{C^1} N^{-1}$.

Boundary value. Fix $B_0 < \infty$, let $\mathcal{B} = [-B_0\lambda q_*^{1/2}, B_0\lambda q_*^{1/2}]$, fix $b_{*,N} \to b_*$ in $\mathcal{B}$, and take q_* large depending on λ, B_0. Throughout, the lattice coefficients are those of the system (11.8)–(11.9) with $\mu_{q_*}^N = b_{*,N}$, the centred kernels integrated from infinity; by Proposition 20.6 the level masses $M = \mu I + \Theta^{[j]}$ lie in the ball $\|M\|_* \leqslant R = B_0 + C_*$, where

$$\|M\|_* = \max\Big(\sup_{q\geqslant q_*} \frac{|\mu_q|}{\lambda q^{1/2}}, \sup_{q\geqslant q_*} \frac{v_1(\Theta_q)}{\lambda^2 q^{-1/2}}\Big), \qquad d_*(M,\hat M) = \|M - \hat M\|_*. \tag{23.1}$$

Constants may depend on λ, B_0 and on the compact scale interval $I \subset [q_*, \infty)$ when so indicated, and never on N. The bounds of Proposition 11.2, of (12.2), of Lemma 12.1 and of Proposition 12.2 hold in this ball.

Measures. We use signed Radon measures, that is, signed measures with finite variation on compact sets. For such a σ, $|\sigma|$ is its total variation measure and $\|\sigma\|_{\mathrm{TV}} = |\sigma|(E)$ on its underlying space E. δ_z has unit mass at z. Weak convergence on a compact space means convergence against every continuous test; vague convergence off zero uses continuous tests with compact support in the punctured torus. An annular test is supported a positive distance from zero.

24 The topology of the centred quadratic kernel

Let $\mathcal{Q}_1$ be the space of even signed Radon measures on $\mathbb{T}^3 \setminus 0$ with norm

$$v_1(\Theta) = \int d(0,z)\,|\Theta|(\,\mathrm{d}z) < \infty.$$

Such a measure acts by the compensated formula

$$(\Theta f)(x) = \frac{1}{2}\int [f(x+z) + f(x-z) - 2f(x)]\,\Theta(\,\mathrm{d}z). \tag{24.1}$$

For a lattice kernel the off-diagonal measure assigns mass $N^{-3}\Theta_N(0,z)$ to $z\neq 0$; its diagonal is supplied by centring, $\Theta_N 1 = 0$. There is no independent diagonal mass in Q_1: that coefficient belongs to μI. The multiplier $\hat{\Theta}(k)$ is characterised by $\Theta e_k = \hat{\Theta}(k) e_k$.

Lemma 24.1. (Compensated kernels and weak paths) *The space Q_1 is complete. The norm v_1 is lower semicontinuous for vague convergence off zero. If $\Theta_N \to \Theta$ vaguely off zero with bounded first moments, then for every smooth f,*

$$\iota_N \Theta_N j_N f \to \Theta f \quad \textit{uniformly}, \quad |\hat{\Theta}_N(k)| \leqslant C\, v_1(\Theta_N)|k|,$$

and $\hat{\Theta}_N(k) \to \hat{\Theta}(k)$ for each k. The same conclusions are locally uniform in scale when convergence against compactly supported tests off zero is locally uniform in scale. Such a topology, with the scalar paths μ^N converging locally uniformly, is called the ***weak path topology****.*

Proof. Multiplication by $d(0,z)$ identifies Q_1 isometrically with the even finite signed measures on the punctured torus, a Banach space in total variation. On each closed annulus the weighted variation is the supremum over continuous tests dominated by d. Taking increasing annuli gives lower semicontinuity.

For $d(0,z) < \delta$ the second difference of a smooth function satisfies

$$|f(x+z) + f(x-z) - 2f(x)| \leqslant C\|f\|_{C^2} d(0,z)^2 \leqslant C\delta \|f\|_{C^2} d(0,z).$$

The small displacement contribution is therefore uniformly negligible, as the first moments are bounded. Away from $d(0,z) < \delta$ use a continuous cutoff and the vague convergence; a finite net in x and the same bound for derivatives give uniformity in x. Cell averages commute with translations by lattice displacements, so replacing f by its cell averages gives the same limit. There is no large distance tail on the torus.

For a Fourier character the multiplier is $\int (\cos(2\pi k\cdot z) - 1)\Theta(\mathrm{d}z)$, and $|\cos(2\pi k\cdot z) - 1| \leqslant C|k| d(0,z)$ proves the bound. This also justifies a Fourier approximation of a smooth f, uniformly over the moment bounded family. The function $\cos(2\pi k\cdot z) - 1$ vanishes to second order at $z = 0$, so the first part with $f = e_k$ gives the convergence of the multipliers. The scale uniform claims follow by the identical estimates. Bounded first moments prevent a hidden second order differential operator from surviving at zero. □

The strong metric d_* of (23.1), restricted to scalar paths, is the one used for the fixed point; on the centred kernels it only records their bounds. It is different from the weak path topology: lattice measures need not converge in total variation to their continuum limits. Both lattice and continuum paths are defined on $[q_*, \infty)$, so the same metric is used for all their contraction estimates. Weak convergence is still only local in scale.

25 Physical vertex measures and their derivatives

If $D^k f_N(y)$ has physical tensor kernel T_N, associate the atomic measure

$$\mathfrak{v}_N = N^{-3k} \sum_{x_1,\dots,x_k \in \Lambda_N} T_N(x_1,\dots,x_k)\, \delta_{(x_1,\dots,x_k)}.$$

Integrating smooth products against it evaluates the derivative on their lattice samples. Evaluation on cell averages differs by $o(1)$, since samples and averages of a fixed smooth function differ uniformly by $O(N^{-1})$ and the tensor variation is bounded. We use atomic measures for weak convergence and cell kernels for operator comparisons. As in Section 20.1, $s(T)$ is the rooted norm.

Lemma 25.1. (**Root bounds pass to limits**) *A family with $s(T_N) \leqslant C$ has weakly convergent subsequences as signed measures on the compact product torus. Its limiting variation has every one-slot marginal bounded by $C\,\mathrm{d}x$. Spatial first moment bounds also pass to the limit. In particular a two-slot limit defines a bounded operator on H of norm at most C.*

Proof. The total variation is at most C, since the root variable has volume one. Weak compactness of bounded measures gives a subsequence. For a continuous nonnegative root test a and a continuous product space test ψ with $|\psi| \leqslant 1$, the rooted estimate gives

$$\Big|\int a(x_i)\psi \,\mathrm{d}\mathfrak{v}_N\Big| \leqslant CN^{-3}\sum_{x\in\Lambda_N} a(x).$$

Pass to the limit and take the supremum over ψ to obtain the marginal variation bound. Insert a distance in the tests for the moment version. A symmetric two-slot measure T therefore obeys

$$\Big|\int u(x)v(z)\mathsf{T}(\,\mathrm{d}x,\,\mathrm{d}z)\Big| \leqslant \Big(\int |u(x)|^2|\mathsf{T}|\Big)^{1/2}\Big(\int |v(z)|^2|\mathsf{T}|\Big)^{1/2} \leqslant C\|u\|_2\|v\|_2.$$

This defines its bounded operator, first on smooth functions, then by density on H. □

Lemma 25.2. (**Recovering derivatives by finite differences**) *Suppose scalar actions $f_N(j_N\phi)$ converge for smooth profiles in an open bounded field domain, and their derivatives through order $m+1$ have uniform rooted bounds there. Then, on smaller compact finite-dimensional families of smooth profiles, all tested derivatives through order m converge. Their physical tensor measures converge weakly to the corresponding derivative measures.*

Proof. Restrict to $\phi + \sum_{i=1}^{d} t_i u_i$, with smooth directions u_i and parameters a positive distance inside the domain. A mixed finite difference of order $k \leqslant m$ and step δ approximates the corresponding derivative with error $C\delta$, by the bound on the derivative of order $k+1$. At fixed δ its finitely many scalar values converge. Let $N \to \infty$, then $\delta \to 0$. The next derivative bound and finite parameter nets give uniformity on compact parameter sets.

Products of smooth one-variable tests determine a measure on the product torus: finite Fourier sums approximate every continuous product space test uniformly, and the common variation bound removes that approximation. This identifies the full weak vertex limit, not just a selected collection of tensor contractions. The limiting scalar action on these families has the stated derivatives, since the fundamental theorem of calculus passes to the limit on segments. □

For even corrections with zero quadratic jet, Taylor's formula also gives

$$D^2h(\phi) = \int_0^1 (1-t)D^4h(t\phi)[\phi,\phi]\,\mathrm{d}t. \tag{25.1}$$

Thus the Hessian Schur bounds follow from the bounds on the fourth derivative on every bounded field set. Local diagonal terms cause no difficulty: they are diagonal measures, not ordinary functions of two independent variables.

26 Compact resolvents receive weak Hessians

The four lemmas of this section (compact sandwich, and contraction against a moving covariance) are the second and third tools listed in Section 18.1 of Lecture 7.

Lemma 26.1. (**Free and quadratic background consistency**) *For paths converging in the weak path topology,*

$$\iota_N(K_q + M_q^N)^{-1}j_N \to (q - \Delta + \mu_q + \Theta_q)^{-1} \quad \textit{in } \mathrm{HS},$$

locally uniformly in q. The same holds after adding $\lambda(j_N\phi)^2$ for a fixed smooth bounded profile ϕ. In particular $\iota_N C_q^N j_N \to C_q$ in HS.

Proof. For each fixed Fourier mode the multipliers converge, by Lemma 24.1 and the convergence of the lattice Laplacian symbol. Their absolute values are bounded by $C(q+c|k|^2)^{-1}$, by the small relative form bound: $|\mu_q| \leqslant R\, g_q q$ and $|\hat{\Theta}_q(k)| \leqslant C\, R\, g_q \lambda |k| \leqslant C\, R\, g_q (g_q^2 q + |k|^2)$. The squared tail beyond $|k| = L$ is therefore at most $C_I \sum_{|k|>L} (1+|k|^2)^{-2} \leqslant C_I / L$. The embedded lattice Fourier characters converge uniformly to their continuum characters on each fixed finite set of modes. The low and high frequency split proves Hilbert–Schmidt convergence.

For the multiplication potential put $V_N = \lambda(\iota_N j_N \phi)^2$ and $V = \lambda\phi^2$, so $\|V_N - V\|_\infty \to 0$. If A_N, A denote the preceding background inverses and G_N, G the inverses with V_N, V, the exact identity is

$$G_N - G = (I - G_N V_N)(A_N - A)(I - VG) + G_N(V - V_N)G.$$

All outer factors have bounded operator norms on the fixed profile, and the middle difference is Hilbert–Schmidt. The identity follows by expanding and using $G_N = (I - G_N V_N) A_N$ and $G = A(I - VG)$. This proves the assertion without an expansion in the field amplitude. □

Lemma 26.2. (Compact sandwich) *Let $G_N \to G$ in Hilbert–Schmidt norm. Let B_N be a uniformly bounded symmetric sequence converging weakly as operators to B, meaning $\langle f, \mathsf{B}_N g\rangle \to \langle f, \mathsf{B} g\rangle$ for all $f, g \in H$. Assume*

$$\max(\|G_N \mathsf{B}_N\|_{2\to 2}, \|G\mathsf{B}\|_{2\to 2}) \leqslant \theta < 1.$$

Then

$$R_N = (I + G_N \mathsf{B}_N)^{-1} G_N \to R = (I + G\mathsf{B})^{-1} G \quad \text{in HS}.$$

Weak convergence of physical Hessian measures with uniform Schur bounds supplies the weak operator hypothesis.

Proof. A uniformly bounded weakly convergent sequence $E_N \to 0$ satisfies $A\, E_N D \to 0$ in Hilbert–Schmidt norm whenever A, D are Hilbert–Schmidt. Indeed, approximate both outer factors by finite rank operators: the finite matrix elements converge, and the two errors are bounded using $\|XYZ\|_{\mathrm{HS}} \leqslant \|X\|_{\mathrm{HS}} \|Y\| \|Z\|$. Apply this to $G(\mathsf{B}_N - \mathsf{B})R$. Also $(G_N - G)(\mathsf{B}_N - \mathsf{B})R \to 0$ in Hilbert–Schmidt norm. Now use

$$R_N - R = (I - R_N \mathsf{B}_N)(G_N - G)(I - \mathsf{B}R) + R_N(\mathsf{B} - \mathsf{B}_N)R,$$

where $R_N = (I + G_N \mathsf{B}_N)^{-1} G_N$ in the last term. The inverse factors are uniformly bounded by $(1-\theta)^{-1}$, so both terms vanish. Weak measure convergence gives weak operator convergence on smooth product tests, and the Schur bound extends it to arbitrary L^2 tests by density. □

In the applications G_N includes the quadratic background and the nonnegative local potential. Here $\phi_N = j_N \phi$ for a fixed smooth bounded profile ϕ, h_N is the nonlinear correction at whichever finite depth is being considered, and $\mathsf{B}_N = D^2 h_N(\phi_N)$. The finite hierarchy gives $\|\mathsf{B}_N\| \leqslant C g_q q$, and $\|G_N\| \leqslant C/q$, so the same large q_* ensures the smallness required in the lemma. The limit inherits these bounds and defines a positive inverse before the convergence argument is used.

Lemma 26.3. (A continuous covariance kernel) *On bounded field sets and compact scale intervals, for every $0 < a < \frac{1}{2}$, the rows of the lattice inverses R_N above satisfy*

$$\|R_N(x,\cdot) - R_N(x',\cdot)\|_2 \leqslant C_{I,a}\, d(x,x')^a. \tag{26.1}$$

Their squares have bounded kernels with the corresponding modulus in both variables. If $R_N \to R$ in Hilbert–Schmidt norm, the squared physical kernels converge uniformly, under cell embedding, to a continuous kernel $R^2(x,z)$.

Proof. For $R_{0,N} = (K_q + M_q)^{-1}$, Parseval and the multiplier bound give

$$\|R_{0,N}(x,\cdot) - R_{0,N}(x',\cdot)\|_2^2 \leqslant C\, d(x,x')^{2a} \sum_k \frac{|k|^{2a}}{(q + c|k|^2)^2} \leqslant C_{I,a} d(x,x')^{2a}.$$

The sum converges because $a < \frac{1}{2}$. At a bounded field, write $R_N = R_{0,N} - R_{0,N}(\lambda\phi_N^2 + \mathsf{B}_N)R_N$; the factor to the right of $R_{0,N}$ is bounded on L^2. Taking row differences proves (26.1). Since $R_N^2(x,z)$ is the inner product of the two inverse rows, their common L^2 bound proves the covariance modulus.

For cell embeddings that modulus has an additional error CN^{-a}, as each point is replaced by its cell centre. The kernels are precompact in the uniform norm up to this vanishing error, for instance by interpolating their grid values continuously. On the other hand Hilbert–Schmidt convergence implies $R_N^2 \to R^2$ in Hilbert–Schmidt norm, by the bounded operator norms. Thus every uniform subsequential limit equals the same L^2 kernel. This proves uniform convergence and continuity. □

Lemma 26.4. (Contraction against a moving covariance) *Let physical vertex measures T_N converge weakly with bounded variation, and let R_N be as in the preceding lemma. Any contraction of two slots of T_N against R_N^2 converges weakly in the remaining slots. In particular the scalar trace converges when T_N is a Hessian.*

Proof. After testing the remaining slots by a continuous function, split the difference as

$$\int (R_N^2 - R^2)\, \mathrm{d}T_N + \int R^2\, \mathrm{d}(T_N - T).$$

The first term tends to zero by uniform convergence and bounded variation; the second by weak convergence against the fixed continuous kernel. The use of atomic vertices and cell covariances is consistent: the former evaluate the latter at their grid values, and the uniform cell error tends to zero. For a Hessian the trace identity follows first for finite rank approximations of the smoothing inverses, then by the same kernel bound. Equivalently, the Schur factorisation of its symmetric measure as $M_{\sqrt{a}} \mathsf{E} M_{\sqrt{a}}$, $\|\mathsf{E}\| \leqslant 1$, makes the double sandwich trace class; here $a(x)\,\mathrm{d}x$ is its variation marginal. Thus no trace of a single, unsquared divergent inverse is taken. □

27 Convergence of the seed and the two trace steps

Lemma 27.1. (The seed has a cutoff independent limit) *The scalar seed actions converge on smooth bounded profiles, locally uniformly in scale and on finite-dimensional profile families. Their tested derivatives through order nine converge. Their physical derivative measures converge weakly at these profiles.*

Proof. Subtract before taking a diagonal. The exact force identity is

$$D\mathcal{L}_q(\phi)(x) = -\lambda^2 \phi(x) \int C_q(x,z)\, \phi(z)^2 G_q(\phi)(z,x)\, \mathrm{d}z. \tag{27.1}$$

It follows from $D\mathcal{L} = \lambda\phi \mathrm{diag}(G - C)$ and $G - C = -C\lambda\phi^2 G$. Both inverses converge in Hilbert–Schmidt norm by Lemma 26.1. Cauchy–Schwarz in their two kernel variables proves L^1 convergence of this force. Its uniform bound $C_I \|\phi\|_\infty^3$ upgrades convergence to every finite spatial L^p. Integrate $\langle D\mathcal{L}_q(t\phi), \phi\rangle$ over $t \in [0,1]$ to identify the scalar action, with value zero at the origin. The ten derivative bounds of Proposition 11.2 and Lemma 25.2 give the nine tested derivatives. All estimates are uniform on compact profile families and scale intervals. In particular neither $\mathrm{diag} C_q$ nor $\mathrm{diag} G_q$ is assigned a separate continuum limit. □

Proposition 27.2. (Consistency of the finite nonlinear recursion) *For prescribed level masses converging in the weak path topology, the functions $h^{[j],N}$ of (11.8) converge on smooth profiles inside their field balls. Tested derivatives converge through orders $9,7,5$ at depths $j=1,2,3$, respectively. All convergences are uniform on compact scale and admissible parameter sets. Corresponding inverse operators converge in Hilbert–Schmidt norm, including all interpolated inverses used below. This is the passage of the trace steps to the limit announced in Section 18.1 of Lecture 7.*

Proof. The assertion at depth one is Lemma 27.1. Suppose scalar and tested convergence hold at depth j. The Hessians converge weakly with uniform Schur bounds. Lemmas 26.1 and 26.2 identify the inverses with Hessian $\theta D^2h^{[j]}$, $0\leqslant\theta\leqslant 1$. The convergence is uniform in θ, since the resolvent identity gives a common Lipschitz bound in that parameter.

