A stochastic analysis perspective on Euclidean fields

(St. Flour's Summer School 2025)

by Massimiliano Gubinelli

Mathematical Institute
University of Oxford

Table of contents

1 Introduction . . . 1

2 Gaussian Free Field . . . 5

2.1 Besov spaces . . . 6
2.2 Regularity of the GFF . . . 9
2.3 Regularity of the Wick powers . . . 10
2.4 Nelson's approach to φ^4_2 . . . 12

3 The variational approach . . . 13

3.1 An example . . . 15
3.2 Euclidean fields as minimisers . . . 16

4 The quartic interaction . . . 25

4.1 Large deviations . . . 33

5 Exponential interaction . . . 37

6 A scale-by-scale interpolation . . . 51

6.1 Two dimensions . . . 54

7 The quartic interaction in three dimensions . . . 57

7.1 Commutators . . . 64
7.2 Bounds . . . 67
7.3 Removal of the UV cutoff . . . 71
7.4 Large deviations . . . 83
7.5 Stochastic estimates . . . 89

Bibliography . . . 96

1 Introduction

The theme of these lectures is the use of stochastic analysis to study a class of random fields defined on a set $\Lambda = \mathbb{T}^d$ (a d-dimensional torus) or $\mathbb{R}^d$ with $d = 2, 3$. Formally, the fields we are interested in, and which we collectively call Euclidean quantum fields (EQF), are expressed via an integration over all configurations $\varphi : \Lambda \to \mathbb{R}$ with weight:

$$\exp\Big(-\int_\Lambda \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}|\varphi(x)|^2 + v(\varphi(x))\mathrm{d}x \Big)\mathcal{D}\varphi \tag{1}$$

where $\mathcal{D}\varphi$ is some kind of infinite-dimensional flat measure (like Lebesgue's, which however does not exist).

This formula, which seems quite arbitrary, has its origin in the fundamental property one would expect these fields to have, i.e. the domain Markov property, which is a natural multidimensional generalisation of the usual Markov property for stochastic processes. Intuitively a Euclidean quantum field theory (EQFT) is a "Markov field" in some d-dimensional Riemannian manifold. This point of view has been introduced by Nelson [47, 48] and exploited by G. Segal to give an axiomatization of (E)QFT as functors (Pickrell [52] provides a formulation accessible to the probabilist).

The Markovian picture is difficult to make precise in relevant examples, and it has been long understood that an appropriate replacement for Markovianity, which still gives enough structure to the problem, is to assume that EQFTs are *reflection positive* and *Euclidean invariant* random fields. For the scope of these lectures I decided to skip the detailed discussion of these properties, suffice it to say that relevant examples we know of random fields satisfying these two properties have the form (1).

A good starting point to study these fields is to start with the case when $v=0$. Then the functional integral becomes quadratic and it can be interpreted as a Gaussian measure, which we denote μ, with covariance given by $(1-\Delta)^{-1}$ where Δ is the Laplacian on Λ. This is the (massive) Gaussian Free Field, an object which has been well studied in the mathematical literature. The Gaussian free field in dimension $d\geqslant 2$ has only distributional sample paths, belonging locally to the Besov space $B^{1-d/2-\kappa}_{\infty,\infty}(\Lambda)$ for any $\kappa>0$. Moreover given its explicit nature it is quite easy to show that it has indeed some form of the Markov property. See e.g. the very good recent book of Berestycki and Powell [10].

Following the heuristic formula, other EQFTs can be realised as Gibbsian perturbations of the GFF, i.e. exponential tilts of the general form

$$\nu(\mathrm{d}\varphi)=\frac{e^{-V}}{Z}\mu(\mathrm{d}\varphi),\qquad \varphi\in\mathcal{S}'(\Lambda) \tag{2}$$

where the function V is a local functional of the form

$$V(\varphi)=\int_\Lambda v(\varphi(x))\mathrm{d}x,$$

and $v\colon\mathbb{R}\to\mathbb{R}$ is called the "potential" of the model.

