proofs, programs & stories

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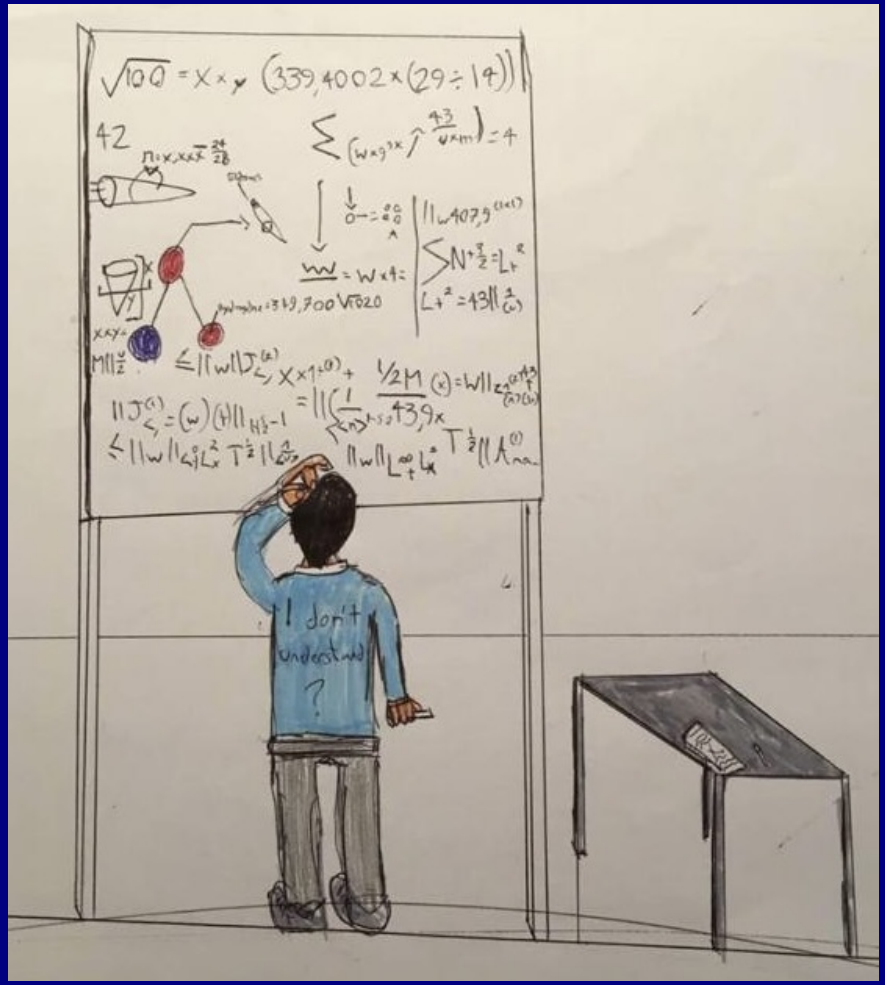
I don't understand
?

What is a businessman? Someone who does business? Not quite. It is someone who sees opportunities for doing business when others might miss them. A mathematician is someone who sees opportunities for doing mathematics.

— Ian Stewart, “Letters to a Young Mathematician”

proofs

I studied theoretical physics. In the courses we were never asked to prove anything, or better to state precisely what we were "proving" and what were the assumptions.

A lot of (mathematical) thinking in physics proceed in this way, by implicitly introducing simplifying assumptions on the way. The risk is always that they are incoherent with each other, but that we do not "see it".

I felt uneasy with this lack of control.

This forces a young shy explorer to stay on the beaten paths, in order not to be lost in the woods without even noticing. One felt reassured by mimicking others.

The problems I was interested, quantum and statistical field theories, non-equilibrium thermodynamics, in were too complex to allow precise statements.

Signs (path integrals, correlation functions) were written on the page and by discussions we agreed with friends, colleagues, supervisors, that other signs would follow from those.

All these signs are associated with strong intuitions about the underlying entities, glympses of an hidden world.

If any doubt, then we tried to find a more elementary formulation of the sign assembly, discretize, in order to assign a numerical value to some approximation of what we really were interested in.

As I approached mathematical research, working with senior mathematicians on some research problem in probability linked to my statistical physics background, I (re-)learned to proceed slowly, learned to weigh every step.

(I was not sure about “weigh” so I asked ChatGPT. I started with “weage” and it suggested me that was a mix of “weigh” and “gauge”, which sounds about right)

Recently, with a colleague we were discussing about a step in a paper we were writing, and it was clear that we were looking at the same math in different ways, so that while the math on the paper was one and we were ok with it, we were disagreeing on what its “real meaning” was, what was the bigger picture around that fragment of math.