Let $G_M=(K_q+M_q+\lambda\phi^2)^{-1}$ and $R_{\theta h}=(K_q+M_q+\lambda\phi^2+\theta D^2h^{[j]}(\phi))^{-1}$. The scalar trace map decomposes exactly as

$$\operatorname{Tr}(G_M-R_{0,q})=-\operatorname{Tr}(R_{0,q}\lambda\phi^2G_M),\quad \operatorname{Tr}(R_h-G_M)=-\int_0^1 \operatorname{Ctr}_{R^2_{\theta h}}D^2h^{[j]}(\phi)\,\mathrm{d}\theta. \tag{27.2}$$

The first term converges as a product of Hilbert–Schmidt operators. The second converges by Lemma 26.4. Thus the scalar trace map converges at every admissible smooth profile. The finite derivative bounds and finite differences also identify its quadratic jet at zero, so its projection by Π converges.

It remains to pass the improper scale integral in (11.8). A fixed profile admissible at scale q remains admissible at all later scales. The initial trace comparison (Lemma 20.4), the common coefficient ball, and Taylor's formula to fourth order give

$$|\Pi(\widetilde{\mathcal{J}}_s[M^{[j]},h^{[j]}]-\partial_s\mathcal{L}_s)(\phi)|\leqslant C\lambda^3s^{-2}\|\phi\|_4^4. \tag{27.3}$$

Consequently the omitted tail beyond L is at most $C\lambda^3L^{-1}\|\phi\|_4^4$, uniformly in the cutoffs. Pass to the limit first on $[q,L]$, then send $L\to\infty$. This proves scalar convergence at depth $j+1$. Its derivative budget is eight at depth two and six at depth three, so finite differences identify derivatives through seven and five. These orders include the fourth zero-field vertex needed by the mass equation. There is one genuinely unused derivative at the final step; it cannot be omitted.

The identities $\partial_qh^{[j+1]}=\Pi\widetilde{\mathcal{J}}_q[M^{[j]},h^{[j]}]$ and the corresponding differentiated estimates give local scale equicontinuity. Compact parameter nets give local uniformity. Only the two stated steps are used; no infinite derivative hierarchy or infinite field Taylor series has been introduced. □

28 The scalar fixed point converges

The continuum source is a measure valued kernel. Weak convergence does not supply an integral in the total variation norm. The following construction is sufficient and avoids that assumption.

Lemma 28.1. (Integrating weak measure sources) *Let $\mathfrak{C}_s\in Q_1$ be even measures off zero, measurable when tested against every compactly supported continuous function off zero, and suppose $v_1(\mathfrak{C}_s)\leqslant f(s)$ with $f\in L^1([q_*,\infty))$. Then the weak prescription*

$$\Theta_q=-\int_q^\infty \mathfrak{C}_s\,\mathrm{d}s \tag{28.1}$$

defines a unique element of Q_1, with

$$v_1(\Theta_q)\leqslant\int_q^\infty f(s)\,\mathrm{d}s,\quad v_1(\Theta_t-\Theta_q)\leqslant\int_q^t f(s)\,\mathrm{d}s\quad(t\geqslant q).$$

In particular the path is continuous in its first moment norm. If $\mathfrak{C}_s^N \to \mathfrak{C}_s$ locally uniformly in scale against annular tests, with the same integrable bound, then $\Theta^N \to \Theta$ in the weak path topology.

Suppose in addition that $\mathfrak{S}_s$ are even finite signed measures on the whole torus, weakly continuous in scale and locally bounded in total variation, that $\mathfrak{C}_s$ are their restrictions off zero, with the centring supplied as in (24.1), so that $\mathfrak{S}_s^\circ$ is the centred convolution they define, and let β_s be any continuous scalar function. If $\mu_q = b_* + \int_{q_*}^q \beta_s \,\mathrm{d}s$ and $M_q = \mu_q I + \Theta_q$, then for smooth u,

$$(M_t - M_q)u = \int_q^t (\beta_s u + \mathfrak{S}_s^\circ u)\,\mathrm{d}s \quad \text{in } H, \qquad \frac{\mathrm{d}}{\mathrm{d}q}(M_q u) = \beta_q u + \mathfrak{S}_q^\circ u \quad \text{in } H. \tag{28.2}$$

The scalar β need not be the constant mode of $\mathfrak{S}$; in the scheme of (20.4) it comes from the final level and $\mathfrak{S}$ from the previous one. Here $\mathfrak{S}_s^\circ u$ means convolution. This does not assert differentiability of the measure path in total variation.

Proof. Write $d(z) = d(0,z)$. On the punctured torus the functional

$$\psi \mapsto -\int_q^\infty \int d(z)\psi(z)\mathfrak{C}_s(\,\mathrm{d}z)\,\mathrm{d}s$$

is bounded on $C_0(\mathbb{T}^3 \setminus 0)$ with the uniform norm: the space of continuous ψ such that the set where $|\psi| \geqslant \varepsilon$ is compact in the punctured torus for every $\varepsilon > 0$. The norm of the functional is at most $\int_q^\infty f$. The inner integrals are measurable: first take compactly supported tests and then approximate in the uniform norm, using the bound on $d|\mathfrak{C}_s|$. The Riesz representation theorem gives a finite signed measure σ_q for this functional. Define $\Theta_q = d^{-1}\sigma_q$ off zero. It is a locally finite signed Radon measure with $\int d|\Theta_q| \leqslant \int_q^\infty f$ and the required test integrals. Uniqueness follows from compactly supported tests. The identical argument on $[q,t]$ proves the difference bound and continuity. For cutoff convergence, first integrate on $[q,L]$ and then bound the common tail by $\int_L^\infty f$ times the weighted test norm. This is uniform for q in a compact interval. Evenness passes through the test prescription.

For the final assertion, with $\delta_z^2 u = u(\cdot+z) + u(\cdot-z) - 2u$, centring and evenness give on smooth u

$$\frac{1}{2}\int \delta_z^2 u \mathfrak{C}_s(\,\mathrm{d}z) = \mathfrak{S}_s u - \langle \mathfrak{S}_s \rangle u = \mathfrak{S}_s^\circ u,$$

where $\langle \mathfrak{S}_s \rangle = \mathfrak{S}_s(\mathbb{T}^3)$; the scalar β_s enters only the scalar part. This integral is legitimate even for an infinite first moment measure: $\|\delta_z^2 u\|_2 \leqslant C_u d(z)^2$. The preceding weak integral and Fubini, with the integrable first moment bound, give the first identity in (28.2), initially against smooth tests. The constant b_* cancels in the difference. To justify the strong H form, convolution by $\mathfrak{S}_s$ has norm at most $\|\mathfrak{S}_s\|_{\mathrm{TV}}$. For a finite Fourier sum, $\mathfrak{S}_s u$ is continuous in H by weak continuity of $\mathfrak{S}_s$. Approximation by Fourier sums and the locally uniform total variation bound give this continuity for every $u \in H$. Thus the right side is the integral of a continuous H-valued function, and its fundamental theorem of calculus proves the derivative assertion. On each compact scale interval $M_t - M_q$ consequently extends to a bounded operator on H, even when the individual M_q are only defined on a dense domain. No Banach space differentiability theorem for measures is used. □

Remark 28.2. (Norm continuity does not imply a measure derivative) Let $e_1 = (1,0,0)$ and, for $0 < a < t < c < \frac{1}{4}$, set $\mathfrak{C}_t = \frac{1}{2}(\delta_{te_1} + \delta_{-te_1})$ off zero. Let $F_t = \int_a^t \mathfrak{C}_s \,\mathrm{d}s$ in the weak sense, with diagonal supplied by centring. Then F is Lipschitz in Q_1, but

$$v_1\Big(\frac{F_{t+h} - F_t}{h} - \mathfrak{C}_t\Big) = 2t + \frac{h}{2} \quad (0 < h < c - t).$$

Indeed the difference quotient is distributed along two short arcs, whereas $\mathfrak{C}_t$ consists of their endpoint atoms. They are mutually singular, so the first moments add. The quotient converges weakly to $\mathfrak{C}_t$, so there is no derivative in the v_1 norm. On a Fourier mode the centred multiplier is instead $\cos(2\pi k_1 t) - 1$, and differentiating the integrated multiplier is valid. This illustrates exactly the distinction in the lemma.

Theorem 28.3. (Consistency of the quadratic path) *The scalar paths μ^N selected by (11.9) with boundary values $b_{*,N} \to b_*$ in $\mathcal{B}$ converge, locally uniformly, to the unique scalar path in the ball $\|\mu\|_* \leqslant R$ solving the continuum problem*

$$\mu_q = b_* + \int_{q_*}^{q} \beta_s \, \mathrm{d}s, \quad \beta_s = \langle \mathfrak{S}_s[M^{[3]}, h^{[3]}] \rangle, \tag{28.3}$$

where the level masses $M^{[j]} = \mu I + \Theta^{[j]}$ and the corrections $h^{[j]}$ are those of the continuum recursion (20.4) on this path. The centred kernels $\Theta^{[j],N}$ converge in the weak path topology to $\Theta^{[j]}$, and the corrections as in Proposition 27.2. The sources $\mathfrak{S}_q[M^{[j],N}, h^{[j],N}]$ converge weakly as measures on the relative variable, locally uniformly in q, to $\mathfrak{S}_q[M^{[j]}, h^{[j]}]$; so $\beta_q^N \to \beta_q$ locally uniformly, and $\hat{\Theta}_q^{[j],N}(k) \to \hat{\Theta}_q^{[j]}(k)$ for each fixed k. Uniqueness here concerns deterministic coefficients, not a limiting Gibbs law or stochastic equation. This is the first tool of Section 18.1 of Lecture 7, and it is the convergence used in Theorem 22.2.

Proof. Step 0: reduction to a fixed boundary value. For fixed N and $b'_* \in \mathcal{B}$ write μ^{N,b'_*} for the scalar path with boundary value b'_*, and M^{N,b'_*} for any of its level masses. By Proposition 20.8, uniformly in N,

$$d_*(M^{N,b_{*,N}}, M^{N,b_*}) \leqslant C|b_{*,N} - b_*|. \tag{28.4}$$

Thus it is enough to prove the theorem for the fixed value b_*: the tests below are Lipschitz in d_*, and the assertions on the corrections follow from Proposition 27.2, which only requires weak path convergence of its input. We write $\mathcal{A}_N$ for the scalar map defined by (11.9) with boundary value b_*, $\mathcal{A}_N(\mu) = b_* + \int_{q_*}^{\cdot} \beta_s[\mu] \, \mathrm{d}s$.

Step 1: the map on continuum paths. The continuum map must be defined on its whole ball, not just on a particular extracted sequence. Its argument is a scalar path, which is its own lattice approximation; the centred kernels are not arguments but outputs, constructed level by level in Step 2. Proposition 27.2 and Step 2 therefore define the continuum recursion and $\mathcal{A}(\mu)$ for every continuous scalar path in the ball, with no approximation of measure-valued inputs. Apply the finite cutoff contraction estimate of Proposition 20.6 and pass to the limit: after exhausting scale intervals this gives the same contraction estimate for $\mathcal{A}$.

Step 2: one application of the map. Let μ^N be scalar paths in the ball converging locally uniformly to μ. Proceed level by level, starting from $\Theta^{[1],N} = 0$ and the seed, which converges by Lemma 27.1. Suppose the level masses $M^{[j],N}$ converge in the weak path topology to $M^{[j]}$, and $h^{[j],N}$ as in Proposition 27.2, which identifies the fourth zero-field vertex. Its contraction with R_0^2 converges by Lemma 26.4; the local contribution $-\lambda \operatorname{Tr} R_0^2$ converges by Hilbert–Schmidt convergence. It follows that $\mathfrak{S}_q[M^{[j],N}, h^{[j],N}]$ converges weakly as a two-slot measure, locally uniformly in scale. The limit is invariant under every torus shift: approximate that shift by lattice shifts in each continuous test. It is therefore also a convolution measure on the relative spatial variable. Its uniform row bound gives the limiting total variation bound $\|\mathfrak{S}_q\|_{\mathrm{TV}} \leqslant C(g_q + g_q^2)$. The uniform weak convergence and finite cutoff continuity give weak continuity of this source.

For the centred integral toward infinity, the bound of Proposition 20.6 is

$$v_1(\mathfrak{S}_s^\circ) \leqslant C\lambda^2 s^{-3/2}.$$

Its upper tail in the first moment norm is at most $2C\lambda^2 L^{-1/2}$. Lemma 28.1 constructs this integral without a strong measurability assumption in the measure norm. It proves convergence against annular tests; the small displacement estimate of Lemma 24.1 gives compensated tests, uniformly on compact scale intervals. So $\Theta^{[j+1],N} \to \Theta^{[j+1]}$ in the weak path topology, and Proposition 27.2 gives the next primitive. After the two levels, testing both external slots of the final source by the constant function one identifies β_q^N, since the volume is one. Hence the scalar integral from q_* to a fixed q converges, and the value b_* at q_* passes through unchanged. Lemma 28.1 gives local Lipschitz continuity of the centred kernels in the first moment norm; the scalar source gives it for μ. The identity $M' = \beta I + \mathfrak{S}_q[M^{[2]}, h^{[2]}]^\circ$ in the continuum has the strong vector meaning (28.2). In particular the term $M'y$ in the complete gradient (29.1) below means a bounded convolution, also on nonsmooth H inputs. No derivative in total variation norm is needed to define the drift. Lower semicontinuity keeps the limits in the same balls. There is no infinite upper scale integral for the scalar mass.

Step 3: iteration. Iterate $\mathcal{A}_N$ from the constant path $\mu \equiv b_*$, which lies in the ball as $b_* \in \mathcal{B}$, and denote the rth iterate by $\mu_N^{(r)}$. Write $\mu_N = \mu^{N,b_*}$. For each fixed r, induction using Step 2 proves convergence to a continuum iterate $\mu^{(r)}$. The contraction estimate has a single $\theta < 1$, independent of N, and gives

$$\|\mu_N - \mu_N^{(r)}\|_* \leqslant C\theta^r, \quad \|\mu_N^{(r+\ell)} - \mu_N^{(r)}\|_* \leqslant C\theta^r.$$

These inequalities pass to the limit on compact scale intervals followed by exhaustion of the half-line. The weighted space of continuous scalar paths is complete. Therefore $\mu^{(r)}$ tends to a path μ, with error $C\theta^r$, and $\mu_N \to \mu$ locally uniformly. The continuum map is Lipschitz by the difference estimates, so $\mu = \mathcal{A}(\mu)$ and the contraction proves uniqueness in the ball. Step 2 with input μ_N then gives the convergence of the centred kernels, of the corrections and of the sources, and the bounds $v_1(\Theta_q^{[j]}) \leqslant C\lambda^2 q^{-1/2}$ pass to the limit.

Step 4: conclusion along the moving boundary. By (28.4), the scalar paths and the level masses with boundary values $b_{*,N}$ converge to the same limits. Step 2 with these inputs gives the convergence of the sources and of β_q^N, and Lemma 24.1 that of the multipliers $\hat{\Theta}_q^{[j],N}(k)$. □

29 Clipping and the complete drift

We keep all quadratic, covariance and scale-clipping terms. The exact gradient of the defect for $q \geqslant q_*$, computed in Lemma 21.2, is

$$De_q(y) = M_q' y + D\partial_q \tilde{h}_q(y) + DY_q(y), \quad DY_q(y) = \lambda y \mathrm{diag} H_q(y)^2 + \frac{1}{2}\mathrm{Ctr}_{H_q(y)^2} D^3 \tilde{h}_q(y), \tag{29.1}$$

with $DY_q = D\big(-\frac{1}{2}\mathrm{Tr}[H_q - H_q(0)]\big)$ the trace part. Write $\mathrm{Y} = \mathrm{Y}_{q_*}$. Following Section 12.4, for $0 < q < q_*$ the coefficients are frozen:

$$H_q = H_{q_*}, \quad De_q(y) = -y + D\mathrm{Y}(y), \quad b_q(y) = H_{q_*}(y)(y - D\mathrm{Y}(y)). \tag{29.2}$$

The term $D\partial_q \tilde{h}_q$ and the mass derivative do not appear below q_*, which is why b_q jumps at q_*.