There are three fundamental problems that a mathematician needs to address to construct a measure ν which can "rightfully" express the "intentions" of the heuristic formula (1).

Renormalisation problem. Due to the distributional character of the sample paths of the measure μ, the definition of V does not really make sense and an approximation procedure is needed, which in turn will involve a "renormalisation", i.e. the need to "tune" carefully the function v as we remove the regularisation. Sometimes this procedure creates a measure which is singular wrt. the GFF, so the formula (2) loses its meaning.

Large field problem. The renormalisation procedure involves several expansions, or other manipulation of the approximate Gibbs measure, which in turn create quantities which are difficult to control probabilistically. Intuitively, the large fluctuations of the field φ should be suppressed both by the Gaussian measure and the "stability" of the interaction potential, but, in concrete proofs, it is usually very difficult to keep track of the stability of the model and one risks losing tightness of the approximate measures.

Infinite volume limit. The natural setting for EQFTs is $\mathbb{R}^d$, the full Euclidean space with its Euclidean group of transformations (rotations + translations). The massive GFF is well defined on $\mathbb{R}^d$ (even in $d=2$ where the GFF without a mass term is not), but the interaction potential has to be integrated over the full space and this gives rise to a singular measure. Again the formula (2) reveals itself as a poor proxy for the object we try to construct.

Regularisation is usually achieved by introducing a smoothing operator $R_T\colon \mathcal{S}'(\Lambda) \to C^\infty(\Lambda)$, which is a symmetric operator which removes fluctuations at spatial scales $\lesssim 1/T$ and such that $R_T \to \mathrm{Id}$ as $T \to \infty$. Usually it can be realised via convolution with a sequence of functions approximating the δ function as $T \to \infty$, or equivalently, by applying a low-pass filter in Fourier space up to frequencies of the order of T. The scale T is called usually the "ultraviolet cutoff" (UV cutoff).

Using the smoothing operator, we can construct a *regularised* measure

$$\nu^T(\mathrm{d}\varphi) = Z_T^{-1} \exp\Big(-\lambda \int_\Lambda v_T(R_T\varphi(x))\mathrm{d}x \Big) \mu^{\mathrm{GFF}}(\mathrm{d}\varphi)$$

for a T-dependent modification v_T of the interaction potential. This is needed (a posteriori) since usually we must "tune" certain components of the potential in order to have a non-trivial limit as we remove the UV cutoff.

There are various models for which we know this procedure can be achieved when $d \geqslant 2$:

1. The $P(\varphi)_2$ models for $d=2$ where v_T is a polynomial bounded below:
$$v_T(\varphi) = \varphi^n + r_{n-1,T}\varphi^{n-1} + \cdots + r_{1,T}\varphi + r_{0,T}$$
where we must consider renormalisation constants $r_{k,T}$ which diverge as $T \to \infty$.
2. The φ^4_3 model in $d=3$, where v_T is a quartic polynomial bounded below. Usually taken to be symmetric:
$$v_T(\varphi) = \varphi^4 - r_T\varphi^2 + c_T,$$
where both r_T, c_T have to be taken to diverge as $T \to \infty$.
3. The sine-Gordon model $(\cos(\beta\varphi))_2$ in $d=2$, with
$$v_T(\varphi) = c_T \cos(\beta\varphi)$$
where $c_T \geqslant 0$ has to be taken to diverge as $T \to \infty$.
4. The exponential model $(\exp(\beta\varphi))_2$ in $d=2$, with
$$v_T(\varphi) = c_T \exp(\beta\varphi)$$
where instead $c_T \geqslant 0$ has to be taken to vanish as $T \to \infty$. Combinations of exponentials with different β are also treatable.

The aim of these notes is to illustrate how stochastic analysis can be used to address the problem of existence of the limit $T \to \infty$ and of the infinite volume limit $\Lambda \nearrow \mathbb{R}^d$ for Euclidean fields.

Stochastic analysis of Euclidean fields is a very recent development and there are various ways to interpret it, some of them already being object of courses and lecture notes:

Parabolic SQ. This is the classical approach of Parisi–Wu via a Langevin equation [51]. The target measure is realised as the invariant measure of a stochastic dynamics.