Discussing with one of my children, I told him on arXiv there were many proofs of Riemann's hypothesis. He was intrigued and went to look for them, found one and started skimming over it. He was surprised to see that it looked like “good maths”: a lot of equations, pages of discussions, pretty indiscernible from a gold-standard proof, like those in other books.

But I still believed there were a mistake in that paper, without reading. I acknowledge with him that, however, I could not exclude it was a correct proof.

It happened to me to write papers whose only purpose was to build a concrete and rigorous “small-scale”, toy model of a vague idea I had in mind. My interest in the paper was not its theorems in themselves, but as a test of the ideas I was pondering at that time.

The paper has no mention of these ideas, as they were too vague to be communicated with some clarity and precision.

Still I think they were mathematical ideas, and the results in the paper, a mathematical artifact, was not what was really at stake.

There are many uninteresting and sterile correct papers, and there are some very interesting and fruitful wrong (or more often, non-rigorous) papers.

Dynamic Scaling of Growing Interfaces

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(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 81.15.Jj

Many challenging problems are associated with growth patterns in clusters[1] and solidification fronts.[2] Several models have been proposed recently to describe the growth of smoke and colloid aggregates, flame fronts, tumors, etc.[1] It is generally recognized that the growth process occurs mainly at an "active" zone on the surface of the cluster, with interesting scaling properties.[3] However, a systematic *analytic* treatment of the static and dynamic fluctuations of the growing interface has been lacking so far.

In this paper we propose a model for the time evolution of the profile of a growing interface, and examine

The interface profile, suitably coarse-grained, is described by a height $h(\mathbf{x},t)$. As usual, it is convenient to ignore overhangs so that h is a single-valued function of $\mathbf{x}$. The simplest nonlinear Langevin equation for a local growth of the profile is given by[12]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \tag{1}$$

The first term on the right-hand side describes relaxation of the interface by a surface tension ν. The second term is the lowest-order nonlinear term that can appear in the interface growth equation, and is

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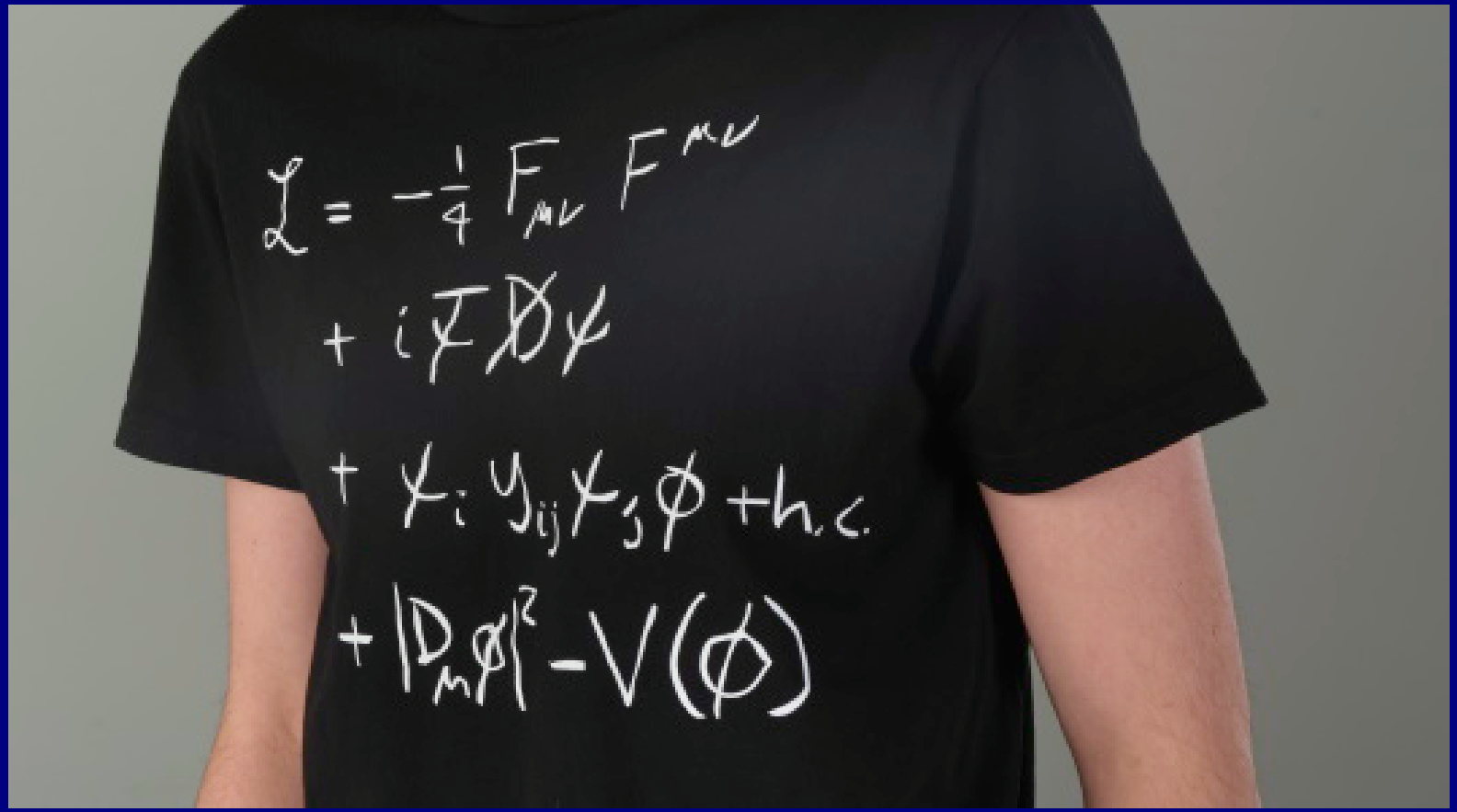
Solving the KPZ equation