Proposition 29.1. (Consistency at smooth profiles) *Let $q \geqslant q_*$ and let ϕ be smooth and bounded on $\mathbb{T}^3$. Then, as $N \to \infty$ along $b_{*,N} \to b_*$:*

a) $\iota_N H_q^N(\phi_N) j_N \to H_q(\phi)$ in HS*, where $H_q(\phi)$ is the limiting inverse, of the form $(I + G\mathsf{B})^{-1}G$ with $G = (K_q + M_q + \lambda\phi^2)^{-1}$ and B the limit of the clipped Hessians;*

b) $\iota_N D e_q^N(\phi_N)$ *and* $\iota_N D\mathrm{Y}_q^N(\phi_N)$ *converge weakly in* H*, to limits called* $D e_q(\phi)$, $D\mathrm{Y}_q(\phi)$;

c) $\iota_N b_q^N(\phi_N) \to b_q(\phi) = -H_q(\phi) D e_q(\phi)$ *in* H*, and* $\iota_N \varrho_q^N(\phi_N) \to \varrho_q(\phi)$ *in* H*, where* $\varrho_q(\phi) = C_q(M_q\phi + D\tilde{h}_q(\phi))$ *and* $D\tilde{h}_q(\phi)$ *is the weak limit of* $\iota_N D\tilde{h}_q^N(\phi_N)$.

Consequently, for every smooth bounded space–scale profile ϕ*, on each compact* $I \subset [q_*, \infty)$ *the clipped inverses converge in* $L^2(I;\mathrm{HS})$*, and their complete drifts and the coordinate corrections* $\varrho_q^N(\phi_N)$ *converge in* $L^2(I;H)$*. These limits include all quadratic, covariance and scale-clipping terms in the exact formulas, and this is the consistency of the coefficients quoted in the proof of Lemma 18.3. The same holds for* $0 < q < q_*$ *with the frozen coefficients* (29.2).

Proof. For q in a compact I, the profile $\kappa_q\phi$ and its finitely many smooth parameter perturbations stay a positive distance inside the unclipped field ball, since $\sup|\tau| < 2$. On the lattice $\kappa_q(\phi_N)$ and $j_N(\kappa_q\phi)$ differ uniformly by $O(N^{-1})$, an error which the next derivative bound removes. The finite chain rule expresses clipped jets through the converging unclipped jets, multiplied by smooth functions of the external spatial variables; repeated slots give diagonal measures. Thus the clipped Hessians and third and fourth vertices converge weakly with their Schur bounds. Lemmas 26.2 and 26.3 identify the clipped inverses, giving (a) at each q.

The scale derivative must also be retained. Use

$$\partial_q \tilde{h}_q(\phi) = (\Pi \widehat{\mathcal{J}}_q[M^{[2]}, h^{[2]}])(\kappa_q\phi) + \langle D h_q^{[3]}(\kappa_q\phi), \partial_q\kappa_q\phi\rangle.$$

The first term and its tested derivatives converge by the trace argument of Proposition 27.2; the second by the finite chain rule. The mass derivative $M' = \beta_q I + \mathfrak{S}_q[M^{[2]}, h^{[2]}]^\circ$ is identified in Theorem 28.3. In the complete gradient (29.1), both covariance terms converge weakly after testing the open slot, by Lemma 26.4. All terms are bounded in L^∞ on this fixed bounded profile. Thus $D e_q^N(\phi_N)$ and its trace part converge weakly in L^2 at each scale, which is (b).

The inverse turns this into a strong drift limit. If f_N is bounded and weakly convergent in H, and $R_N \to R$ in Hilbert–Schmidt norm, then $R_N f_N - R f = (R_N - R) f_N + R(f_N - f) \to 0$: the second term uses the compactness of R. Apply this to $-D e^N$. For ϱ_q, the quadratic part converges on smooth tests by Lemma 24.1; apply the same compact argument with C_q to the weakly converging clipped first derivative. This is (c).

The bounds $\|H_q^N(y)\|_{\mathrm{HS}}^2 \leqslant C a_2(q)$ and $\|b_q^N(y)\|_2 \leqslant C\lambda q^{-3/2}\|y\|_2$, of Lemma 12.1 and Proposition 12.2, and $\|\varrho_q^N(y)\|_2 \leqslant \varepsilon_q\|y\|_2$ are uniform. So for a space–scale profile we integrate the convergence at each scale over I by dominated convergence.

For $q < q_*$, (a) and (b) at the single scale q_* give $H_{q_*}^N(\phi_N) \to H_{q_*}(\phi)$ and the weak convergence of $D\mathrm{Y}^N(\phi_N)$; adding the strongly convergent ϕ_N and applying the same compact argument gives $b_q^N(\phi_N) \to H_{q_*}(\phi)(\phi - D\mathrm{Y}(\phi))$. The domination is (12.9). □

The proof gives the convergence at every fixed scale, not only in $L^2(I)$. This matters: for the frozen coefficients, the scale q_* is a single point of the coefficient interval.

Lemma 29.2. (Continuity in the actual field) *For arbitrary lattice fields* y, z *and* $q \in I \subset [q_*, \infty)$*, with* $a = y^2 - z^2$*,*

$$\|H_q(y) - H_q(z)\|_{\mathrm{HS}} \leqslant C_I(\|y - z\|_2 + \|a\|_2), \tag{29.3}$$

$$\|D e_q(y) - D e_q(z)\|_2 + \|b_q(y) - b_q(z)\|_2 \leqslant C_I(1 + \|z\|_\infty)(\|y - z\|_2 + \|a\|_2), \tag{29.4}$$

$$\|\varrho_q(y) - \varrho_q(z)\|_2 \leqslant \varepsilon_q\|y - z\|_2. \tag{29.5}$$

The constants are uniform in N*, and only* z *is assumed bounded in* (29.4)*; this is the estimate quoted in the proof of Lemma 18.3. The estimate* (29.4) *also holds for the trace part* $D\mathrm{Y}_q$*, and so, at* $q = q_*$*, for the frozen drift of* (29.2) *at every* $q < q_*$.

Proof. Write $\Delta H = H_q(y) - H_q(z)$, $H_y = H_q(y)$, $H_z = H_q(z)$, and $\mathsf{D}_y = D^2\tilde{h}_q(y)$, $\mathsf{D}_z = D^2\tilde{h}_q(z)$. The exact identity is

$$\Delta H = -H_y[\lambda M_a + \mathsf{D}_y - \mathsf{D}_z]H_z.$$

Every row of H_z has bounded L^2 norm on I, by the diagonal bound (12.3), so $\|M_a H_z\|_{\mathrm{HS}} \leqslant C_I\|a\|_2$. Also

$$\mathsf{D}_y - \mathsf{D}_z = \int_0^1 D^3\tilde{h}_q(z + t(y-z))\,[y-z]\,\mathrm{d}t.$$

Bound a row of the last inverse in this product by Minkowski, then collapse the third tensor over one internal slot: for a three-tensor T, $S(x,u) = \int |T(x,v,u)|\,\mathrm{d}v$ has both Schur masses at most $s(T)$. Hence

$$\|(\mathsf{D}_y - \mathsf{D}_z)H_z\|_{\mathrm{HS}} \leqslant C_I\mathsf{S}_3\|y-z\|_2, \quad \|(\mathsf{D}_y - \mathsf{D}_z)f\|_2 \leqslant \mathsf{S}_3\|f\|_\infty\|y-z\|_2, \tag{29.6}$$

where $\mathsf{S}_3 = \sup s(D^3\tilde{h}_q)$ is finite by (12.2). With $\|H_y\| \leqslant C/q$ this proves (29.3).

Let $d(x) = \|\Delta H(x,\cdot)\|_2$, so $\|d\|_2 = \|\Delta H\|_{\mathrm{HS}}$. Since $H^2(u,v)$ is the inner product of the two rows, symmetry gives

$$|\mathrm{diag}(H_y^2 - H_z^2)(x)| \leqslant C_I d(x), \quad |(H_y^2 - H_z^2)(u,v)| \leqslant C_I(d(u) + d(v)).$$

In (29.1), split $y\mathrm{diag}H_y^2 - z\mathrm{diag}H_z^2 = (y-z)\mathrm{diag}H_y^2 + z\mathrm{diag}(H_y^2 - H_z^2)$. For the third vertex term, the vertex difference is an integral of $D^4\tilde{h}_q[y-z]$, and the covariance difference is bounded using the last display and the rooted third tensor. The terms $M_q' y$ and $D\partial_q\tilde{h}_q$ are linear, respectively Lipschitz, with constant $C g_q$, by $\|\mathfrak{S}_q\|_{\mathrm{TV}} \leqslant C g_q$ and (12.2). These give

$$\|D e_q(y) - D e_q(z)\|_2 + \|D\mathrm{Y}_q(y) - D\mathrm{Y}_q(z)\|_2 \leqslant C_I(1 + \|z\|_\infty)(\|y-z\|_2 + \|a\|_2).$$

Finally use

$$b_q(y) - b_q(z) = -H_y(D e_q(y) - D e_q(z)) + H_y[\lambda M_a + \mathsf{D}_y - \mathsf{D}_z]H_z D e_q(z).$$

The global linear bound of Proposition 12.2 gives $\|H_z D e_q(z)\|_\infty = \|b_q(z)\|_\infty \leqslant C_I\|z\|_\infty$, and the second bound of (29.6) handles the remaining Hessian term. The coordinate estimate is the bound on $D\varrho_q$ in (12.5).

For the frozen drift, $b(y) - b(z) = H_y(y - z - D\mathrm{Y}(y) + D\mathrm{Y}(z)) - H_y[\lambda M_a + \mathsf{D}_y - \mathsf{D}_z]b(z)$, with H_y, D_y at scale q_* and $\|b(z)\|_\infty \leqslant C\|z\|_\infty$ by (12.9). The same bounds apply. □

Proposition 29.3. (Coefficients at bounded fields) *For each $q > 0$ the maps $H_q, D e_q, b_q$, for $q \geqslant q_*$ also ϱ_q (and $D\mathrm{Y}$), defined on smooth bounded profiles by Proposition 29.1, extend uniquely to $L^\infty(\mathbb{T}^3)$, continuously for the H topology on each ball $\|y\|_\infty \leqslant L$, with (29.3)–(29.5). If $z_N \in H_N$, $\sup_N \|z_N\|_\infty < \infty$ and $\iota_N z_N \to z$ in H, then $z \in L^\infty$ and*

$$\iota_N H_q^N(z_N) j_N \to H_q(z) \quad \text{in HS}, \quad \iota_N b_q^N(z_N) \to b_q(z), \quad \iota_N \varrho_q^N(z_N) \to \varrho_q(z) \quad \text{in } H, \tag{29.7}$$

while $\iota_N D e_q^N(z_N) \to D e_q(z)$ and $\iota_N D\mathrm{Y}_q^N(z_N) \to D\mathrm{Y}_q(z)$ weakly in H. Moreover $\varrho_q(y) = C_q(M_q y + D\tilde{h}_q(y))$ for $q \geqslant q_*$, where $C_q M_q$ is the Fourier multiplier and $D\tilde{h}_q$ is Lipschitz on H with constant $C g_q q$. Finally $(q,y) \mapsto H_q(y), b_q(y), \varrho_q(y)$ are jointly Borel on $(0,\infty) \times L^\infty$, the balls carrying the H topology.

Proof. Fix $q \geqslant q_*$ and $L < \infty$. On the ball $\|y\|_\infty \leqslant L$ we have $\|y^2 - z^2\|_2 \leqslant 2L\|y-z\|_2$, so (29.3)–(29.5) give uniform Lipschitz continuity for the H norm, uniformly in N. Smooth profiles bounded by L are dense in this ball, by mollification, and the ball is closed in H. For smooth ϕ, ϕ' bounded by L, the bounds pass to the limit from ϕ_N, ϕ_N', in (29.4) by the lower semicontinuity of the norm for weak limits. This defines the extension and gives the estimates.

For the convergence, a limit z of $\iota_N z_N$ is bounded, as balls are closed. Take a smooth ϕ, bounded by $L = \sup \|z_N\|_\infty$, with $\|\phi - z\|_2 \leqslant \eta$. Since $\|\iota_N z_N - \iota_N \phi_N\|_2 \leqslant \|\iota_N z_N - z\|_2 + \eta + \|\phi - \mathsf{P}_N \phi\|_2$, the three term split

$$H_q^N(z_N) - H_q(z) = [H_q^N(z_N) - H_q^N(\phi_N)] + [H_q^N(\phi_N) - H_q(\phi)] + [H_q(\phi) - H_q(z)]$$

(with embeddings omitted) and Proposition 29.1 give the claim as $N \to \infty$ and then $\eta \to 0$. The same split, after testing with $f \in H$, gives the weak convergence of De and DY, and without testing that of b and ϱ.

The form of ϱ_q follows by the Lipschitz bound $\|D\tilde{h}_q^N(y) - D\tilde{h}_q^N(z)\|_2 \leqslant s(D^2\tilde{h}_q)\|y - z\|_2 \leqslant C g_q q \|y - z\|_2$ and (12.2), passed to the weak limit, together with the fact that the multiplier $(q + 4\pi^2|k|^2)^{-1}(\mu_q + \hat{\Theta}_q(k))$ is bounded.

For the measurability, on smooth ϕ the maps $q \mapsto H_q(\phi), b_q(\phi), \varrho_q(\phi)$ are limits of continuous functions of q (the second piecewise), so are Borel. For explicit jointly measurable versions, choose in each supremum ball a countable dense collection of smooth profiles and select the first profile within H distance 2^{-n} of the input. This is a Borel selection. The continuity estimates identify the limit of the resulting countably valued maps. Supremum balls are closed in H, so their increasing union gives Borel coefficient maps on bounded fields. Fixed values may be assigned outside this union, which does not affect the actual integrals. For $q < q_*$ everything reduces to scale q_*, and $y \mapsto y$ is continuous. □

30 Random inputs

The conclusion below is the statement of Section 18.1 of Lecture 7, used in the proofs of Theorems 18.4 and 18.6. We work on one probability space. All fields Y^N are measurable functions on $\Omega \times I$ with values in H_N, and Y is a random element of $L^2(I \times \mathbb{T}^3)$; I is a compact interval in $(0, \infty)$, and the coefficients for $q < q_*$ are the frozen ones (29.2). If I contains q_* in its interior, split it there.

Theorem 30.1. (Coefficient identification along random fields) *Suppose that $\iota_N Y^N \to Y$ in $L^2(I \times \mathbb{T}^3)$ in probability and that $(\sup_{q \in I} \|Y_q^N\|_\infty)_N$ is tight. Then Y is almost surely bounded on $I \times \mathbb{T}^3$ and, in probability,*

$$\iota_N H_q^N(Y_q^N) j_N \to H_q(Y_q) \quad \text{in } L^2(I; \mathrm{HS}), \quad \iota_N b_q^N(Y_q^N) \to b_q(Y_q) \quad \text{in } L^2(I; H), \tag{30.1}$$

and, for $I \subset [q_*, \infty)$, $\iota_N \varrho_q^N(Y_q^N) \to \varrho_q(Y_q)$ in $L^2(I; H)$. In particular this holds if, for almost every $q \in I$, $\iota_N Y_q^N \to Y_q$ uniformly on $\mathbb{T}^3$ in probability, with $\sup_I \|Y^N\|_\infty$ bounded in probability.

For $q < q_*$ the limits are $H_q = H_{q_*}$ and $b_q(y) = H_{q_*}(y)(y - DY(y))$, with Y the limit of $-\frac{1}{2}\mathrm{Tr}[H_{q_*}^N(y) - H_{q_*}^N(0)]$, given by (30.3) below.

Proof. Bounded fields. Let $L < \infty$. Consider nonrandom fields with $\|z\|_\infty \leqslant L$ on $I \times \mathbb{T}^3$. Integrating (29.3)–(29.5) over I, on this set $\|y^2 - z^2\|_2 \leqslant 2L\|y - z\|_2$ gives

$$\|H^N(y) - H^N(z)\|_{L^2(I;\mathrm{HS})} + \|b^N(y) - b^N(z)\|_{L^2(I;H)} \leqslant C_{I,L} \|y - z\|_{L^2(I;H_N)}, \tag{30.2}$$

and the same for the continuum maps of Proposition 29.3: uniform continuity in strong space–time L^2. Smooth space–scale profiles bounded by L are dense in this set: convolve in space and scale, using a bounded extension at the scale endpoints. Cell averages preserve the supremum bound and converge strongly.

Reduction to a net. Fix $\varepsilon>0$. Choose L so that the probability that $\sup_I\|Y^N\|_\infty>L$ is below ε for all N. The closed ball S_L of radius L of $L^\infty(I\times\mathbb{T}^3)$ is closed in $L^2(I\times\mathbb{T}^3)$, so the convergence in probability gives $\mathbb{P}(Y\notin S_L)\leqslant\liminf\mathbb{P}(\iota_N Y^N\notin S_L)\leqslant\varepsilon$. As ε is arbitrary, Y is almost surely bounded. The law of the single random element Y of the Polish space $L^2(I\times\mathbb{T}^3)$ is tight, so there is a compact $\mathcal{K}\subset S_L$ with $\mathbb{P}(Y\notin\mathcal{K})\leqslant 2\varepsilon$.