Elliptic SQ. In the elliptic variant, the target measure is realised as a marginal of the solution to an elliptic SPDE. This method has been recently introduced in [1, 2] via the supersymmetric dimensional reduction of Parisi–Sourlas [50].

Canonical SQ. In this case the relevant SPDE is a wave equation, see e.g. [29, 31, 30, 16].

To provide a different perspective we decided to focus here on the **variational approach** to Euclidean quantum fields initiated in [5] using as basic examples the $(\varphi^4)_{2,3}$ EQFTs on a bounded domain $\mathbb{T}^d$, and subsequently to suggest which form it takes in the case of the infinite volume problem on $\mathbb{R}^2$. We will not consider the case of $\mathbb{R}^3$ since it has not been worked out in detail. Moreover, we would also like to discuss the case of $(\exp(\beta\varphi))_2$ as in this case we can prove uniqueness of the infinite volume limit and it will allow us also to showcase the construction of other models than those with polynomial interactions.

The variational method has been recently used in various contexts to provide novel results: Bringmann used it to construct singular Gibbs measures [14] and study their invariance under dispersive dynamics [15], Chandra, Gunaratnam and Weber used it to show the existence of phase transition in the φ^4_3 model [17], Oh, Okamoto and Tolomeo [49] to prove existence and non-existence of certain Gibbs measures.

The PhD thesis of N. Barashkov [6] describes also how to study the infinite volume limit of the Sine–Gordon model (up to the first threshold) via the variational method including a full verification of the Osterwalder–Schrader axioms. The joint work of S. J. Meyer and the author [32] extends these results to the second threshold using a renormalisation group point of view coupled with the forward-backward stochastic dynamics for the optimal control problem. Moreover in [7] together with Barashkov we provide another approach to describe the most singular part of the φ^4_3 measure by approximating the optimal drift of the variational description.

Similar ideas relying on a coupling between the Gaussian free field and an interacting field have been used by Bauerschmidt and Hofstetter to control the maximum of the Sine–Gordon model [8]. More generally, mapping a Brownian motion into a measure of interest has revealed itself a useful method also in the context of functional inequalities, see e.g. the work of Borell [11], Lehec [42] and the recent paper of Mikulincer and Shenfeld [43] on the "Brownian transport map".

Let us finally mention the *manifesto* [4] where a (speculative) proposal for a new class of stochastic equations, Wilson–Ito diffusions, describing local random fields, including EQFTs is laid out.

The plan of the lectures is the following:

1. We introduce the Gaussian Free field as a generalised Gaussian Markov process and discuss its regularity and a simple construction of φ^4_2 in finite volume, due to Nelson.
2. We introduce the variational characterisation of an EQFT in terms of a stochastic control problem, via the formula of Boué–Dupuis. This will allow us to transform a problem in probability theory into a problem of stochastic analysis amenable to functional analytic techniques.
3. We will illustrate these analytic techniques via the construction of φ^4_2 in finite volume, providing an alternative proof to that of Nelson.

4. We shift to the infinite volume limit problem via variational techniques, where new ideas are needed which shift the attention to the Euler–Lagrange equations satisfied by the minimisers of the variational problem. We will discuss how to obtain a-priori information on the EQFTs via these equations in the case of the φ_2^4 and exponential models.
5. We will finish by focusing on the more challenging situation of the φ_3^4 model in finite volume where the variational problem has to be "renormalised" to provide a description which is stable in the UV limit.