By Martin Hairer

Abstract

We introduce a new concept of solution to the KPZ equation which is shown to extend the classical Cole-Hopf solution. This notion provides a factorisation of the Cole-Hopf solution map into a "universal" measurable map from the probability space into an explicitly described auxiliary metric space, composed with a new solution map that has very good continuity properties. The advantage of such a formulation is that it essentially provides a pathwise notion of a solution, together with a very detailed approximation theory. In particular, our construction completely bypasses the Cole-Hopf transform, thus laying the groundwork for proving that the KPZ equation describes the fluctuations of systems in the KPZ universality class.

As a corollary of our construction, we obtain very detailed new regularity results about the solution, as well as its derivative with respect to the initial condition. Other byproducts of the proof include an explicit approximation to the stationary solution of the KPZ equation, a well-posedness result for the Fokker-Planck equation associated to a particle diffusing in a rough space-time dependent potential, and a new periodic homogenisation result for the heat equation with a space-time periodic potential. One ingredient in our construction is an example of a non-Gaussian rough path such that the area process of its natural approximations needs to be renormalised by a diverging term for the approximations to converge.


$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
$+ i \not{\psi} \not{D} \psi$
$+ \psi_i y_{ij} \psi_j \phi + h.c.$
$+ |D_\mu \phi|^2 - V(\phi)$