Given $\eta>0$, choose finitely many smooth space–scale profiles $\phi^1,\dots,\phi^m$, bounded by L, so that every point of $\mathcal{K}$ is within η of one of them in $L^2(I\times\mathbb{T}^3)$. On the event that $Y\in\mathcal{K}$, that $\sup_I\|Y^N\|_\infty\leqslant L$ and that $\|\iota_N Y^N-Y\|_{L^2}\leqslant\eta$, whose probability is at least $1-4\varepsilon$ for large N, pick i with $\|Y-\phi^i\|_{L^2}\leqslant\eta$. Then

$$\|H^N(Y^N)-H(Y)\|\leqslant\|H^N(Y^N)-H^N(\phi^i_N)\|+\max_{j\leqslant m}\|H^N(\phi^j_N)-H(\phi^j)\|+\|H(\phi^i)-H(Y)\|,$$

in $L^2(I;\mathrm{HS})$, with embeddings omitted. The first term is at most $C_{I,L}(2\eta+\max_j\|\phi^j-\mathsf{P}_N\phi^j\|_{L^2})$ by (30.2), the last at most $C_{I,L}\eta$, and the middle term tends to zero by Proposition 29.1, deterministically. So the probability that the left side exceeds $4C_{I,L}\eta$ is at most 5ε for large N. Letting $\eta\to0$ and then $\varepsilon\to0$ proves (30.1) for H. The same argument applies to b and ϱ. The limits are the maps of Proposition 29.3, evaluated at Y_q for almost every q; these are measurable by the last assertion of that proposition.

The uniform hypothesis. Assume now that for almost every $q\in I$, $\|\iota_N Y^N_q-Y_q\|_\infty\to0$ in probability, with $\sup_I\|Y^N\|_\infty$ tight. By Fubini and dominated convergence, $\mathbb{E}\int_I\min(1,\|\iota_N Y^N_q-Y_q\|_\infty)\,\mathrm{d}q\to0$. Along a subsequence, almost surely, $\iota_N Y^N_q\to Y_q$ uniformly for almost every q. Let A_N be the event that $\sup_I\|Y^N\|_\infty\leqslant L$, with $\mathbb{P}(A_N)\geqslant1-\varepsilon$. The event that A_N occurs infinitely often along the subsequence has probability at least $1-\varepsilon$, and on it $\|Y_q\|_\infty\leqslant L$ for almost every q. So $\mathbb{P}(\|Y\|_\infty\leqslant L)\geqslant1-\varepsilon$, and on A_N intersected with this event,

$$\int_I\|\iota_N Y^N_q-Y_q\|_2^2\,\mathrm{d}q\leqslant(1+2L)^2\int_I\min(1,\|\iota_N Y^N_q-Y_q\|_\infty)^2\,\mathrm{d}q,$$

whose expectation tends to zero. So $\iota_N Y^N\to Y$ in $L^2(I\times\mathbb{T}^3)$ in probability, and the preceding part applies.

The frozen coefficients. The continuity and the consistency used above were established for the frozen drift in Lemma 29.2 and Proposition 29.1. Since $\mathrm{Y}^N(0)=0$,

$$\mathrm{Y}^N(y)=-\frac{1}{2}\mathrm{Tr}[H^N_{q_*}(y)-H^N_{q_*}(0)]=\int_0^1\langle D\mathrm{Y}^N(ty),y\rangle\,\mathrm{d}t,\quad \mathrm{Y}(y)=\int_0^1\langle D\mathrm{Y}(ty),y\rangle\,\mathrm{d}t,\tag{30.3}$$

and this limit, for fields converging as in Proposition 29.3, is proved in Theorem 31.2 below. $D\mathrm{Y}$ is the trace part in the gradient (29.1) of e_{q_*}, as in (12.8). □

Remark 30.2. (**Which hypotheses are used**) The theorem needs less than Section 18.1 of Lecture 7 assumes: uniform convergence in space is not used, only strong $L^2(I\times\mathbb{T}^3)$ convergence, and the sup bound need only be tight, with no moments. The estimate (29.5) requires no bound at all, and (29.3) requires only the L^4 convergence of the field, but we do not use this. Moments of the sup norm such as those of Lemma 17.1 are one way of obtaining the tightness.

31 The coordinate, high modes and stochastic integrals

We now prove Lemma 18.3 and the statement on stochastic integrals in Section 18.1 of Lecture 7. Let τ_a denote the lattice translation by $a\in\Lambda_N$.

Lemma 31.1. **(Discrete compactness)** *Let $u_N \in H_N$ be bounded in L^∞ with $\operatorname{limsup}_N \sup_{|a|\leqslant\delta} \|\tau_a u_N - u_N\|_\infty \leqslant \varpi(\delta)$, where $\varpi(\delta)\to 0$ as $\delta\to 0$. Then every subsequence has a further subsequence along which $\iota_N u_N$ converges uniformly to a continuous function. If moreover $\iota_N u_N \to u$ in H, then u is continuous and $\iota_N u_N \to u$ uniformly along the whole sequence.*

Proof. Let ρ_δ be a mollifier supported in the ball of radius δ. If z lies in the cell of x and $|w|\leqslant\delta$, then $z-w$ lies in the cell of a lattice point within $\delta + C/N$ of x. So $\|\iota_N u_N - \rho_\delta * \iota_N u_N\|_\infty \leqslant \sup_{|a|\leqslant\delta+C/N} \|\tau_a u_N - u_N\|_\infty$, whose limsup is at most $\varpi(2\delta)$. For each fixed δ, the functions $\rho_\delta * \iota_N u_N$ are bounded in C^1, so by the Ascoli theorem and a diagonal argument over $\delta = 1/m$ they converge uniformly along a subsequence for every m. By the last bound $\iota_N u_N$ is uniformly Cauchy along it, and the limit is a uniform limit of the continuous limits of $\rho_{1/m} * \iota_N u_N$. If $\iota_N u_N \to u$ in H, then every such limit equals u. □

The continuum coordinate. Applying C_s, the coordinate equation $K_s y + D\bar\Gamma_s(y) = K_s x$ reads, by (12.5),

$$y + \varrho_s(y) + \frac{\lambda}{3} C_s y^3 = x \quad (s \geqslant q_*). \tag{31.1}$$

We take this equation, with the maps of Proposition 29.3, as the continuum coordinate equation, for $y \in L^\infty$. For $s < q_*$ we use (12.7), that is $y_s(x) = y_{q_*}(x - (q_* - s)C_{q_*}x)$.

The continuum defect is defined on L^∞ by

$$e_s(y) = \int_0^1 \langle D e_s(ty), y\rangle \,\mathrm{d}t \quad (s \geqslant q_*), \qquad e_s(y) = -\frac{1}{2}\|y\|_2^2 + \Upsilon(y) \quad (s < q_*). \tag{31.2}$$

Theorem 31.2 shows that it is the limit of the lattice defects.

Theorem 31.2. **(Consistency)** *This is Lemma 18.3 of Lecture 7. Let $s > 0$, $x_N \in H_N$, let x be continuous on $\mathbb{T}^3$, and suppose $\iota_N x_N \to x$ uniformly. Then:*

a) There is a unique $y_s(x) \in L^\infty$ solving the continuum coordinate equation, it is continuous, and $\iota_N y_s^N(x_N) \to y_s(x)$ uniformly. Moreover $\|y_s(x) - y_s(x')\|_\infty \leqslant 3\|x - x'\|_\infty$ for $s \geqslant q_*$, with the constant 6 for $s < q_*$.

b) If $z_N \in H_N$ are bounded in L^∞ and $\iota_N z_N \to z$ in H, then $e_s^N(z_N) \to e_s(z)$ and $\Upsilon^N(z_N) \to \Upsilon(z)$. In particular $e_s^N(y_s^N(x_N)) \to e_s(y_s(x))$.

c) For $s \geqslant q_*$, uniformly in N, $|e_s^N(y)| \leqslant C g_s \|y\|_2^2$; for $s < q_*$, $|e_s^N(y)| \leqslant C\|y\|_2^2$.

Proof. Let first $s \geqslant q_*$.

Uniqueness. Let $y, y' \in L^\infty$ solve (31.1), and put $\delta = y - y'$, $f = D\tilde h_s(y) - D\tilde h_s(y')$ and $w = \frac{\lambda}{3}(y^3 - (y')^3)$, which lie in H. By the form of ϱ_s in Proposition 29.3, in Fourier variables,

$$m_s(k)\hat\delta(k) = -\hat f(k) - \hat w(k), \qquad m_s(k) = s + 4\pi^2|k|^2 + \mu_s + \hat\Theta_s(k).$$

By the small relative form bound used in the proof of Lemma 26.1, $m_s(k) \geqslant \frac{1}{2}(s + 4\pi^2|k|^2)$. So $\delta \in H^2$, and $y - x \in H^1$ by (31.1). Multiplying by the conjugate of $\hat\delta(k)$ and summing,

$$\frac{s}{2}\|\delta\|_2^2 \leqslant \sum_k m_s(k)|\hat\delta(k)|^2 = -\langle \delta, f\rangle - \langle \delta, w\rangle \leqslant C g_s s \|\delta\|_2^2,$$

as $\langle \delta, w\rangle \geqslant 0$ by the monotonicity of the cube. For q_* large this gives $\delta = 0$.

Convergence. Put $y_N = y_s^N(x_N)$. As $\bar\Gamma_s^N$ is even, $y_s^N(0) = 0$, so Lemma 16.3 gives $\|y_N\|_\infty \leqslant 3\|x_N\|_\infty$, which is bounded. As $\bar\Gamma_s^N$ is translation invariant, the uniqueness of the coordinate gives $y_s^N(\tau_a x_N) = \tau_a y_s^N(x_N)$, so the same lemma gives

$$\|\tau_a y_N - y_N\|_\infty \leqslant 3\|\tau_a x_N - x_N\|_\infty \leqslant 6\|\iota_N x_N - x\|_\infty + 3\varpi_x(|a| + C/N),$$

with ϖ_x the modulus of continuity of x. By Lemma 31.1, along a subsequence $\iota_N y_N \to y$ uniformly, with y continuous. Pass to the limit in the lattice form of (31.1). Then $\iota_N(y_N^3) \to y^3$ uniformly and $\iota_N C_s^N j_N \to C_s$ in HS, by Lemma 26.1 with vanishing path, so that $\iota_N C_s^N y_N^3 \to C_s y^3$ in H. By (29.7), $\iota_N \varrho_s^N(y_N) \to \varrho_s(y)$ in H. Thus y solves (31.1). So a bounded solution exists, it is continuous, and by uniqueness every subsequential limit equals it: the whole sequence converges. The Lipschitz bound passes to the limit, using $x_N = j_N x$ and $x'_N = j_N x'$.

Defect. The lattice defect satisfies $e_s^N(0) = 0$, as $\tilde{h}_s^N$ and its scale derivative vanish at 0, and so $e_s^N(y) = \int_0^1 \langle D e_s^N(ty), y\rangle \, dt$. Let z_N, z be as in (b). For each t, $\iota_N(tz_N) \to tz$ in H and is bounded, so by Proposition 29.3, $\iota_N D e_s^N(tz_N) \to D e_s(tz)$ weakly in H, and $\iota_N z_N \to z$ strongly; the pairing converges. As $D e_s^N(0) = 0$ by evenness, (29.4) with second field 0 gives $\|D e_s^N(t z_N)\|_2 \leqslant C(\|z_N\|_2 + \|z_N^2\|_2)$, so dominated convergence in t gives $e_s^N(z_N) \to e_s(z)$. The same proof applies to $\mathrm{Y}^N(z_N)$ with $D\mathrm{Y}^N$, by (30.3). This also proves the limit claim in (30.3). Part (a) makes the choice $z_N = y_s^N(x_N)$ admissible.

The bound (c). The proof of the global linear bound of Proposition 12.2 (Lemma 21.2) gives $\|D e_s^N(y)\|_p \leqslant C g_s \|y\|_p$ for all y, and so $|e_s^N(y)| \leqslant C g_s \|y\|_2^2$. For $s < q_*$ this is (12.9).

The frozen case $s < q_*$. By (12.7), $y_s^N(x_N) = y_{q_*}^N(x'_N)$ with $x'_N = x_N - (q_* - s) C_{q_*}^N x_N$. The operator $C_{q_*}^N$ has norm $1/q_*$ on L^∞ and commutes with lattice translations, so $u_N = C_{q_*}^N x_N$ is bounded and $\|\tau_a u_N - u_N\|_\infty \leqslant q_*^{-1} \|\tau_a x_N - x_N\|_\infty$. Also $\iota_N u_N \to C_{q_*} x$ in H by Lemma 26.1. So by Lemma 31.1, $\iota_N x'_N \to x' = x - (q_* - s) C_{q_*} x$ uniformly, with x' continuous. The case $s = q_*$ gives (a), with Lipschitz constant $3\|I - (q_* - s) C_{q_*}\|_{\infty\to\infty} \leqslant 6$. By definition $e_s^N(y) = -\frac{1}{2}\|y\|_2^2 + \mathrm{Y}^N(y)$; $\|\iota_N z_N\|_2^2 \to \|z\|_2^2$ by the strong convergence, and $\mathrm{Y}^N(z_N) \to \mathrm{Y}(z)$ as shown. This proves (b). □

The following gives the second claim of Section 18.1 of Lecture 7: the identity $H^N = (I - H^N D^2 \bar{\Gamma}^N) C^N$ and its consequence on high modes. Let $F_{>m}^N$ be the Fourier projection on H_N onto the modes $k \in \Lambda_N^*$ with $|k| > m$, and $F_{\leqslant m}$ the Fourier projection on H onto the modes $|k| \leqslant m$.

Lemma 31.3. (High modes of the response) *For $q \geqslant q_*$, $y \in H_N$ with $\|y\|_\infty \leqslant L$, and $m \geqslant 1$,*

$$H_q^N(y) = (I - H_q^N(y) D^2 \bar{\Gamma}_q^N(y)) C_q^N, \quad \|H_q^N(y) D^2 \bar{\Gamma}_q^N(y)\|_{2\to 2} \leqslant C(1 + \lambda L^2/q), \tag{31.3}$$

and hence, uniformly in N,

$$\|H_q^N(y) F_{>m}^N\|_{\mathrm{HS}} \leqslant C(1 + \lambda L^2/q) m^{-1/2}. \tag{31.4}$$

For $q < q_*$ the bound (31.4) holds with $H_q^N = H_{q_*}^N$.

Proof. The identity is $H^N(K_q + D^2\bar{\Gamma}^N) = I$. By (12.1), $D^2\bar{\Gamma}^N = \lambda y^2 + \mu_q + \Theta_q + D^2\tilde{h}_q(y)$. By Lemma 12.1, $\|H^N\|_{2\to 2} \leqslant C/q$. So $H^N(\lambda y^2 + \mu_q + D^2\tilde{h}_q)$ has norm at most $C(\lambda L^2/q + g_q)$, by (11.12) and (12.2). The kernel Θ_q is not bounded on L^2 uniformly in N, and we argue by the other resolvent identity. By Lemma 24.1, $\Theta_q C_q$ has the multiplier $\hat{\Theta}_q(k)(q + c_k)^{-1}$, of size at most $C v_1(\Theta_q) q^{-1/2} \leqslant C R \lambda^2 q^{-1} \leqslant \frac{1}{2}$. From $H^N = C_q - C_q D^2 \bar{\Gamma}^N H^N$,

$$(I + \Theta_q C_q)\Theta_q H^N = \Theta_q C_q (I - (\lambda y^2 + \mu_q + D^2\tilde{h}_q) H^N),$$

so that $\|\Theta_q H^N\|_{2\to 2} \leqslant C\lambda^2 q^{-1}(1 + \lambda L^2/q)$, and $H^N \Theta_q$ is its adjoint. This proves (31.3). Next

$$\|C_q^N F_{>m}^N\|_{\mathrm{HS}}^2 = \sum_{k \in \Lambda_N^*, |k| > m} (q + c_k)^{-2} \leqslant C \sum_{|k| > m} (1 + |k|^2)^{-2} \leqslant C/m,$$

and multiplying by the bounded factor of (31.3) gives (31.4). The frozen case uses the same identity at q_*. □

The projected noise. Let W be a cylindrical Brownian motion on H. As in Section 18.3 of Lecture 7, its **Fourier projection** is $W^N = E_N W$, where

$$E_N f = \sum_{k \in \Lambda_N^*} \langle f, e_k \rangle e_k^N,$$

a real operator $H \to H_N$ with $E_N E_N^* = I$, as N is odd and the e_k^N are orthonormal in H_N. The alternative is $W^N = j_N W$, with $j_N j_N^* = j_N \iota_N = I$. Both are cylindrical on H_N. We show that both give the same limit.

Theorem 31.4. (Stochastic integrals against the projected noise) *Let W be a cylindrical Brownian motion on H for a filtration $(\mathcal{F}_q)$, and let Y^N and Y be progressively measurable, and satisfy the hypotheses of Theorem 30.1 on a compact interval I. Let W^N be either projection of W. Then*

$$\mathbb{E} \sup_{t \in I} \| \int_{I, q \leqslant t} \iota_N H_q^N(Y_q^N) \, \mathrm{d} W_q^N - \int_{I, q \leqslant t} H_q(Y_q) \, \mathrm{d} W_q \|_2^2 \to 0. \tag{31.5}$$

In particular $\int_I \iota_N H^N(Y^N) \, \mathrm{d} W^N \to \int_I H(Y) \, \mathrm{d} W$ in H, in probability. For $q < q_*$ this holds with the frozen coefficients (29.2), and I may contain q_*.