2 Gaussian Free Field

To discuss the Gaussian free field, I find it interesting to start with its one dimensional version, which is the Gaussian Markov process known as the Ornstein–Uhlenbeck process, i.e. the stationary solution to the SDE:

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}B_t, \qquad t \in \mathbb{R},$$

which can be expressed as the Wiener integral

$$X_t = \int_{-\infty}^{t} e^{-(t-s)}\mathrm{d}B_s, \qquad t \in \mathbb{R},$$

or as the Gaussian process with covariance

$$C(t-s) = \mathbb{E}[X_t X_s] = \frac{1}{2}e^{-|t-s|}, \qquad t, s \in \mathbb{R}.$$

This process is Markov and its covariance has the Fourier transform given by

$$C(t) = \int_{\mathbb{R}} \frac{e^{i\omega t}}{1+\omega^2}\frac{\mathrm{d}\omega}{2\pi}$$

as can be checked via an application of the residue formula. This means that the covariance C is a fundamental solution for the operator $(1-\partial_t^2)$.

By analogy, the multidimensional version $(\varphi(x))_{x\in\mathbb{R}^d}$ of this random field can be defined as the Gaussian field on $\mathbb{R}^d$ whose covariance $C(x)$ has the Fourier transform given by

$$C(x) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot x}}{1+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d}$$

which gives a fundamental solution of the operator $(1-\Delta)$ where Δ is the Laplacian on $\mathbb{R}^d$. However, when $d \geqslant 2$ we have that $C(0) = +\infty$ and more generally that

$$C(x) \approx \begin{cases} -\log|x| & (d=2) \\ |x|^{2-d} & (d>2) \end{cases}, \qquad |x| \ll 1,$$

which means that the random field φ does not have continuous sample paths and it must be realised as a generalised function taking values a.s. in $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

The rigorous definition of GFF is therefore that of a Gaussian process indexed by $\mathcal{S}(\mathbb{R}^d)$ and such that

$$\mathbb{E}[\varphi(f)^2] = \langle f, (1-\Delta)^{-1} f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d)$$

where we denoted by $(1-\Delta)^{-1}$ the operator whose L^2-kernel is given by C. We denote by $\mu = \mu^{\mathrm{GFF}}$ the law of the GFF as a probability measure on $\mathcal{S}'(\mathbb{R}^d)$. Later on we will consider also the GFF on $\mathbb{T}^d = [-1/2, 1/2]^d$ (with periodic b.c.) which has a similar definition.

If we denote by C_Λ the covariance of the GFF on $\Lambda = \mathbb{T}^d$ or $\mathbb{R}^d$ then they are related by the method of images:

$$C_{\mathbb{T}^d}(x) = \sum_{k\in\mathbb{Z}^d} C_{\mathbb{R}^d}(x+k)$$

where the sum converges as $C_{\mathbb{R}^d}$ has exponential decay for large distances.

2.1 Besov spaces

Besov spaces provide a convenient setting to measure regularity of random fields. For simplicity we discuss the case of $\mathbb{R}^d$ and discuss later how to adapt it to the case of $\mathbb{T}^d$. The reader can find full details and proofs in [3, 28] and for weighted spaces in [26, 45].

Let $\langle\alpha\rangle := (1+|\alpha|^2)^{1/2}$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of test functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ the corresponding space of tempered distributions. For any $\alpha\in\mathbb{R}$ we define fractional derivatives $\langle \mathrm{D}\rangle^\alpha = (1+\mathrm{D}^2)^{\alpha/2}$ on $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \mathrm{D}\rangle^\alpha f = \mathcal{F}^{-1}(\xi \mapsto \langle\xi\rangle^\alpha (\mathcal{F}f)(\xi)), \qquad f\in\mathcal{S}'(\mathbb{R}^d)$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^d$. More generally if $h\colon \mathbb{R}^d\to\mathbb{R}$ is a nice radial function we let

$$h(\mathrm{D})f = \mathcal{F}^{-1}(\xi\mapsto h(\xi)(\mathcal{F}f)(\xi)), \qquad f\in\mathcal{S}'(\mathbb{R}^d).$$

To localise functions in frequency space, we need a partition of unity whose existence can be proved but we shall take this for granted here. Let χ, ϱ be smooth radial functions $\mathbb{R}^d\to\mathbb{R}$ such that

- $\mathrm{supp}\,\chi \subseteq B(0,4/3)$, $\mathrm{supp}\,\varrho\subseteq\{3/4\leqslant|\xi|\leqslant 8/3\}$;
- $0\leqslant\chi,\varrho\leqslant 1$, $\chi(\xi)+\sum_{j\geqslant 0}\varrho(2^{-j}\xi)=1$ for any $\xi\in\mathbb{R}^d$;
- $\mathrm{supp}\,\varrho(2^{-j}\cdot)\cap\mathrm{supp}\,\varrho(2^{-i}\cdot)=\varnothing$ if $|i-j|>1$.