The renormalization group and critical phenomena*

Kenneth G. Wilson

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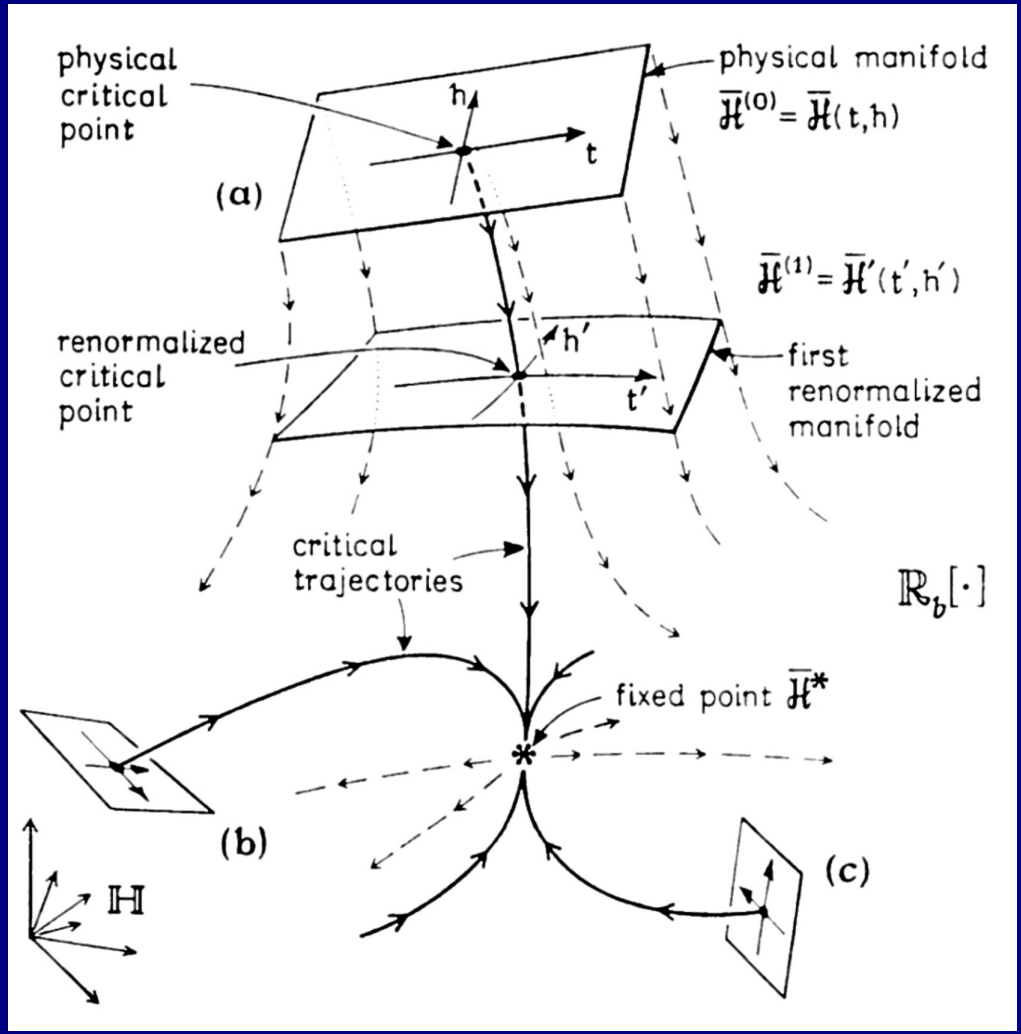
Nobel prize in 1982

The possible types of cooperative behavior, in the renormalization group picture, are determined by the possible fixed points $\mathcal{H}^*$ of T. Suppose for example that there are three fixed points $\mathcal{H}_A^*$, $\mathcal{H}_B^*$, and $\mathcal{H}_C^*$. Then one would have three possible forms of cooperative behavior. If a particular system has an initial interaction $\mathcal{H}_0$, one has to construct the sequence $\mathcal{H}_1$, $\mathcal{H}_2$, etc. in order to find out which of $\mathcal{H}_A^*$, $\mathcal{H}_B^*$, or $\mathcal{H}_C^*$ gives the limit of the sequence. If $\mathcal{H}_A^*$ is the limit of the sequence, then the cooperative behavior resulting from $\mathcal{H}_0$ will be the cooperative behavior determined by $\mathcal{H}_A^*$. In this example the set of all possible initial interactions $\mathcal{H}_0$ would divide into three subsets (called "domains"), one for each fixed point. Universality would now hold separately for each domain. See section 12 for further discussion.

This is how one derives a form of universality in the renormalization group picture. It is not so bold as previous formulations [9]. Experience with soluble examples of the renormalization group transformation for critical phenomena shows that it generally has a number of fixed points, so one has to define domains of initial Hamiltonians associated with each fixed point, and only within a given domain is the critical behavior independent of the initial interaction.

There is no *a priori* requirement that the sequence $\mathcal{H}_l$ approach a fixed point for $l \to \infty$. In