Proof. Let $E_N^\sharp$ be either E_N or j_N. Then $\int \iota_N H^N(Y^N) \, \mathrm{d} W^N = \int \mathcal{J}^N \, \mathrm{d} W$, with the integrand $\mathcal{J}_q^N = \iota_N H_q^N(Y_q^N) E_N^\sharp$, which is progressively measurable. So by Doob's inequality and the Itô isometry the left side of (31.5) is at most $4 \mathbb{E} \int_I \| \Phi_q^N - H_q(Y_q) \|_{\mathrm{HS}}^2 \, \mathrm{d} q$. By (12.3) and (12.9), $\| \Phi_q^N \|_{\mathrm{HS}}^2 \leqslant \operatorname{Tr} H_q^N(Y_q^N)^2 \leqslant C_I$ deterministically, and the same bound holds for the limit by lower semicontinuity. It is therefore enough to show that $\Phi^N \to H(Y)$ in $L^2(I; \mathrm{HS})$ in probability.

For $E_N^\sharp = j_N$ this is Theorem 30.1. For $E_N^\sharp = E_N$, it remains to show that $\| \iota_N H_q^N(Y_q^N)(E_N - j_N) \|_{\mathrm{HS}} \to 0$ in $L^2(I)$, in probability. Work on the event that $\sup_I \| Y^N \|_\infty \leqslant L$ and split with $F_{\leqslant m}$ and $I - F_{\leqslant m}$.

Low modes. $E_N e_k = e_k^N$, while $j_N e_k = \mathfrak{s}_N(k) e_k^N$ with $\mathfrak{s}_N(k) = \prod_i \operatorname{sinc}(\pi k_i / N)$. So for m fixed, $\| (E_N - j_N) F_{\leqslant m} \|_{\mathrm{HS}} \to 0$, and $\| H^N \|_{2 \to 2} \leqslant C$.

High modes, Fourier. Since $E_N(I - F_{\leqslant m}) = F_{>m}^N E_N$ and $\| E_N \| \leqslant 1$, (31.4) gives $\| H^N E_N (I - F_{\leqslant m}) \|_{\mathrm{HS}} \leqslant C_L m^{-1/2}$, uniformly in N and in $q \in I$.

High modes, cell average. This operator does not preserve high modes exactly, because of aliases, but

$$\| \iota_N H_q^N(Y_q^N) j_N (I - F_{\leqslant m}) \|_{\mathrm{HS}} \leqslant \| \iota_N H_q^N(Y_q^N) j_N - H_q(Y_q) \|_{\mathrm{HS}} + \| H_q(Y_q)(I - F_{\leqslant m}) \|_{\mathrm{HS}}.$$

The first term tends to 0 in $L^2(I)$ in probability by Theorem 30.1. The second tends to 0 as $m \to \infty$, for each q, and it is dominated, so it tends to 0 in $L^2(I)$ almost surely. Sending $N \to \infty$ and then $m \to \infty$ and $L \to \infty$ gives the claim.

For the final assertions, the frozen case is included in Theorem 30.1. If I contains q_*, split it there; the integrand is defined for all q, and the formula for H_q is continuous across q_*, as the drift is not involved. □

Remark 31.5. (Across the auxiliary scale) $\bar{\Gamma}_q$ is continuous in q at q_*, and $y_{q_* -} = y_{q_*}$ by (12.7), as $C_{q_*} K_{q_*} = I$. So $Y_q = y_q(x_q)$ is continuous across q_*, though b_q jumps there. Its forward equation is obtained by integrating on each side and adding; the jump of b does not matter for convergence in $L^2(I; H)$. The adaptedness of Y^N and Y in Theorem 31.4 is a genuine hypothesis, not implied by the convergence. In Section 18.3 of Lecture 7 it holds since the explicit flow is a strong solution, adapted to W.

32 Complements not used in the lectures

The lectures obtain the limit law from an explicit density, and the FBSDE from it by Girsanov's theorem. This section records the older routes and some stronger statements. References to the lectures are to their present numbering. In the subsection on compactness, the convergence is the stronger one of $C(I;H)$ and strong $L^2(I;\mathrm{HS})$, which the lectures do not claim. These routes predate the density of Section 18 of Lecture 7; where they differ from it, the lectures are authoritative.

In this section V_q without argument is the field (13.4), and $V_q(x)$ the potential; $\chi_q(y)$ is the pointwise large-field indicator, while $\chi_q(x)$ in the uniqueness proofs is a cut-off of the norm and χ in the penalty a profile; τ in the cut-offs is not the clip profile; ξ denotes the free flow or $Y-y$ as stated; A is the penalty constant, $A(q,x)$ the Feynman–Kac exponent; J is $D\kappa$, J_q the filter, J_∞ the energy.

32.1 Regularity in space and scale

On a finite scale interval $I=[q_*,Q]$ write the field equation (13.5) as the linear elliptic equation

$$K_{q_0}V_q=f_q,\qquad f_q=(q-q_0)\,(\zeta_q-\varrho_q(Y_q)+Z_q)-\frac{\lambda}{3}Y_q^3. \tag{32.1}$$

The energies control $\mathbb{E}\int_I\|Y_q\|_6^6\mathrm{d}q$ (since $|Y|^6\leqslant r_q^6+|t_q|^6$) and $\mathbb{E}\int_I\|Z_q\|_2^2\mathrm{d}q$, and $\mathbb{E}\|\zeta_q\|_2^2\leqslant C\sqrt{q}$. Hence $\mathbb{E}\int_I\|f_q\|_2^2\leqslant C_I$, and testing with V,

$$\mathbb{E}\int_I(\|\nabla_N V_q\|_2^2+\|V_q\|_2^2)\mathrm{d}q\leqslant C_I.$$

So the part V of the approximate forecast is in H^1, and $Y=V+\zeta-\varrho(Y)$ inherits the regularity of the noise ζ. This is that of the free field, whatever the interaction: by Lemma 13.2, $\mathbb{E}\|\zeta_q\|_{H^\sigma}^2\leqslant C\,(q-q_0)\sum_k(1+|k|^2)^\sigma\,(q+c|k|^2)^{-2}$, finite uniformly in N for $\sigma<\frac12$.

Proposition 32.1. (Forward compactness) *For $0<\sigma,\theta<\frac12$,*

$$\sup_N\mathbb{E}\Big[\int_I\|\iota_N Y_q^N\|_{H^\sigma}^2\mathrm{d}q+\|\iota_N Y^N\|_{H^\theta(I;H)}^2\Big]<\infty.$$

The embedded approximate forecasts are tight in every space–time L^p, and polynomials of them converge along strongly convergent subsequences. The forward equation gives the Hölder bound $\sup_N\mathbb{E}\,\|\iota_N Y^N\|_{C^{1/4}(I;H)}^8<\infty$, and averaging in time against the H^σ bound gives uniform Fourier tails $\lim_m\sup_N\mathbb{E}\sup_{q\in I}\|(I-\mathsf{F}_m)\,\iota_N Y_q^N\|_2^2=0$ (F_m the projection on frequencies $|k|\leqslant m$); hence they are also tight in $C(I;H)$.

The spatial bound combines the H^1 bound on V, the H^σ regularity of ζ ($\sigma<\frac12$, for which the cell embedding is uniformly bounded), and the contraction property of ϱ: since ϱ commutes with lattice translations τ_a, (12.5) gives $\|\tau_a\varrho(Y)-\varrho(Y)\|_2\leqslant\varepsilon_q\|\tau_a Y-Y\|_2$, so the increments of Y are controlled by those of $V+\zeta$; the scale regularity uses the forward equation, whose drift $b(Y)+H(Y)\,Z$ has bounded integrated square by Proposition 12.2 and the energies, and whose martingale part has bounded bracket density. The supremum bound of Lemma 17.1 upgrades L^2 compactness to every L^p.

32.2 The limit by compactness: the noise

The delicate point is the backward martingale. Fields converging in law do not imply that the stochastic integrals converge to stochastic integrals of the limits.

Remark 32.2. (Hidden noise) Let B be a real Brownian motion and $L^n_t = \int_0^t (1+\sin(ns))\mathrm{d}B_s$. Then (L^n, B) converges in law to $(B + 2^{-1/2}\tilde{B}, B)$, with $\tilde{B}$ an independent Brownian motion: the integrands converge weakly to 1, but their squares converge weakly to $\frac{3}{2}$. A limiting FBSDE obtained this way would have an extra martingale, orthogonal to the driving noise.

To exclude this one needs *strong* compactness of the integrands. For the forward equation the integrand $H(Y)$ is an explicit function of Y, and strong convergence follows from the coefficient limits. For the backward equation the integrand is $\Xi = \mathrm{Cov} - H(Y)$, and we need strong compactness of the posterior covariance Cov_q. It comes from its exact dynamics.

Lemma 32.3. (Riccati equation of the posterior covariance) *At each N, the conditional covariance $\mathrm{Cov}_q > 0$ satisfies*

$$\mathrm{dCov}_q = -\mathrm{Cov}_q^2\,\mathrm{d}q + \mathfrak{k}_q[\mathrm{d}W_q], \tag{32.2}$$

where $\mathfrak{k}_q$ is the third conditional cumulant of Φ_∞ given $\mathcal{F}_q$.

Proof. The conditional law is an exponential family in $p = K_q\Phi_q$ (proof of Proposition 10.4 of Lecture 6), with log-partition function $\mathcal{K}_q(p) = \log\int \mathrm{e}^{\langle p,\varphi\rangle - \frac{q-q_0}{2}\|\varphi\|_2^2}\nu_N(\mathrm{d}\varphi)$, so $\mathrm{Cov}_q = D^2\mathcal{K}_q(p)$ and $\mathfrak{k}_q = D^3\mathcal{K}_q(p)$. Differentiating, $\partial_q\mathcal{K} + \frac{1}{2}\mathrm{Tr}D^2\mathcal{K} + \frac{1}{2}\|D\mathcal{K}\|_2^2 = 0$; differentiate this twice in p: $(\partial_q + \frac{1}{2}\Delta_p + D\,\mathcal{K}\cdot D_p)\,D^2\,\mathcal{K} = -(D^2\,\mathcal{K})^2$, and apply Itô's formula along $\mathrm{d}p = D\,\mathcal{K}(p)\mathrm{d}q + \mathrm{d}W$, which follows from (10.6), $\partial_q K_q = I$ and $\psi_q = \Phi_q + C_q F_q$. □

This is (8.9) of Lecture 4 for $\partial_q\mathcal{R} = 1$: conditional variance is resolved at the rate Cov^2. For a smooth test f, the process $\langle f, \mathrm{Cov}_q f\rangle \geqslant 0$ is a nonnegative supermartingale, with drift $-\|\mathrm{Cov}_q f\|^2$, and initial value $\mathrm{Var}_{\nu_N}\langle\Phi_\infty, f\rangle$, bounded uniformly in N by Corollary 17.2. A nonnegative supermartingale has controlled oscillations: after stopping at the first time it exceeds a level K (an event of probability at most C_f/K), its decreasing part A and the bracket of its martingale part J satisfy $\mathbb{E}\,A_Q \leqslant C_f$ and $\mathbb{E}[J]_Q \leqslant K^2 + 2KC_f$, which gives time regularity in $W^{a,1}$, $a < \frac{1}{2}$, and hence strong compactness of each finite block $\langle f_i, \mathrm{Cov}\, f_j\rangle$. In this subsection $I = [q_0, Q]$, since ψ, W and the covariance do not involve Y. Spatial tails come from the isometry (recall $\psi_{q_0} = \mathbb{E}\,\Phi_\infty = 0$, ν_N being even), with symmetry exchanging input and output tails, $\mathbb{E}\int_I \|(I - \mathsf{F}_m)\mathrm{Cov}_q\|_{\mathrm{HS}}^2\mathrm{d}q = \mathbb{E}\|(I-\mathsf{F}_m)\,\psi_Q\|_2^2$, and from the terminal averaging $\psi_Q = \mathbb{E}_Q\int_L^{L+1}(Y_s + Z_s)\mathrm{d}s$, using the H^σ bound of Proposition 32.1 and $\mathbb{E}\|Z_s\|_2^2 \leqslant C\,s^{-5/2}$ (first $m\to\infty$, then $L\to\infty$). Altogether Cov^N is tight in strong $L^2(I;\mathrm{HS})$. Since $S \mapsto \int S\,S^*$ is continuous from $L^2(I;\mathrm{HS})$ to $C(I;\text{trace class})$, the brackets of ψ^N are tight in $C(I)$; a continuous martingale with tight brackets and uniformly small maximal Fourier tails (from the terminal averaging above and Doob's inequality) is tight in $C(I;H)$. Hence ψ^N, and $Z^N = \psi^N - Y^N$, are tight in $C(I;H)$. Strong convergence of the integrands gives convergence of the brackets $\int \mathrm{CovCov}^*$ and of the covariations with W, and then:

Proposition 32.4. (Identification of the martingales) *Along a subsequence, on a common filtered probability space, the limit of the forecast satisfies $\psi_q = \int_{q_0}^q \mathrm{Cov}_s\,\mathrm{d}W_s$ with W a cylindrical Brownian motion for the joint filtration, $X_q = \int_{q_0}^q C_s\,\mathrm{d}W_s$, and for $q \geqslant q_0$ (for $q < q_*$ see the paragraph after Theorem 32.5)*

$$Y_q = \int_{q_0}^q (b_s(Y_s) + H_s(Y_s)Z_s)\,\mathrm{d}s + \int_{q_0}^q H_s(Y_s)\,\mathrm{d}W_s, \qquad Z_q = -\int_{q_0}^q (b_s(Y_s) + H_s(Y_s)Z_s)\,\mathrm{d}s + \int_{q_0}^q \Xi_s\,\mathrm{d}W_s. \tag{32.3}$$

The argument is: the limiting ψ is a continuous square-integrable martingale for the joint filtration (fixed-time martingale identities pass to the limit with uniform integrability); its bracket and its covariation with W are those of $\int \mathrm{Cov}\,\mathrm{d}W$; so $\psi - \int \mathrm{Cov}\,\mathrm{d}W$ has zero bracket and vanishes. No extra martingale survives. The same argument for the forward equation identifies its noise with the same W, and identifies the limit of the auxiliary Gaussian field X as $\int C\,\mathrm{d}W$, not a new Gaussian field.

32.3 The limit by compactness: the result

Theorem 32.5. (Construction of $\boldsymbol{\Phi^4_3}$) *Let $\lambda, q_0 > 0$, $r \in \mathbb{R}$, with the counterterm of Theorem 11.3 (or q_* large and $b_{*,N} \to b_*$, $|b_{*,N}| \leqslant B_0 \lambda q_*^{1/2}$). Along a subsequence of lattices (the whole sequence, by Theorem 18.6):*

a) the embedded processes (ψ^N, X^N, Y^N, Z^N), with $X^N_q = \int_{q_0}^q C_s\,\mathrm{d}W^N_s$ the auxiliary Gaussian field, converge in law in $C(I;H)$ for compact $I \subset [q_0,\infty)$, together with the embedded innovations W^N in $C(I;H^{-\vartheta})$, $\vartheta > \frac{3}{2}$, and Cov^N, Ξ^N in strong $L^2(I;\mathrm{HS})$; all jointly and with an almost sure representation;

b) the limit solves the FBSDE (32.3) on $[q_0,\infty)$, $Y_{q_0} = Z_{q_0} = 0$, with the deterministic continuum coefficients b, H, limits of those of Lecture 6 and unique in their ball, an H-cylindrical Brownian motion W for the joint filtration, obtained as the limit of the lattice innovations, a symmetric nonnegative integrand $\mathrm{Cov} = H(Y) + \Xi$, and $\mathbb{E}\|Z_q\|_2^2 \leqslant C q^{-5/2}$ for large q;

c) there is a terminal field Φ with $Y_q \to \Phi$ in H^{-s}, almost surely and in mean square, for every $s > \frac{1}{2}$. Its law ν is the weak limit of the lattice measures ν_N, and for smooth h,

$$\langle \psi_q, h\rangle = \mathbb{E}[\langle \Phi, h\rangle | \bar{\mathcal{F}}_q]: \tag{32.4}$$

the forecast remains the conditional mean of the terminal field;

d) the limiting thresholded energies E_∞, J_∞ satisfy the bounds of Theorem 16.2. Indeed, the lattice identities $V_q = C_q\big[(q-q_0)\psi_q - \frac{\lambda}{3} Y_q^3\big]$ and $\zeta_q = X_q + C_q \int_{q_0}^q (\psi_s - X_s)\,\mathrm{d}s - (q-q_0) C_q \psi_q$ need neither ϱ nor W, so V^N and ζ^N converge by (a), the L^6 convergence of Y^N and the Hilbert–Schmidt convergence of the free resolvent; the integrands of E and J are lower semi-continuous in these variables, and Fatou's lemma applies.