Introduce the notations $\varrho_{-1}=\chi$, $\varrho_j=\varrho(2^{-j}\cdot)$ for $j\geqslant 0$.

Given such a partition of unity, we define the associated ***Littlewood–Paley–projectors*** as

$$\Delta_j = \varrho_j(\mathrm{D}), \qquad j\geqslant -1,$$

and write

$$K_j = \mathcal{F}^{-1}\varrho_j, \qquad j\geqslant -1,$$

for the resulting convolution kernels in position space.

We can then write, for any $u\in S'(\mathbb{R}^d)$,

$$u = \sum_{j\geqslant -1}\Delta_j u = \sum_{j\geqslant -1} K_j * u. \tag{3}$$

where convergence is in the sense of $S'(\mathbb{R}^d)$.

Note that $\Delta_j u$ is now a holomorphic function since its Fourier transform is supported on a ball (for $j=-1$) or on an annulus (for $j\geqslant 0$), indeed we have for any $u\in S'(\mathbb{R}^d)$

$$(\Delta_j u)(x) := \mathcal{F}^{-1}(\varrho_j \hat{u}(\cdot)).$$

From (3) we finally define the Besov spaces.

Definition 1. *Let $s \in \mathbb{R}, p, q \in [1, \infty]$. For a Schwartz distribution $f \in \mathscr{S}'(\Lambda)$ define the norm*

$$\|f\|_{B^s_{p,q}} = \|(2^{js}\|\Delta_j f\|_{L^p})_{j \geqslant -1}\|_{\ell^q}$$

where $\|\|_{L^p}$ denotes the normalised $L^p(\Lambda)$ norm. The space $B^s_{p,q}$ is the set of functions $f \in \mathscr{S}'(\Lambda)$ such that $\|f\|_{B^s_{p,q}} < \infty$ moreover $H^s = B^s_{2,2}$ are the usual Sobolev spaces, and we denote by $\mathscr{C}^s$ the closure of smooth functions in the $B^s_{\infty,\infty}$ norm.

Remark 2.

- The specific norm $\|\cdot\|_{B^s_{p,q}}$ depends on the partition of unity, but all such norms are equivalent and the space $B^s_{p,q}$ does not depend on the specific choice.
- The space $(B^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{B^s_{p,q}})$ is a Banach space.

Let us quickly note that the parameter $s \in \mathbb{R}$ indeed measures regularity in the usual sense.

Theorem 3. *For any $s, \alpha \in \mathbb{R}$ and $u \in S'(\mathbb{R}^d)$*

$$\|u\|_{B^s_{p,q}} \approx \|(1-\Delta)^{\alpha/2} u\|_{B^{s-\alpha}_{p,q}}.$$

Remark 4. There is a relation between Besov spaces and Sobolev and Holder spaces: in particular one can prove that,

$$B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d), \qquad s \in \mathbb{R}_+ \setminus \mathbb{Z} \qquad B^\alpha_{\infty,\infty}(\mathbb{R}^d) = C^\alpha(\mathbb{R}^d), \qquad \alpha \in \mathbb{R} \setminus \mathbb{Z}.$$

Easy embeddings among Besov spaces are as follows:

- Since $\ell^{q_1}(\mathbb{Z}) \hookrightarrow \ell^{q_2}(\mathbb{Z})$ for $q_1 \leqslant q_2$, we directly see $B^s_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^d)$.
- Moreover, for $s_2 \leqslant s_1$, $2^{qs_2 j} \leqslant 2^{qs_1 j}$ so that $B^{s_1}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^d)$.