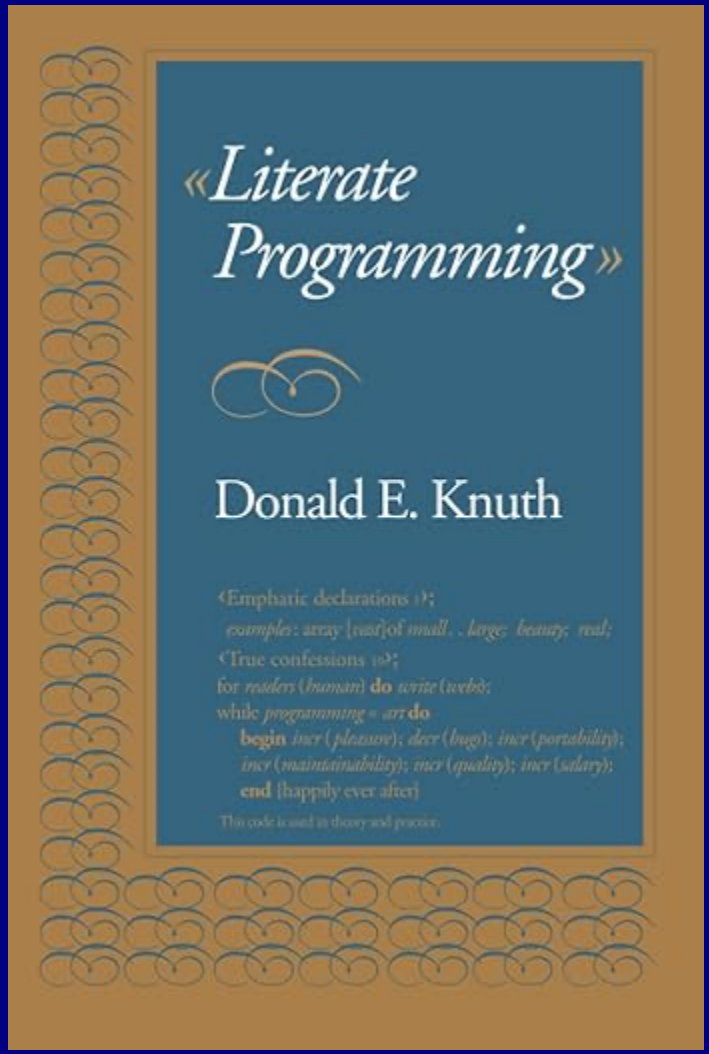

physical
critical
point
physical manifold
$\overline{\mathcal{H}}^{(0)} = \overline{\mathcal{H}}(t,h)$
h
t
(a)
$\overline{\mathcal{H}}^{(1)} = \overline{\mathcal{H}}'(t',h')$
renormalized
critical
point
h'
t'
first
renormalized
manifold
critical
trajectories
$\mathbb{R}_b[\cdot]$
fixed point $\overline{\mathcal{H}}^*$
(b)
(c)
$\mathbb{H}$

Craindre l'erreur et craindre la vérité est une seule et même chose. Celui qui craint de se tromper est impuissant à découvrir. C'est quand nous craignons de nous tromper que l'erreur qui est en nous se fait immuable comme un roc. Car dans notre peur, nous nous accrochons à ce que nous avons décrété « vrai » un jour, ou à ce qui depuis toujours nous a été présenté comme tel. Quand nous sommes mûs, non par la peur de voir s'évanouir une illusoire sécurité, mais par une soif de connaître, alors l'erreur, comme la souffrance ou la tristesse, nous traverse sans se figer jamais, et la trace de son passage est une connaissance renouvelée.

Alexandre Grothendieck, extrait de « Récoltes et Semailles », pp 128-129

Fearing error and fearing truth are one and the same thing. He who fears being wrong is powerless to discover. It is when we fear being wrong that the error within us becomes as immutable as a rock. For in our fear, we cling to what we once decreed to be "true," or to what has always been presented to us as such. When we are moved, not by the fear of seeing an illusory security vanish, but by a thirst for knowledge, then error, like suffering or sadness, passes through us without ever settling, and the trace of its passage is a renewed knowledge.

programs


«Literate Programming»
Donald E. Knuth
‹Emphatic declarations 1›;
examples: array [vast]of small . . large; beauty; real;
‹True confessions 10›;
for readers{human} do write{web};
while programming = art do
begin incr(pleasure); decr(bugs); incr(portability);
incr(maintainability); incr(quality); incr(salary);
end {happily ever after}
This code is used in theory and practice.

A traditional computer program consists of a text file containing program code. Scattered in amongst the program code are comments which describe the various parts of the code.

In literate programming the emphasis is reversed. Instead of writing code containing documentation, the literate programmer writes documentation containing code. No longer does the English commentary injected into a program have to be hidden in comment delimiters at the top of the file, or under procedure headings, or at the end of lines. Instead, it is wrenched into the daylight and made the main focus. [...]

The effect of this simple shift of emphasis can be so profound as to change one's whole approach to programming. Under the literate programming paradigm, the central activity of programming becomes that of conveying meaning to other intelligent beings rather than merely convincing the computer to behave in a particular way. It is the difference between performing and exposing a magic trick.

Another surprising thing that I learned while using WEB was that traditional programming languages had been causing me to write inferior programs, although I hadn't realized what I was doing. My original idea was that WEB would be merely a tool for documentation, but I actually found that my WEB programs were better than the programs I had been writing in other languages.

—Donald E. Knuth, *Literate Programming*

Software is a mess.