Below the auxiliary scale. Above q_* the proof is that of Section 32.2, Propositions 32.1 and 32.4, with the moments of Y_{q_*} from Section 17, and spatial compactness at q_* by one-sided averaging on $[q_*, q_* + \delta]$. The processes ψ, W, Cov never involve Y: for $q \leqslant Q$, $\psi_q = \mathbb{E}_q \psi_Q$ gives their moments, and the Riccati argument starts at q_0 from the deterministic $\mathrm{Var}_{\nu_N}\langle \Phi^N_\infty, j_N f\rangle$. On $I = [q_0, q_*]$, $C_{q_*} K_s \Phi^N_s = C_{q_*} W^N_s + \int_{q_0}^s C_{q_*} \psi^N_u\,\mathrm{d}u$ is tight in $C(I;L^\infty)$: the first term by Kolmogorov, the second as $C_{q_*}: L^2 \to C^{1/2}$ and ψ^N converges in $C(I;H)$. By (12.7) and Lemmas 16.3, 18.3, $Y^N_s = y^N_{q_*}(C_{q_*} K_s \Phi^N_s) \to Y$ uniformly on $I \times \mathbb{T}^3$, so $Z^N \to Z$ in $C(I;H)$. The convergence stated at the end of Section 18.1, at the single scale q_*, gives $H(Y^N) \to H(Y)$ in $L^2(I;\mathrm{HS})$ and $b(Y^N) \to b(Y)$ in $L^2(I;H)$, so $\Xi^N \to \Xi$ and the limit solves the system on I. At q_0, $\Phi_{q_0} = 0$: so $Y_{q_0} = 0$ since $\bar{\mathcal{F}}$ is odd, and $Z_{q_0} = \psi_{q_0} = \mathbb{E}_\nu \Phi = 0$.

So the posterior structure survives the continuum limit: $\psi = Y + Z$ is a closed observable, the conditional mean of the terminal field; Y is computed from the explicit action; and the error Z is small at high resolution. This is the FBSDE of Lecture 4, in forecast coordinates, for the limit ν of the renormalised lattice measures, with the full noise retained.

32.4 Density at a fixed scale

The following is not used in the lectures: it gives the law at a fixed high scale, and it is where the second chaos of $\bar V_q$ becomes visible.

Proposition 32.6. (Density at a fixed scale) *Let q be large enough, beyond the auxiliary scale and such that $\sup_N \|Q_q^{1/2}\Theta_q^N Q_q^{1/2}\| \leqslant \frac14$. There are constants $c_N(q)$ with: $\mathrm{e}^{-\bar V_q + c_N}$ is bounded in $L^p(\gamma_{Q_q})$, $1<p<2$, uniformly in N, and converges in probability along the coupling to a function $\mathrm{e}^{-\bar V_q^\circ}$. The measure*

$$\rho_q(\mathrm{d}x) = \mathrm{e}^{-\bar V_q^\circ(x) - A(q,x)}\gamma_{Q_q}(\mathrm{d}x) \tag{32.5}$$

is finite, and for bounded continuous cylinder O,

$$\int O\,\mathrm{d}\nu = \frac{\int \mathrm{e}^{-\bar V_q^\circ(x)}\gamma_{Q_q}(\mathrm{d}x)\,\mathbb{E}^{\mathbb{R}^{q,x}}\big[O(Y_\infty)\mathrm{e}^{-\int_q^\infty e_s(Y_s)\,\mathrm{d}s}\big]}{\int \mathrm{e}^{-\bar V_q^\circ(x)}\gamma_{Q_q}(\mathrm{d}x)\,\mathbb{E}^{\mathbb{R}^{q,x}}\big[\mathrm{e}^{-\int_q^\infty e_s(Y_s)\,\mathrm{d}s}\big]}. \tag{32.6}$$

The law of Φ_q under ν is ρ_q normalised, the convergence holds for the whole sequence without the limit $q\to\infty$, and the renormalised partition functions $\int \mathrm{e}^{-V_{q,N}+C_N(q)+c_N}\,\mathrm{d}\gamma_{Q_q}$ converge to the total mass of ρ_q.

Proof. Bounds on $\bar V$. On the unit volume, $\bar V_q(x) = \min_y\big[\bar\Gamma_q(y) + \frac12\langle x-y, K_q(x-y)\rangle\big]$, the functional being strictly convex for $q\geqslant q_*$. As $U + \frac12\mu_q y^2 + \tilde h_q \geqslant -C_q$ (the quartic dominates, as $\mathsf{S}_4(h_q)$ is small, and $|\tilde h_q(y)| \leqslant C\,\mathsf{S}_4(h_q)\|y\|_4^4$), $\bar V_q(x) \geqslant -C_q + \frac12\langle x, \Theta_q' x\rangle$, with $\Theta_q' = K_q(K_q+\Theta_q)^{-1}\Theta_q$. With $c_N = \frac12\mathrm{Tr}(Q_q\Theta_q')$, the determinant formula for the second chaos gives the L^p bound, since the operator has norm $\leqslant\frac12$ and is Hilbert–Schmidt, uniformly in N; here the multiplier of Θ_q is at most $C|k|$ and that of Q_q at most $C|k|^{-4}$. Convergence in probability is Step 4 of the proof of Theorem 32.13, without χ_q: the second chaos in L^2, the other terms by continuity for the uniform norm.

Cut-off. For fixed N, the law of Φ_q is proportional to $\mathrm{e}^{-\bar V_q + c_N - A_N(q,\cdot)}\gamma_{Q_q}$ by the definition of A_N. For χ_R a continuous cut-off of $\|x\|_\infty$, on its support A_N is bounded and continuously convergent (Theorem 18.4), and $\iota_N x_N \to x$ uniformly in probability; with the L^p bound,

$$I_N(R) := \int \chi_R \mathrm{e}^{-\bar V_q + c_N - A_N}\,\mathrm{d}\gamma_{Q_q} \to I(R) := \int \chi_R\,\mathrm{d}\rho_q,$$

with $I(R)>0$, and the same with a bounded continuous function under the integral. By Step 1 of the proof of Theorem 32.13, that is (12.5) and Lemma 17.1, $\sup_N \mathbb{P}(\|\Phi_q^N\|_\infty > R/2) \to 0$ as $R\to\infty$. So for R large, $I_N(R)/Z_N \geqslant \frac12$, where Z_N is the total mass, and hence $\limsup_N Z_N \leqslant 2I(R) < \infty$; ρ_q is finite by Fatou. Letting $R\to\infty$ the law of Φ_q^N converges to ρ_q normalised, as does Z_N to its total mass.

The formula. The kernel form follows by disintegrating ν_N at scale q, and using (18.12) with the uniform convergence in probability: the cylinder function $\Pi_q^N O(x_N)$ is bounded, and converges for continuously convergent arguments. So $\int O\,\mathrm{d}\nu = \int \Pi_q O\,\mathrm{d}\rho_q/\rho_q(1)$, which by (18.12) and (32.5) is (32.6), the factor e^A cancelling. □

32.5 Feynman–Kac comparison along the explicit flow

Let $\bar V_q(x) = \Gamma^\sharp(y) + \frac12\langle x-y, K_q(x-y)\rangle$, where $y = y_q(x)$ solves $K_q y + D\Gamma^\sharp(y) = K_q x$, for $\Gamma^\sharp$ one of the actions below. Let $e^\sharp = \partial_q\Gamma^\sharp - \frac12\mathrm{Tr}[H^\sharp - H^\sharp(0)]$ and $c(q) = \frac12\mathrm{Tr}(C_q - H_q(0))$.

Lemma 32.7. (Hamilton–Jacobi defect) $\partial_q\bar V+\frac12\mathrm{Tr}(C_q^2D^2\bar V)-\frac12\|C_qD\bar V\|_2^2=e^\sharp\circ y+c(q)$, *provided* $H^\sharp(0)=H_q(0)$.

Proof. By the envelope theorem $\partial_q\bar V=\partial_q\Gamma^\sharp(y)+\frac12\|x-y\|_2^2$. Further $D\bar V=K_q(x-y)$, and since $Dy=H^\sharp K_q$, $D^2\bar V=K_q-K_qH^\sharp K_q$. So $\mathrm{Tr}(C_q^2D^2\bar V)=\mathrm{Tr}(C_q-H^\sharp)$ and $\|C_qD\bar V\|_2=\|x-y\|_2$. □

The penalty. Let $\chi\in C^{1,1}$, smooth off $\frac{15}{16}$ and nondecreasing, with $\chi(u)=0$ for $u\leqslant\frac{15}{16}$, $\chi(u)=1$ for $u\geqslant1$, and $\chi(u)^2=\big(u-\frac{15}{16}\big)^4$ near $\frac{15}{16}$. For a constant A put $\mathfrak{g}_s(y)=g_s\int y^2\chi^2(|y|/r_s)$ and

$$\delta_q(y)=A\int_q^\infty\mathfrak{g}_s(y)\,\mathrm{d}s=\int F_q(y(x))\,\mathrm{d}x,\quad\Gamma^\pm=\bar\Gamma\pm\delta.$$

F_q is even and C^4, vanishes on $|u|\leqslant\frac{15}{16}r_q$, and the s integral runs up to $s_u=\frac{256}{225}\lambda u^2$. Let $v=|u|/r_q-\frac{15}{16}$. Then for $v>0$, with constants independent of A and q,

$$|F_q''(u)|\leqslant CAg_q(q+\lambda u^2)1_{v>0},\quad|F_q''(u)|\leqslant CAq^{1/2}\lambda\min(v_+^3,1)\frac{|u|}{r_q}\quad\text{for all }u,\tag{32.7}$$

the second from differentiating $u^2\int_q^{s_u}\lambda s^{-1/2}\chi^2(|u|/r_s)\,\mathrm{d}s$ twice in u: near the threshold the integral is of order $u^2g_qqv^5$, and for $|u|\gg r_q$ it is of order $A\lambda^{3/2}|u|^3$.

Lemma 32.8. (Global bounds for the comparison actions) *For A fixed and $q\geqslant q_A$, $\Gamma^\pm$ satisfy the conclusions of Lemma 12.1, $H^\pm(0)=H_q(0)$, and $\|b^\pm(y)\|_2\leqslant C_A\lambda q^{-3/2}\|y\|_2$.*

Proof. $F''\leqslant CAg_q\lambda y^2$ on its support, as $q\leqslant C\lambda y^2$ there. So $\lambda y^2\pm F''(y)\geqslant0$ for Ag_q small, and the proof of Lemma 12.1 applies with this potential: killing gives $0\leqslant(K_q+\lambda y^2\pm F'')^{-1}\leqslant C_q$ entrywise, and the Neumann series handles $M+D^2\tilde h$. For the drift, the new terms in $De^\pm$ are $\mp AD\mathfrak{g}_q$ with $|D\mathfrak{g}_q|\leqslant Cg_q|y|$, and $\pm\frac12F'''(y)\mathrm{diag}(H^\pm)^2$ together with the change of H in the terms of De. Since $|F'''(u)|\leqslant CA\lambda^{3/2}1_{v>0}$ and $\mathrm{diag}(H^\pm)^2\leqslant Ca_2$, the former is at most $CAg_q^2|y|$. Multiplication by $H^\pm$ costs C/q, and the global linear bound for b gives the rest. □

The linear large-field bound. Write $De=D\kappa De^{\mathrm o}\circ\kappa+\mathfrak{D}$ as in Lemma 21.2, $\mathfrak{D}=De-D\kappa De^{\mathrm o}\circ\kappa$.

Lemma 32.9. (The linear large-field bound) $\|\mathfrak{D}_s(z)\|_2\leqslant Cg_s\|t_s(z)\|_2$ *for all z and $s\geqslant q_*$.*

Proof. The ledger (21.10) has six rows, of which four are linear in t. Only rows four and six carry $y^2\chi$, through $\mathcal{D}_2=H^2-H^{\mathrm o2}=-H^2\mathsf{D}H^{\mathrm o}-H\mathsf{D}H^{\mathrm o2}$ with $\mathsf{D}=\lambda M_{y^2-z^2}+\mathsf{D}'$, where z is now the clipped field and D' the nonlocal insertions. The latter cost $Cgq\|\chi\|_2$, and $r\chi\leqslant|t|$. For the local insertion, put $e_0=(y^2-z^2)/|y|\leqslant C|t|$ and $Q=\lambda^{1/2}M_{|y|}H$. By Lemma 21.1, $\|Q\|_{2\to2}\leqslant Cq^{-1/2}$ and $\mathrm{diag}H^2,\mathrm{diag}H^{\mathrm o2}\leqslant Ca_2$. Let T be a symmetric 3-tensor and $\varphi\in L^2$. Then $X=H^2\lambda M_{y^2-z^2}H^{\mathrm o}$ reads $\lambda^{1/2}(HQ^*)M_{e_0}H^{\mathrm o}$, and Cauchy–Schwarz in all variables gives

$$\Big|\int\varphi(x)T(x,u,v)X(u,v)\Big|\leqslant\lambda^{1/2}\Big(\int\varphi(x)^2|T|(HQ^*)(u,a)^2\Big)^{1/2}\Big(\int|T|e_0(a)^2H^{\mathrm o}(a,v)^2\Big)^{1/2}.$$

Integrating a and using $HQ^*QH\leqslant\|Q\|^2H^2$, the first factor is at most $(Ca_2s(T)/q)^{1/2}\|\varphi\|_2$; integrating x,u and then v, the second is at most $(Ca_2s(T))^{1/2}\|e_0\|_2$. So $\|\mathrm{Ctr}_XT\|_2\leqslant Cr^{-1}a_2s(T)\|t\|_2$. For $X=H\lambda M_{y^2-z^2}H^{\mathrm o2}$ write it as $\lambda^{1/2}(Q^*M_{e_0}H^{\mathrm o})H^{\mathrm o}$ and put φ^2 on the factor with the outer $H^{\mathrm o}$: the same bound results. With $s(f_3)\leqslant Cgq/r$ this is row six, $Cg^2\|t\|_2$. With the local tensor, $s(T)=1$, it gives $\|\mathrm{diag}\mathcal{D}_2\|_2\leqslant Cr^{-1}a_2\|t\|_2$, hence row four, $C\lambda ra_2r^{-1}\|t\|_2\leqslant Cg\|t\|_2$. □

Lemma 32.10. (Sub- and supersolutions) *With π_s the pointwise projection on $[-r_s, r_s]$, for A large and $s \geqslant q_A$,*

$$e_s^- \geqslant -C(\lambda^5 s^{-3}\|\pi_s y\|_4^4 + \lambda^4 s^{-2}\|\pi_s y\|_2^2), \quad e_s^+ \leqslant C(\lambda^5 s^{-3}\|\pi_s y\|_4^4 + \lambda^4 s^{-2}\|\pi_s y\|_2^2). \tag{32.8}$$

Proof. The action $\bar\Gamma$. On $\|z\|_\infty \leqslant r_s$ the clip is the identity, so $e_s(\pi_s y) = e_s^{\mathrm{o}}(\pi_s y)$, and by evenness, the zero scalar jets and (11.14), $|e_s^{\mathrm{o}}(\pi_s y)| \leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4 + C\lambda^4 s^{-2}\|\pi_s y\|_2^2$, the quadratic term coming from the centred quadratic jet $\frac{1}{2}\langle y, \mathcal{E}_s^c y\rangle$. With $z_\theta = \pi y + \theta(y - \pi y)$,

$$e_s(y) - e_s(\pi_s y) = \int_0^1 \langle D e_s(z_\theta), y - \pi_s y\rangle \, \mathrm{d}\theta.$$

The vector $y - \pi y$ is supported on the set $\mathcal{A}$ where $|y| > r_s$ and its norm is at most $\|t_s(y)\|_2$; also $\|t_s(z_\theta)\|_2 \leqslant \|t_s(y)\|_2$. The first term of De is bounded pointwise by $\sup|De^{\mathrm{o}}| \leqslant \mathsf{S}_4(e^{\mathrm{o}})(2r_s)^3/6 + \|\mathcal{E}_s^c\|_{\mathrm{row}} 2r_s \leqslant C\lambda^{7/2}s^{-3/2}$, and $|\mathcal{A}|^{1/2} \leqslant r_s^{-1}\|t_s\|_2$, so it contributes at most $C\lambda^4 s^{-2}\|t_s\|_2^2 \leqslant C g_s \|t_s\|_2^2$. By Lemma 32.9 so does the second. Since $\chi = 1$ on $\mathcal{A}$, $g_s\|t_s\|_2^2 \leqslant \mathfrak{g}_s(y)$. Thus $|e_s(y)| \leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4 + C\lambda^4 s^{-2}\|\pi_s y\|_2^2 + C_1\mathfrak{g}_s(y)$ with C_1 independent of A.

The penalty. $\partial_s \delta_s = -A\mathfrak{g}_s$, so $e^\pm = e \mp A\mathfrak{g}_s - \frac{1}{2}\mathrm{Tr}(H^\pm - H)$. As $H^\pm - H = \mp H^\pm D^2\delta H$ and $D^2\delta$ is diagonal, $|\mathrm{Tr}(H^\pm - H)| \leqslant C\, a_2 \int |F_s''(y)|$. Pointwise, with $u = y(x)$ and v as above: by (32.7), the ratio of $a_2|F''(u)|$ to the density $A\, g_s u^2 \chi^2(|u|/r_s)$ of $A\mathfrak{g}_s$ is at most $C\, g_s / \min(v, 1)$. So for $v \geqslant C_2 g_s$, this term is at most half of the density of $A\mathfrak{g}_s$. For $0 < v < C_2 g_s$, the thin layer, $a_2|F''| \leqslant C\, A\, q^{-1/2} q^{1/2} \lambda v^3 \leqslant C A \lambda g_s^3$. There $|\pi_s u| \geqslant \frac{15}{16} r_s$, and $\lambda^5 s^{-3} r_s^4 = \lambda^3 s^{-1}$, while $A\lambda g_s^3 = A\lambda^4 s^{-3/2} \leqslant \lambda^3 s^{-1}$ for s large depending on A. Hence $\frac{1}{2}|\mathrm{Tr}(H^\pm - H)| \leqslant \frac{A}{2}\mathfrak{g}_s(y) + C\lambda^5 s^{-3}\|\pi_s y\|_4^4$. Taking $A \geqslant 2C_1$ gives (32.8). □

The flows. Fix x and q. Let $\bar\Phi$ solve $\mathrm{d}\bar\Phi_s = -C_s^2 D\bar V_s^\pm(\bar\Phi_s)\,\mathrm{d}s + C_s\,\mathrm{d}W_s$, $\bar\Phi_q = x$. The drift is K_s-Lipschitz at fixed N with linear growth, as $\|Dy\|$ is bounded. Its coordinate $Y_s = y_s^\pm(\bar\Phi_s)$ satisfies $\mathrm{d}Y = b^\pm(Y)\,\mathrm{d}s + H^\pm(Y)\,\mathrm{d}W$, by the computation of Theorem 10.6 with no error. No Z appears.