More generally, functions with localised Fourier transform have equivalent Lebesgue norms (this is called Bernstein's theorem) and this generates embeddings similar to the well-known Sobolev embeddings:

Lemma 5. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. If $s_2 \leqslant s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)$, then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d).$$

Besov spaces on $\mathbb{T}^d$ have a similar definition using the LP decomposition on the torus.

All along the paper we need to consider Banach subspaces of $\mathcal{S}'(\mathbb{R}^d)$ associated with a positive spatial weight ρ.

Definition 6. *A polynomial weight ρ is a function $\rho : \mathbb{R}^d \to \mathbb{R}_+$ of the form $\rho(x) = c\langle \ell x \rangle^{-\sigma}$ for $\sigma, c, \ell \geqslant 0$. For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}^d} |f(x)\rho(x)|^p \, \mathrm{d}x \right)^{1/p}$$

and denote by $L^p(\rho)$ the space of functions for which this norm is finite. For functions defined on a torus in $\mathbb{R}^d$ we consider their periodic extensions on $\mathbb{R}^d$. Similarly we define the weighted Besov spaces $B^s_{p,q}(\rho)$ as the set of elements of $\mathscr{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{B^s_{p,q}(\rho)} = \|(2^{js}\|\Delta_j f\|_{L^p(\rho)})_{j\geqslant -1}\|_{\ell^q}$$

is finite and by $\mathscr{C}^s(\rho)$ those such that the norm

$$\|f\|_{\mathscr{C}^s(\rho)} = \|(2^{js}\|\rho\Delta_j f\|_{L^\infty})_{j\geqslant -1}\|_{\ell^\infty}$$

is finite.

We will fix a weight ρ such that for some sufficiently small $\varepsilon > 0$ we have

$$|\Delta\,\rho^{1/2}| + |\nabla\rho^{1/2}| \leqslant \varepsilon\,\rho^{1/2}, \qquad \rho \lesssim \langle\cdot\rangle^{-32}. \tag{4}$$

The number ε can depend on various parameters but not on the volume cutoffs we introduce to regularise the Euclidean QFTs.

We note that the weighted norm $B^s_{p,q}(\rho^\alpha)$ is equivalent to

$$\|f\|^q_{B^s_{p,q}(\rho^\alpha)} = \sum_{i\geqslant -1} 2^{isq}\|\Delta_i f\|^q_{L^p(\rho^\alpha)} = \sum_{i\geqslant -1} 2^{isq}\|\rho^\alpha\Delta_i f\|^q_{L^p(\mathbb{R}^d)} \approx \sum_{i\geqslant -1} 2^{isq}\|\Delta_i(\rho^\alpha f)\|^q_{L^p(\mathbb{R}^d)},$$

that is, the weight ρ^α can be freely commuted with the LP projectors as far as Besov norms are concerned.

We collect here some basic estimates which will be useful below.

Proposition 7. *Let $\delta > 0$. We have for any $q_1, q_2 \in [1,\infty]$, $q_1 < q_2$*

$$\|f\|_{B^s_{p,q_2}} \leq \|f\|_{B^s_{p,q_1}} \lesssim \|f\|_{B^{s+\delta}_{p,\infty}}.$$

Furthermore, if we denote by $W^{s,p}$ the normalised fractional Sobolev spaces then for any $q \in [1,\infty]$

$$\|f\|_{B^s_{p,q}} \lesssim \|f\|_{W^{s+\delta,p}} \lesssim \|f\|_{B^{s+2\delta}_{p,\infty}}.$$

Proposition 8. *For any $s_1, s_2 \in \mathbb{R}$ such that $s_2 < s_1$, any $p, q \in [1,\infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.*

Definition 9. *A smooth function η is said to be an S^m multiplier if for every multi-index α there exists a constant C_α such that*

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\eta(\xi)\right| \lesssim_\alpha (1+|\xi|)^{m-|\alpha|}, \qquad \xi \in \mathbb{R}^d. \tag{5}$$

We say that a family η_t is a uniformly S^m multiplier if (5) is satisfied for every t with C_α independent of t.