In the last 40 years I and my colleagues have been writing buggy, unspecified, and difficult to maintain software.
By doing so we have created an entire industry. We have collectively created billions of lines of unmaintainable unspecified code that will require millions of programmers to work for thousands of years to correct the mess we've made.
Where did things go wrong? Can we make things work again? Should we admit we were wrong and throw all our legacy software away and start again?
How can we make correct software, that is easy to understand and maintain?

Joe Armstrong - The mess we're in from NDC Conferences (2014)

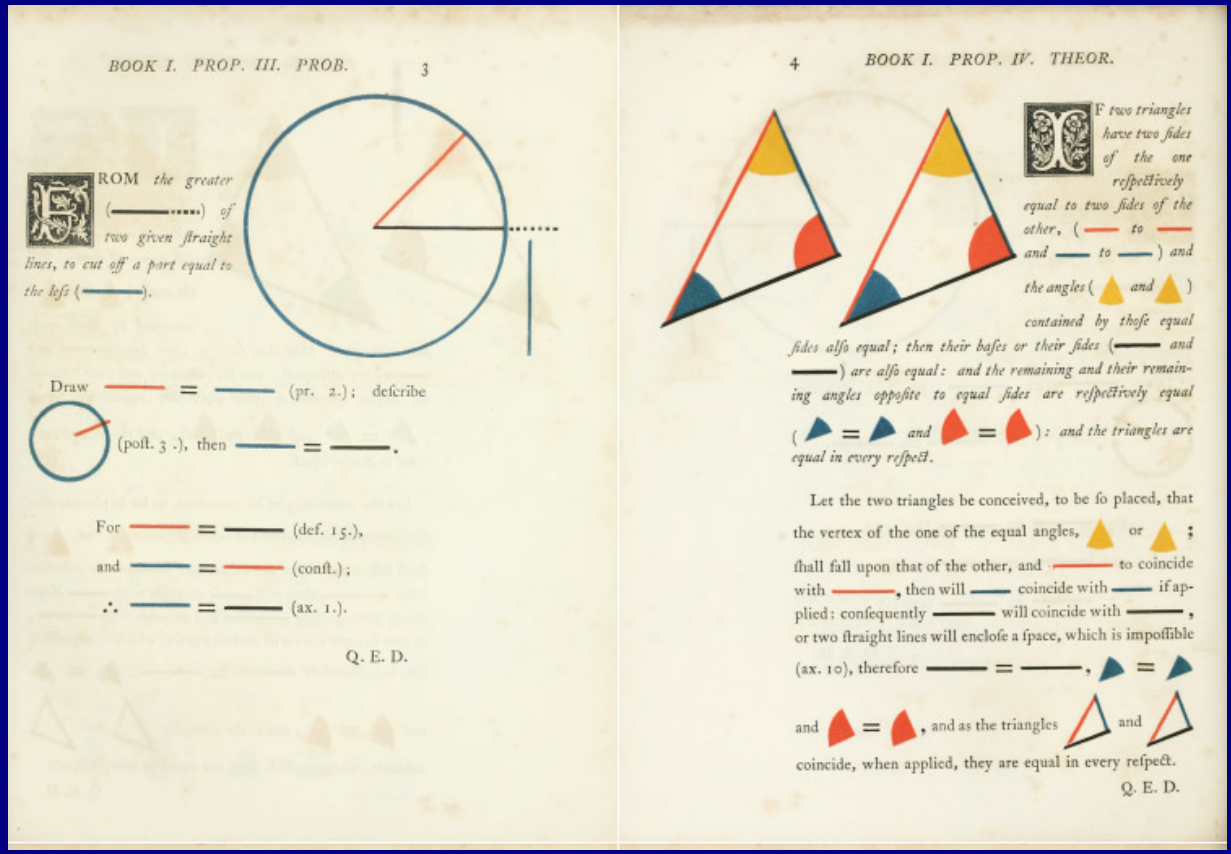
Where is a meaning in a program? We can change every identifier to a random word, that would make a program unintelligible.

It is still the same program? What about a compiled version? It is functionally the same thing, but we certainly lost something (e.g. we commercially distribute compiled programs but not their sources to preserve our “ownership”).