Lemma 32.11. (Riccati bound) *For $y = y_q^\pm(x)$,*

$$\log \bar{\mathbb{E}} \exp\Big(\int_q^\infty C_3(\lambda^5 s^{-3}\|\pi_s Y_s\|_4^4 + \lambda^4 s^{-2}\|\pi_s Y_s\|_2^2)\,\mathrm{d}s\Big) \leqslant C\lambda^5 q^{-2}\|y\|_4^4 + C\lambda^4 q^{-1/2}(1 + \lambda q^{-1}\|y\|_2^2) + C\lambda^4 q^{-1}\|y\|_2^2. \tag{32.9}$$

Proof. Let $\xi = Y - y$. Then $|\pi_s Y - \pi_s y| \leqslant \min(|\xi|, 2r_s)$, so $\|\pi_s Y\|_4^4 \leqslant 8\|\pi_s y\|_4^4 + 32 r_s^2\|\xi\|_2^2$, and $\|\pi_s Y\|_2^2 \leqslant 2\|y\|_2^2 + 2\|\xi\|_2^2$. So the exponent is at most a deterministic term plus $\int \kappa_s w_s\,\mathrm{d}s$, with $w = \|\xi\|_2^2$ and $\kappa_s = C\lambda^4 s^{-2}$. By Lemma 32.8, $\mathrm{diag}(H^\pm)^2 \leqslant C s^{-1/2}$ and $\|H^\pm\| \leqslant (1 + \varepsilon_s)/s$:

$$\mathrm{d}w \leqslant (C\lambda s^{-3/2}(w + \|y\|_2^2) + C s^{-1/2})\,\mathrm{d}s + \mathrm{d}M, \quad \mathrm{d}\langle M\rangle \leqslant 5 s^{-2} w\,\mathrm{d}s.$$

Let $\theta_s = 4C\lambda^4/s$. For s large, $\theta' + C\lambda s^{-3/2}\theta + \frac{5}{2}s^{-2}\theta^2 + \kappa \leqslant 0$, because $\theta' = -4\kappa$ dominates. Then $\exp\big(\theta_s w_s + \int_q^s \kappa w - \int_q^s a\big)$, with $a_s = \theta_s(C\lambda s^{-3/2}\|y\|_2^2 + C s^{-1/2})$, is a local supermartingale by Itô's formula. It starts from 1 as $w_q = 0$, and $\theta w \geqslant 0$. Fatou gives $\bar{\mathbb{E}}\exp\big(\int_q^\infty \kappa w\big) \leqslant \exp\int_q^\infty a$, and $\int_q^\infty a \leqslant C\lambda^5 q^{-3/2}\|y\|_2^2 + C\lambda^4 q^{-1/2}$. The deterministic term is at most $C\lambda^5 \int_q^\infty s^{-3}\|y\|_4^4\,\mathrm{d}s + C\lambda^4 \int_q^\infty s^{-2}\|y\|_2^2\,\mathrm{d}s$. □

Proposition 32.12. (The comparison) *Let $C_N(q) = \int_q^\infty c(s)\,\mathrm{d}s$, which is finite at fixed N. For $q \geqslant \max(q_*, q_A)$, if $\|y_q(x)\|_\infty \leqslant \frac{15}{16} r_q$, then, uniformly in N,*

$$|V_{q,N}(x) - \bar V_{q,N}(x) - C_N(q)| \leqslant C\lambda^5 q^{-2}\|y\|_4^4 + C\lambda^4 q^{-1/2}(1 + \lambda q^{-1}\|y\|_2^2) + C\lambda^4 q^{-1}\|y\|_2^2. \tag{32.10}$$

Proof. Let ξ_s be the free flow, $\mathrm{d}\xi = C_s\,\mathrm{d}W$, $\xi_q = x$. Then $\mathrm{e}^{-V_q(x)} = \mathbb{E}\mathrm{e}^{-V_T(\xi_T)}$ for $T > q$, as V solves the Polchinski equation. By Lemma 32.7 and Itô's formula, $\exp\big(-\bar V_T^{\pm}(\xi_T) + \bar V_q^{\pm}(x) + \int_q^T (e^{\pm}\circ y + c)\big)$ is the Girsanov density of the flow $\bar\Phi$ with respect to ξ on $[q,T]$; it is a true martingale, because the drift has linear growth. Therefore, with $\Delta_T^{\pm} = V_T - \bar V_T^{\pm}$,

$$\mathrm{e}^{-V_q(x)} = \mathrm{e}^{-\bar V_q^{\pm}(x) - \int_q^T c}\bar{\mathbb{E}}^{\pm}\exp(-\Delta_T^{\pm}(\bar\Phi_T) - \int_q^T e^{\pm}(Y_s)\,\mathrm{d}s). \tag{32.11}$$

At fixed N: $|c(s)| \leqslant C_N s^{-2}$ by $H_s(0) - C_s = O(C_s M_s C_s)$ and $a_2 = O_N(s^{-2})$, so C_N is finite.

Upper bound for V_q. Use Γ^+ and Jensen in (32.11): $V_q(x) \leqslant \bar V_q^+(x) + \int_q^T c + \bar{\mathbb{E}}^+\Delta_T^+(\bar\Phi_T) + \bar{\mathbb{E}}^+\int_q^T e^+$. By (32.8) and Jensen again, the last term is bounded by the right side of (32.9). As $\delta \geqslant 0$, $\Delta_T^+ \leqslant V_T - \bar V_T$. This tends to 0 uniformly on $\|x\|_\infty \leqslant T^{1/8}$, by the estimates of the next paragraph. Outside this set it has polynomial domination uniform in T: $V_T \leqslant C_N(1 + \|x\|_\infty^4)$ by Jensen, and $\bar V_T \geqslant -C_N$ since U dominates $\tilde h_T$. And $\bar\Phi_T$ has all moments uniformly in T at fixed N, by the linear bound on b^+ and $\int^\infty \mathrm{Tr}(H^+)^2\,\mathrm{d}s < \infty$ at fixed N. So the limsup of $\bar{\mathbb{E}}^+\Delta_T^+$ is at most 0.

Lower bound. Let $\mathcal{G}_T$ be the event $\|\xi_T\|_\infty \leqslant T^{1/8}$. Since $V_T \geqslant \inf V_N > -\infty$ at fixed N, $\mathbb{E}\mathrm{e}^{-V_T(\xi_T)}1_{\mathcal{G}_T^c} \leqslant C_N\mathbb{P}(\mathcal{G}_T^c) \to 0$. On $\mathcal{G}_T$, $\|\xi_T\|_\infty \ll r_T$, so δ_T vanishes at the coordinate and $\bar V_T^- = \bar V_T$. We claim $\sup_{\mathcal{G}_T}|\Delta_T^-| \to 0$. Indeed at fixed N: $|\mu_T - \mu_N| = O_N(T^{-1})$; $\|\Theta_T\|_{\mathrm{row}} = O_N(T^{-1/2})$; $\mathsf{S}_4(h_T) = O_N(T^{-2})$, by the integrated bound using $a_2(T) = O_N(T^{-2})$; and $|y - x| \leqslant C_N T^{-1}(\|x\|_\infty + \|x\|_\infty^3)$. These give $|\bar V_T(x) - V_N(x)| = O_N(T^{-1/4})$ for $\|x\|_\infty \leqslant T^{1/8}$. Further $|V_T(x) - V_N(x)| = O_N(T^{-1/8})$ on the same set, by Gaussian integration, since the remaining variance per site is $O_N(T^{-1})$. So

$$\mathrm{e}^{-V_q(x)} \leqslant \mathrm{e}^{o(1)}\mathrm{e}^{-\bar V_q^-(x) - \int_q^T c}\bar{\mathbb{E}}^-\exp(-\int_q^T e^-(Y_s)\,\mathrm{d}s) + o(1).$$

By (32.8) and (32.9) the expectation is at most the exponential of the right side of (32.9), and $T \to \infty$.

Conclusion. If $\|y\|_\infty \leqslant \frac{15}{16}r_q$, then δ_q and $D\delta_q$ vanish at y. So y is the critical point of the strictly convex functional defining $\bar V_q^{\pm}(x)$, and $\bar V_q^{\pm}(x) = \bar V_q(x)$. □

32.6 Uniqueness through the comparison

Theorem 32.13. (Uniqueness) *Let $q_* \geqslant \max(\hat q, q_A)$ and $b_{*,N} \to b_*$ in $\mathcal{B}$, in particular for the explicit counterterm (22.2). Then the whole sequence $\iota_N\nu_N$ converges weakly in H^{-s}, $s > \frac12$, to a measure ν. The joint law of Φ and of the Polchinski process $\Phi_q = (q - q_0)C_q\Phi + C_qB_{q-q_0}$, B independent of Φ, is unique. The dynamics are not used.*

Proof. Fix $\delta \in \big(0, \frac{1}{12}\big)$, $\rho_q = q^{5/12+\delta}$, and $\chi_q(x) = \tau(\|x\|_\infty/\rho_q)$ with τ continuous, $\tau = 1$ on $\big[0, \frac12\big]$, $\tau = 0$ on $[1,\infty)$.

Step 1: the good set. By (12.5), $\Phi_q = Y_q + \varrho_q(Y_q) + \frac{\lambda}{3}C_qY_q^3$, so $\|\Phi_q\|_\infty \leqslant 2\|Y_q\|_\infty + \frac{\lambda}{3q}\|Y_q\|_\infty^3$. Proposition 14.4, whose constant involves $\|C_{q_0}\|_{2\to\infty}$, finite with $a_2(q_0) \leqslant q_0^{-2} + C$ uniformly in N, with Lemma 14.5 and Theorem 16.2 gives Lemma 17.1: $\|Y_q\|_\infty \leqslant \mathcal{Y}q^{5/12}\log^{1/6}(2+q)$ for $q \geqslant q_*$, with $\mathcal{Y}$ having moments uniformly in N. Therefore $p_q := \sup_N\mathbb{P}(\|\Phi_q^N\|_\infty > \rho_q/2) \to 0$.

Step 2: the error. If $\|x\|_\infty \leqslant \rho_q$ then $\|y_q(x)\|_\infty \leqslant 3\rho_q$. Indeed with $u = y + \varrho_q(y)$, $K_q(u-x) + \frac{\lambda}{3}y^3 = 0$. At a positive maximum of $u - x$, $K_q(u-x) > 0$, so $y < 0$ there, and $u - x \leqslant \|\varrho_q(y)\|_\infty + \|x\|_\infty$; similarly at a negative minimum. So $\|u\|_\infty \leqslant 2\|x\|_\infty + \varepsilon_q\|y\|_\infty$, and $\|y\|_\infty \leqslant 3\|x\|_\infty$ for ε_q small. For q large, $3\rho_q \leqslant \frac{15}{16}r_q$, and (32.10) gives on $\mathrm{supp}\chi_q$

$$|V_{q,N} - \bar V_{q,N} - C_N(q)| \leqslant \varepsilon_q' := C(q^{-1/3+4\delta} + q^{-1/2} + q^{-2/3+2\delta} + q^{-1/6+2\delta}) \to 0, \tag{32.12}$$

the last term coming from $C\lambda^4 q^{-1}\|y\|_2^2$ with $\|y\|_\infty \leqslant 3\rho_q$; it tends to zero as $\delta < \frac{1}{12}$.

Step 3: total variation. Φ_q^N has law $\mu_{N,q} \propto \mathrm{e}^{-V_{q,N}}\gamma_{Q_q}$, with $Q_q = C - C_q$. Let $\hat\nu_{N,q} \propto \chi_q \mathrm{e}^{-\bar V_{q,N}}\gamma_{Q_q}$. On $\operatorname{supp}\chi_q$ the two unnormalised densities have ratio $\mathrm{e}^{-C_N(q)+\theta}$, $|\theta| \leqslant \varepsilon_q'$; so the density of $\hat\nu_{N,q}$ with respect to the measure $\chi_q\mu_{N,q}/\int \chi_q \,\mathrm{d}\mu_{N,q}$ lies in $[\mathrm{e}^{-2\varepsilon_q'}, \mathrm{e}^{2\varepsilon_q'}]$. The constant cancels, whether or not it converges. Removing χ_q costs at most $2p_q$ in total variation, so

$$d_{\mathrm{TV}}(\mu_{N,q}, \hat\nu_{N,q}) \leqslant d_q := 2p_q + 2(\mathrm{e}^{2\varepsilon_q'} - 1) \to 0,$$

uniformly in N, the convention being the total mass of the difference.

Step 4: the reference law converges at fixed q. Let $x_N \sim \gamma_{Q_{q,N}}$ be coupled through common Fourier Gaussians. The multiplier $(q - q_0)((q_0 + c_k)(q + c_k))^{-1} \lesssim (1 + |k|)^{-4}$ gives, by the Kolmogorov criterion, $\iota_N x_N \to x$ in probability in L^∞. Let $y = y_q(x)$ and $w = y - x = -\varrho_q(y) - \frac{\lambda}{3}C_q y^3$. Then

$$\bar V_q(x) = U(y) + \frac{\mu_q}{2}\|y\|_2^2 + \frac{1}{2}\langle y, \Theta_q y\rangle + \tilde h_q(y) + \frac{1}{2}\langle D\bar\Gamma_q(y), x - y\rangle.$$

On $\operatorname{supp}\chi_q$, $\|y\|_\infty \leqslant 3\rho_q$. Since $|\hat\Theta_q(k)| \leqslant 2|k| v_1(\Theta_q)$ we have $\|C_q\Theta_q\|_{L^2\to H^1} \leqslant C\, v_1$, and the other terms in w are in H^2, so $\|w\|_{H^1} \leqslant C_q$ uniformly in N. So the terms involving Θ split as

$$\langle y, \Theta y\rangle = \langle x, \Theta x\rangle + 2\langle x, \Theta w\rangle + \langle w, \Theta w\rangle, \quad |\langle x, \Theta w\rangle| \leqslant C\|x\|_2 v_1 \|w\|_{H^1},$$

all bounded on the support except the second chaos $\langle x, \Theta_q x\rangle$. Its operator $T = Q_q^{1/2}\Theta_q Q_q^{1/2}$ has multiplier at most $C\min(|k|^{-3}, \lambda^2 q^{-1/2} q_0^{-1/2})$, so it is Hilbert–Schmidt, with $\|T\|_{\mathrm{op}} \leqslant \frac{1}{4}$ for q large. Therefore the centred form converges in L^2 by Proposition 22.1 and dominated convergence in the multipliers, and its exponential is uniformly integrable, by the determinant formula. The trace is a constant, which cancels on normalising. We need that $D\tilde h_q^N$ converges weakly along strongly convergent bounded fields at fixed q. For smooth bounded profiles this is in Proposition 29.1, the finite recursion being uniform on compact sets of scales; it extends to strongly convergent bounded fields by the Lipschitz constant $\mathsf{S}_2 \leqslant C g_q q$ on L^2. The same holds for the term $M_q y$ by the fixed k convergence of Proposition 22.1. With this, L^2 compactness of w_N from the H^1 bound passes the coordinate equation to the limit. Its solution is unique among bounded fields y with $y - x \in H^1$: if d is the difference of two solutions y, y', test $K_q d + \frac{\lambda}{3}(y^3 - (y')^3) + M_q d + D\tilde h_q(y) - D\tilde h_q(y') = 0$ with d, and use $\langle d, M_q d\rangle \geqslant -\varepsilon_q\langle d, K_q d\rangle$ and $\|D\tilde h_q(y) - D\tilde h_q(y')\|_2 \leqslant C g_q q \|d\|_2$, both of which pass from the lattice; so $d = 0$. Thus $\iota_N y_N \to y$ in L^2 and boundedly. Hence $U(y_N)$, $\|y_N\|_2^2$ and $\tilde h_q^N(y_N) = \int_0^1 \langle D\tilde h_q^N(t\, y_N), y_N\rangle \,\mathrm{d}t$ converge; so do the cross terms, pairing strong x_N with weak $\Theta^N w_N$. So $\chi_q \mathrm{e}^{-\bar V_{q,N} + \mathrm{const}_N}$ converges in probability and in $L^1(\lambda)$. The normalisers are bounded below: since $\bar V_q(x) \leqslant \bar\Gamma_q(x)$ by taking $y' = x$ in the infimum, the density is bounded below on the intersection of a small uniform ball with the event that the centred chaos is at most K, and this event has positive mass, uniformly in N, for K large. Thus $\iota_N \hat\nu_{N,q}$ converges weakly to a limit $\hat\nu_q$, along the whole sequence.