What about a proof (which can be encoded in a program)?

```
/--   $$ d[X;Y] \geq 0.$$  -/
lemma rdist_nonneg : 0 ≤ d[ X ; μ # Y ; μ' ] := by
  suffices : 0 ≤ 2 * d[ X ; μ # Y ; μ' ]
  . linarith
  have h : |H[X ; μ] - H[Y ; μ']| ≤ 2 * d[X ; μ # Y ; μ' ] := by
    exact diff_ent_le_rdist
  have h' : 0 ≤ |H[X ; μ] - H[Y ; μ']| := by
    exact abs_nonneg (H[X; μ] - H[Y; μ'])
  exact ge_trans h h'
```

Byrne's Euclid (1847). https://www.c82.net/euclid/



BOOK I. PROP. III. PROB. 3

FROM *the greater* (——·····) *of two given ſtraight lines, to cut off a part equal to the leſs* (——).

Draw —— = —— (pr. 2.); deſcribe (poſt. 3.), then —— = ——.

For —— = —— (def. 15.),

and —— = —— (conſt.);

∴ —— = —— (ax. 1.).

Q. E. D.

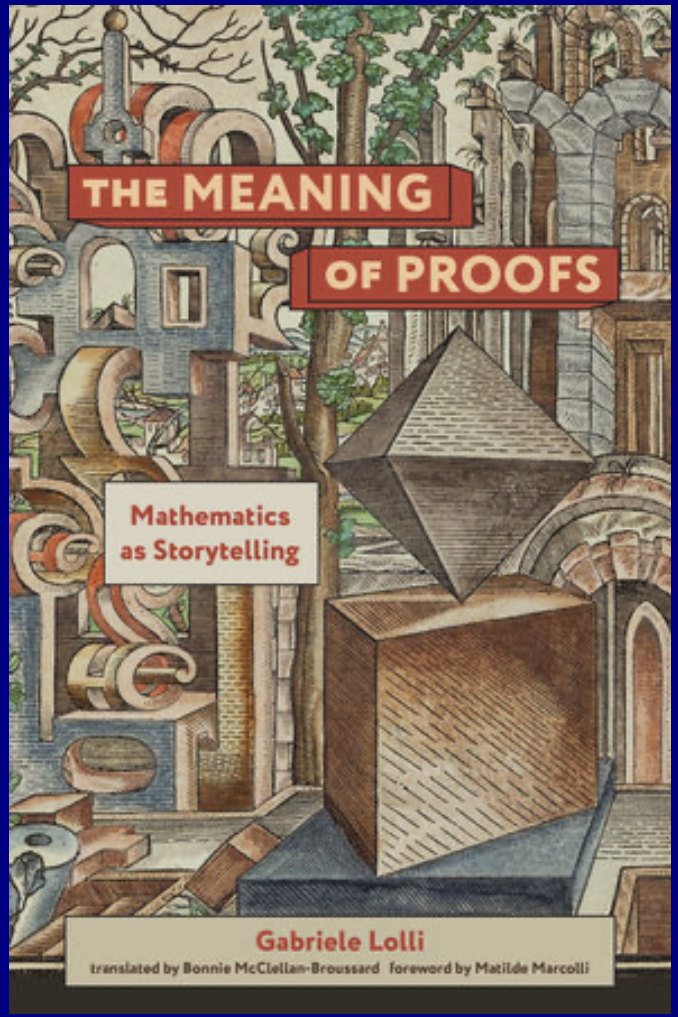
4 BOOK I. PROP. IV. THEOR.

IF *two triangles have two ſides of the one reſpectively equal to two ſides of the other,* (—— *to* —— *and* —— *to* ——) *and the angles* (▲ *and* ▲) *contained by thoſe equal ſides alſo equal; then their baſes or their ſides* (—— *and* ——) *are alſo equal: and the remaining and their remaining angles oppoſite to equal ſides are reſpectively equal* (▲ = ▲ *and* ▲ = ▲): *and the triangles are equal in every reſpect.*

Let the two triangles be conceived, to be ſo placed, that the vertex of the one of the equal angles, ▲ or ▲; ſhall fall upon that of the other, and —— to coincide with ——, then will —— coincide with —— if applied: conſequently —— will coincide with ——, or two ſtraight lines will encloſe a ſpace, which is impoſſible (ax. 10), therefore —— = ——, ▲ = ▲ and ▲ = ▲, and as the triangles ▲ and ▲ coincide, when applied, they are equal in every reſpect.