Step 5: from the scale to the field. For smooth h, as $\Phi_q = J_q\Phi_\infty + C_q B_{q-q_0}$ and $I - J_q = C_q K_{q_0}$,

$$\langle \iota_N(\Phi_\infty - \Phi_q), h\rangle = \langle \Phi_\infty, C_q K_{q_0} j_N h\rangle - \langle C_q B_{q-q_0}, j_N h\rangle.$$

By (17.6), the first has mean absolute value at most $C\|C_q K_{q_0} j_N h\|_* \leqslant C q^{-1}\|h\|_{C^2}$, as C_q has norm q^{-1} on each L^p. The second has variance at most $\|h\|_2^2/q$. So for $G = g(\langle\cdot, h_1\rangle, \dots, \langle\cdot, h_k\rangle)$ with g bounded and Lipschitz,

$$\operatorname{limsup}_{N,N'} |\mathbb{E}\, G(\iota_N \Phi_\infty^N) - \mathbb{E}\, G(\iota_{N'}\Phi_\infty^{N'})| \leqslant 2\|g\|_\infty d_q + C_G q^{-1/2}.$$

Let $q \to \infty$. Cylinder functions separate measures on H^{-s} and Theorem 17.5 gives tightness, so the whole sequence converges. Since B^N is a Brownian motion independent of Φ^N_∞, independence passes to the limit, with $\Phi_q = (q-q_0)C_q\Phi + C_qB_{q-q_0}$. □

What this proof does not give. This argument identifies only the joint law of Φ and the Polchinski process. It does not by itself give uniqueness for the system of Theorem 32.5(b) as an equation, nor that ψ_q is measurable for the filtration of the Polchinski process; these follow from the density in Lecture 7, Theorem 18.6(b) and Theorem 18.11. It does not cover H^{-s} with s near $\frac{1}{2}$ for quartic moments.

32.7 Symmetrised Girsanov comparison

This section gives a second proof of Steps 2–3 of Theorem 32.13, which replaces all of Section 32.5. It follows the idea of Gubinelli and Meyer for sine–Gordon [GM24], Proposition 4.9, where the limit law is written as an explicit density over an explicit SDE, and the backward remainder disappears from the identification. There the smoothed force Q_tF_t and the defect are bounded uniformly in the field [GM24], Proposition 3.13, there is a spatial cut-off, and the flow starts from zero. Here the defect is unbounded, and this older route uses the action only above q_*, without the frozen extension of Section 12.4, so we use the symmetrised entropy at a fixed scale, which needs only expectations of the defect, and let the scale tend to infinity at the end. In particular neither the penalty, nor the sub- and supersolutions, nor the exponential Riccati bound, nor Lemma 32.9 is needed.

Let $y = y_q(x)$ solve $K_qy + D\bar\Gamma_q(y) = K_qx$, and let $\bar V_q(x) = \bar\Gamma_q(y) + \frac{1}{2}\langle x-y, K_q(x-y)\rangle$ be the Moreau transform of the action. Let $e_q = \partial_q\bar\Gamma_q - \frac{1}{2}\mathrm{Tr}[H_q - H_q(0)]$ be its defect, and let V_q be the exact potential, $\mathrm{e}^{-V_q(x)} = \mathbb{E}\mathrm{e}^{-V_N(x+\int_q^\infty C_s\,\mathrm{d}W_s)}$, with force $F_q = -DV_q$. By the envelope theorem and $Dy = H_qK_q$,

$$\partial_q\bar V + \frac{1}{2}\mathrm{Tr}(C_q^2D^2\bar V) - \frac{1}{2}\|C_qD\bar V\|_2^2 = e_q\circ y + c(q), \qquad c(q) = \frac{1}{2}\mathrm{Tr}(C_q - H_q(0)). \tag{32.13}$$

So $\bar V$ solves the Polchinski equation up to its defect, and $\mathrm{d}\bar\Phi = -C_s^2D\bar V_s(\bar\Phi)\,\mathrm{d}s + C_s\,\mathrm{d}W$ is an explicit flow, whose coordinate $Y = y_s(\bar\Phi_s)$ solves $\mathrm{d}Y = b(Y)\,\mathrm{d}s + H(Y)\,\mathrm{d}W$, $b = -HDe$: indeed $\mathrm{d}(K_s\bar\Phi) = Y\,\mathrm{d}s + \mathrm{d}W$, and one applies Itô's formula to $K_sY + D\bar\Gamma_s(Y) = K_s\bar\Phi$. It contains no Z.

Lemma 32.14. (The defect on all fields) *Let π_s be the pointwise projection on $[-r_s, r_s]$. For all y and $s \geqslant q_*$,*

$$|e_s(y)| \leqslant C\lambda^5 s^{-3}\|\pi_sy\|_4^4 + C\lambda^4 s^{-2}\|\pi_sy\|_2^2 + C\lambda^3 s^{-5/2}\|t_s(y)\|_6^6. \tag{32.14}$$

Proof. On the ball $e_s = e_s^{\mathrm{o}}$, so $|e_s(\pi_sy)| \leqslant C\lambda^5 s^{-3}\|\pi_sy\|_4^4 + C\lambda^4 s^{-2}\|\pi_sy\|_2^2$ by the zero scalar jets, the centred quadratic jet and (11.14). With $z_\theta = \pi_sy + \theta(y - \pi_sy)$, $e_s(y) - e_s(\pi_sy) = \int_0^1 \langle De_s(z_\theta), y - \pi_sy\rangle\,\mathrm{d}\theta$, and $y - \pi_sy$ is supported on the set $\mathcal{A}$ where $|y| > r_s$, with norm at most $\|t_s\|_2$. Write $De = D\kappa De^{\mathrm{o}}\circ\kappa + \mathfrak{D}$. Then $|De^{\mathrm{o}}| \leqslant \mathsf{S}_4(e^{\mathrm{o}})(2r_s)^3/6 + \|\mathcal{E}_s^c\|_{\mathrm{row}}2r_s \leqslant C\lambda^{7/2}s^{-3/2}$ and $|\mathcal{A}|^{1/2} \leqslant r_s^{-1}\|t_s\|_2$, so the first term contributes at most $C\lambda^4 s^{-2}\|t_s\|_2^2$. By (21.11), which bounds $\mathfrak{D}$ before multiplication by H (the bound stated for $b^1 = -H\mathfrak{D}$ would not suffice, as H^{-1} is unbounded), $\|\mathfrak{D}_s(z)\|_2 \leqslant Cg_s(\|t_s(z)\|_2 + r_s^{-1}\|z^2\chi_s(z)\|_2)$; since $|z_\theta| \leqslant |y|$ and z_θ has size above r_s only on $\mathcal{A}$, the second term contributes at most $Cg_s(\|t_s\|_2^2 + r_s^{-1}\|t_s\|_4^2\|t_s\|_2)$. On the support of t_s we have $|t_s| \geqslant r_s$, so $\|t_s\|_2 \leqslant r_s^{-2}\|t_s\|_6^3$ and $\|t_s\|_4^2 \leqslant r_s^{-1}\|t_s\|_6^3$. All these are at most $Cg_sr_s^{-4}\|t_s\|_6^6 = C\lambda^3 s^{-5/2}\|t_s\|_6^6$. □

Fix $\delta \in \left(0, \frac{1}{12}\right)$, $\rho_q = q^{5/12+\delta}$, and $\chi_q(x) = \tau(\|x\|_\infty/\rho_q)$, with τ continuous, $\tau = 1$ on $\left[0, \frac{1}{2}\right]$, $\tau = 0$ on $[1, \infty)$. Let $\mu_{N,q} \propto \mathrm{e}^{-V_{q,N}}\gamma_{Q_q}$ be the law of Φ_q^N, and $\hat\nu_{N,q} \propto \chi_q\mathrm{e}^{-\bar V_{q,N}}\gamma_{Q_q}$ the explicit law.

Proposition 32.15. (Comparison by Girsanov) *For q large, so that $p_q \leqslant \frac{1}{2}$ and $r_q \geqslant 6\rho_q$, there are $d_q \to 0$, independent of N, with $\|\mu_{N,q} - \hat{\nu}_{N,q}\|_{\mathrm{TV}} \leqslant d_q$. Precisely, with $p_q = \sup_N \mathbb{P}(\|\Phi_q^N\|_\infty > \rho_q/2)$ and $\sigma_q = \lambda^5 q^{-2}\rho_q^4 + \lambda^4 q^{-1}\rho_q^2 + \lambda^4 q^{-1/2} + \lambda^2 q^{-1}$, one has $d_q \leqslant C(p_q + \sigma_q^{1/2})$, and $p_q \to 0$ by Step 1 of Theorem 32.13.*

Proof. The two path laws. On $[q,\infty)$ let $\mathbb{P}$ be the exact law, conditioned softly: initial law $\chi_q \mu_{N,q} / \int \chi_q \,\mathrm{d}\mu_{N,q}$, drift $C_s^2 F_s$. Let $\mathbb{R}$ be the explicit one: initial law $\hat{\nu}_{N,q}$, drift $-C_s^2 D\bar{V}_s$. Both are absolutely continuous with respect to the free flow $\mathrm{d}\xi = C_s\,\mathrm{d}W$ with initial law γ_{Q_q}. On $[q,T]$ the first has density proportional to $\chi_q(\xi_q)\mathrm{e}^{-V_T(\xi_T)}$, since e^{-V} is harmonic. By (32.13) and Itô's formula, the second has density proportional to

$$\chi_q(\xi_q)\mathrm{e}^{-\bar{V}_q(\xi_q)}\exp(-\bar{V}_T(\xi_T) + \bar{V}_q(\xi_q) + \int_q^T (e_s(y_s(\xi_s)) + c(s))\,\mathrm{d}s).$$

This is a true martingale density: the drift is $-C_s(x - y_s(x))$, Lipschitz in x at fixed N since $Dy = HK$. The factors at ξ_q cancel: this is why no estimate at the initial scale is needed. So $\log \mathrm{d}\mathbb{P} / \mathrm{d}\mathbb{R} = -\Delta_T(\xi_T) - \int_q^T e_s(Y_s)\,\mathrm{d}s + \mathrm{const}$, with $\Delta_T = V_T - \bar{V}_T$. The constant includes $\int c$ and the normalisations, which diverge with N; it does not matter, since **in the symmetrised relative entropy it cancels**:

$$\mathrm{Ent}(\mathbb{P}|\mathbb{R}) + \mathrm{Ent}(\mathbb{R}|\mathbb{P}) = \mathbb{E}^{\mathbb{R}}\Big[\Delta_T + \int_q^T e_s(Y_s)\,\mathrm{d}s\Big] - \mathbb{E}^{\mathbb{P}}\Big[\Delta_T + \int_q^T e_s(Y_s)\,\mathrm{d}s\Big]. \tag{32.15}$$

At fixed N: $|\mu_T - \mu_N| = O_N(T^{-1})$, $\|\Theta_T\|_{\mathrm{row}} = O_N(T^{-1/2})$, $\mathsf{S}_4(h_T) = O_N(T^{-2})$ and $|y - x| \leqslant C_N T^{-1}(\|x\|_\infty + \|x\|_\infty^3)$ give $|\bar{V}_T - V_N| = O_N(T^{-1/4})$ on $\|x\|_\infty \leqslant T^{1/8}$, and Gaussian integration, the remaining variance per site being $O_N(T^{-1})$, gives $|V_T - V_N| = O_N(T^{-1/8})$ there. For all x, $|\Delta_T(x)| \leqslant C_N(1 + \|x\|_\infty^4)$ uniformly in T: $\inf V_N \leqslant V_T(x) \leqslant C_N(1 + \|x\|_\infty^4)$ by Jensen, and $-C_N \leqslant \bar{V}_T(x) \leqslant \bar{\Gamma}_T(x) \leqslant C_N(1 + \|x\|_\infty^4)$, as U dominates $\tilde{h}_T$. The moments of ξ_T are uniform in T: under $\mathbb{P}$ since $\Phi_T = J_T\Phi_\infty + C_T B_{T-q_0}$; under $\mathbb{R}$ by the Ito computation below with $\int^\infty a_2\,\mathrm{d}s < \infty$ at fixed N, and $x = Y + \varrho_T(Y) + \frac{\lambda}{3}C_T Y^3$. So $\mathbb{E}^{\mathbb{P}}\Delta_T, \mathbb{E}^{\mathbb{R}}\Delta_T \to 0$. With the bounds below, $\log \mathrm{d}\mathbb{P} / \mathrm{d}\mathbb{R}$ is integrable under both laws at each T, so both entropies are finite and (32.15) is justified. By Pinsker's inequality the squared total variation at the initial scale is at most the sum of the two expectations of $\int_q^\infty |e_s(Y_s)|\,\mathrm{d}s$, and removing χ_q costs at most $2p_q$. It remains to bound these expectations, by (32.14).

Under the exact law. $\int_q^\infty \lambda^3 s^{-5/2}\|t_s\|_6^6\,\mathrm{d}s \leqslant \lambda^2 q^{-1} J_\infty$. Further $\|\pi_s Y_s\|_4^4 \leqslant r_s^2\|Y_s\|_2^2$, and $\mathbb{E}\|Y_s\|_2^2 \leqslant C(1 + s^{1/2})$ by Itô's formula for $\|Y_s\|_2^2$ along the forward equation of Theorem 10.6, from q_*, with the moments of Y_{q_*} and $\mathbb{E}\|Z_u\|_2^2 \leqslant C u^{-5/2}$ of Theorem 16.2, $\|H\| \leqslant C/u$, the linear bound on b and $\mathrm{Tr}\,H^2 \leqslant C u^{-1/2}$. So the first two terms of (32.14), both at most $C\lambda^4 s^{-2}\|Y_s\|_2^2$, integrate in expectation to at most $C\lambda^4 q^{-1/2}$. The conditioning costs at most a factor 2.

Under the explicit law. Here $Y_q = y$ with $\|y\|_\infty \leqslant 3\rho_q$ by the maximum principle: with $u = y + \varrho_q(y)$, $K_q(u - x) + \frac{\lambda}{3}y^3 = 0$, and at a positive maximum of $u - x$ we have $y < 0$, so $u - x \leqslant \|\varrho_q(y)\|_\infty + \|x\|_\infty$; similarly at a negative minimum. Thus $\|u\|_\infty \leqslant 2\|x\|_\infty + \varepsilon_q\|y\|_\infty$, with ε_q the contraction constant of (12.5), and $\|y\|_\infty \leqslant 3\|x\|_\infty$ for q large. Let $\xi_s = Y_s - y$, which is a different ξ from the free flow. By the global linear bound of Proposition 12.2, $\|b(Y)\|_p \leqslant C\lambda s^{-3/2}\|Y\|_p$ for every p (it follows from the exact gradient of e, without the ledger), and $\mathrm{diag}H^2 \leqslant C s^{-1/2}$, Itô's formula gives, for $p = 2$ and $p = 12$, $\mathbb{E}\|\xi_s\|_p^p \leqslant C(s^{p/4} + (g_q\rho_q)^p)$. So, as $\|\pi_s Y\|_4^4 \leqslant 8\|y\|_4^4 + 32 r_s^2\|\xi\|_2^2$, and $\|\pi_s Y\|_2^2 \leqslant 2\|y\|_2^2 + 2\|\xi\|_2^2$, the first two terms of (32.14) integrate to at most $C\lambda^5 q^{-2}\rho_q^4 + C\lambda^4 q^{-1}\rho_q^2 + C\lambda^4 q^{-1/2}$. For large q, $r_s \geqslant r_q \geqslant 6\rho_q$, so where $|Y| > r_s$ we have $|\xi| > r_s/2$ and $|Y| \leqslant 2|\xi|$. Thus $\|t_s\|_6^6 \leqslant C r_s^{-6}\|\xi_s\|_{12}^{12}$, and the third term integrates to at most $C\lambda^6 q^{-3/2}$. □

The bound on $\mathbb{E}^{\mathbb{P}}$ is where UV stability enters. So the backward equation is used only through Theorem 16.2, for J_∞ and $\mathbb{E}\|Z_s\|_2^2$; under $\mathbb{R}$ the comparison has no Z, no conditional expectation and no exponential moment. What is lost is the pointwise bound on $V_q - \bar{V}_q$ on typical fields which a Feynman–Kac comparison gives: here one gets total variation at a fixed scale, and that is all the theorem needs.

Remark 32.16. (**The backward error alone**) The drifts differ by $C_s^2(F_s - \bar{f}_s)$ with $\bar{f} = -D\bar{V}$, and $C_s(F_s - \bar{f}_s) = Z_s$. So, from the same initial law at scale q, Girsanov's theorem gives directly that the relative entropy of the exact path law with respect to the explicit flow is $\frac{1}{2}\int_q^\infty \mathbb{E}\|Z_s\|_2^2 \,\mathrm{d}s \leqslant C q^{-3/2}$, hence total variation $\leqslant C q^{-3/4}$. This replaces the backward error by a change of measure, but it does not identify the limit: it reduces the terminal law to the law at scale q, which is the unknown, since the flow does not forget the low modes. In [GM24] this does not arise, as the flow starts from a deterministic point; the same holds in Lecture 7, where the frozen action starts the flow at q_0 from zero (Theorem 18.5). The symmetrised entropy above compares the laws at scale q as well, through the cancellation of the factors at ξ_q.

With Proposition 32.15, Steps 1, 4 and 5 of Theorem 32.13 conclude the proof as before: it uses Theorem 16.2, Proposition 14.4, Lemma 14.5, (11.14), Lemma 12.1, and Proposition 12.2 with its ledger. On this route the condition $q_* \geqslant q_A$ of Theorem 32.13 is not needed.

References

[GM24] M. Gubinelli, S.-J. Meyer. The FBSDE approach to sine–Gordon up to 6π. Preprint, arXiv:2401.13648, 2024.