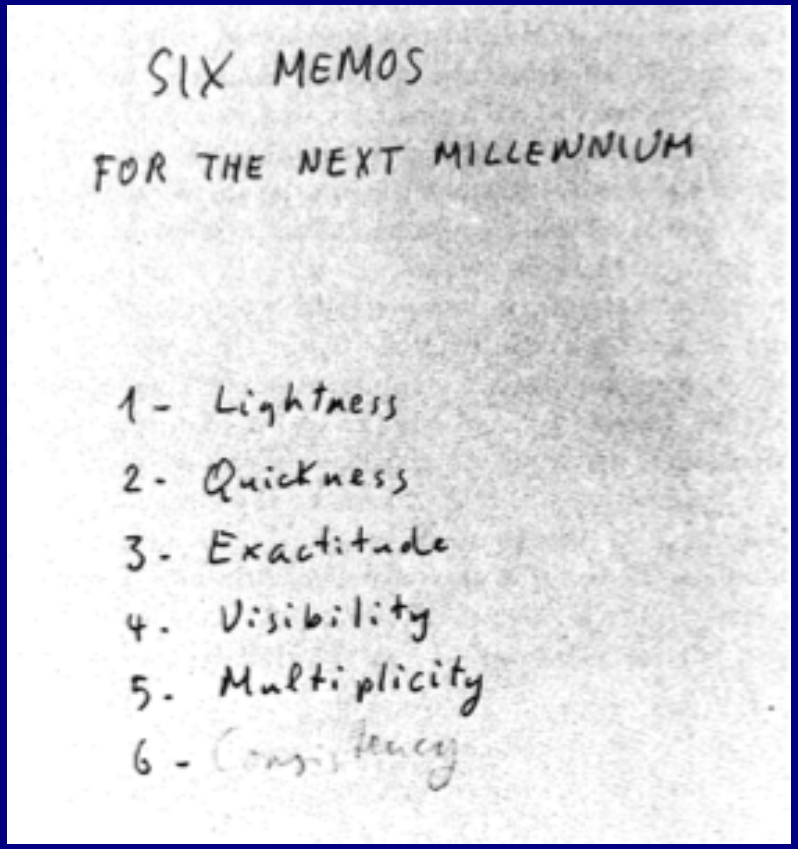
Q. E. D.

stories



Mathematics, imprisoned within its symbols and images, Lolli writes, says nothing if its meaning is not narrated in a story.

The minute mathematicians open their mouths to explain something—the meaning of x, how to find y—they are framing a narrative.



Italo Calvino's Six memos

Exactitude

"To my mind exactitude means three things above all:

(1) a well-defined and well-calculated plan for the work in question;
(2) an evocation of clear, incisive, memorable visual images;
(3) a language as precise as possible both in choice of words and in expression of the subtleties of thought and imagination"

What I admired about Fernando was his capacity to persuade people to be honest with themselves, for when this happens they gain the advantage of surprise. An incomparable tactical advantage in any insurrection.

[J. Berger, From A. to X.]

thinking about the time a prof told us that in real research mathematics it's fine to be slow, speed itself is not essential, as long as you can find it within yourself to make consistent unyielding inexorable forward progress, like the time some guy stole an M60A3 tank and terrorized a suburban neighborhood with it, said guy wasn't going that fast but plowed through cars and telephone poles and shit no problem. i'm not kidding that's what he said, that's the metaphor he used, he told us that the act of mathematics is like the 1995 san diego tank rampage

(From a closed blog at Tumblr)

Questions & Answers

A good student gives me a good answer.

But I prefer a student that gives me a good question.

We discussed only about answers, tests, and proofs.

Less about questions, and definitions, and concept creation.

(Weyl was driven to imagine gauge theory by beauty, not truth)

Ron Silliman, “Sunset Debris”

firmly, repeatedly, until finally you have swallowed what I came to say? Is this where the one-track mind jumps the track? Where is he at? How do you choose the questions? What is the function of this particular sequence? Why did the fire truck cross the road? Doesn’t this bore you? Isn’t it tedious? Isn’t it silly? Isn’t it a waste of time? Is what you fear the idea that who I am talking to is not you, but an image I carry of who you are? Don’t I have a problem? Isn’t the question of existence of time answered by the fact of irreversible processes? How do we get the perfume back into the bottle? How do we unburn down the house? Why are my eyes not blue? Don’t you think it’s unhealthy to put so much of your identity into the hands of others? What do you want for Xmas? How are we to define the poem? What is the name of this leaf? Isn’t it the case that we can apply these terms

J.Y Girard – The ghost of transparence

Logic is a mediation between reason and irrational

1. What is an answer
2. What is a question
3. The doubt

	Raw	Format
Explicit	Constat	Factory
Implicit	Performance	Use

“the poem begins not where the knife enters but where the blade twists.”

— Hanif Abdurraqib, `The Prestige'