Proposition 10. *Let η be an S^m multiplier, $s \in \mathbb{R}$, $p, q \in [1,\infty]$, and $f \in B^s_{p,q}$, then*

$$\|\eta(\mathrm{D})f\|_{B^{s-m}_{p,q}} \lesssim \|f\|_{B^s_{p,q}}.$$

Furthermore the constant depends only on s, p, q, d and the constants C_α in eq. (5).

For a proof see [3] Lemma 2.78.

Proposition 11. *Let $\theta \in [0,1]$, p, p_1, p_2 and s, s_1, s_2 be such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $s = \theta s_1 + (1-\theta)s_2$ and assume that $f \in W^{s_1,p_1} \cap W^{s_2,p_2}$. Then*

$$\|f\|_{W^{s,p}} \leqslant \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}.$$

For a proof see [44].

2.2 Regularity of the GFF

The basic argument to obtain a Besov regularity for the GFF is to compute the following average:

$$\mathbb{E}\Big[\, \|\varphi\|^{2p}_{B^{\alpha}_{2p,2p}(\mathbb{T}^d)} \Big].$$

We have

$$\mathbb{E}\big[\, \|\varphi\|^{2p}_{B^{\alpha}_{2p,2p}} \big] = \mathbb{E}\Bigg[\sum_{j \geqslant -1} 2^{2p\alpha j} \|\Delta_j \varphi\|^{2p}_{L^{2p}} \Bigg] = \sum_{j \geqslant -1} 2^{2p\alpha j} \mathbb{E}[\|\Delta_j \varphi\|^{2p}_{L^{2p}}].$$

Now we observe that, by Fubini theorem and **hypercontractivity** of Gaussian vectors (see e.g. Janson's book on Gaussian measures [38])

$$\mathbb{E}[\|\Delta_j \varphi\|^{2p}_{L^{2p}}] \lesssim (\mathbb{E}[\|\Delta_j \varphi\|^2_{L^2}])^p,$$

so we reduced ourselves to computing covariances. Now we observe that, by standard Fourier analysis

$$\mathbb{E}[\|\Delta_j \varphi\|^2_{L^2}] = \sum_{k,k' \in 2\pi\mathbb{Z}^d} \varrho_j(k)\, \varrho_j(k') \mathbb{E}[\hat{\varphi}(k)\, \overline{\hat{\varphi}(k')}] = \sum_{k \in 2\pi\mathbb{Z}^d} \frac{\varrho_j^2(k)}{1+|k|^2}$$

But now the sum is over 2^{jd} points thanks to the support of $\varrho_j(k)$ and in addition this forces $|k|$ to be of order 2^j, these two remarks give

$$\mathbb{E}[\|\Delta_j \varphi\|^2_{L^2}] \lesssim 2^{jd} 2^{-2j} = 2^{j(d-2)}.$$

This means that

$$\mathbb{E}\big[\, \|\varphi\|^{2p}_{B^{\alpha}_{2p,2p}} \big] \lesssim \sum_{j \geqslant -1} 2^{2p\alpha j} \mathbb{E}[\|\Delta_j \varphi\|^{2p}_{L^{2p}}] \lesssim \sum_{j \geqslant -1} 2^{2p\alpha j} 2^{jp(d-2)} = \sum_{j \geqslant -1} 2^{2pj(\alpha + d/2 - 1)}.$$

So we see that in order to have a convergent series we need

$$\alpha + d/2 - 1 < 0 \qquad \Rightarrow \qquad \alpha < \frac{2-d}{2}.$$

To conclude, we shall use Besov embedding recalling that if $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, $s_2 \leqslant s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)$, then

$$B^{s_1}_{p_1,q_1}(\mathbb{T}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{T}^d).$$

We are interested in the case

$$p_2 = q_2 = \infty, \qquad p_1 = q_1 = 2p, \qquad s_1 = \alpha$$

then we see that

$$B^{\alpha}_{2p,2p}(\mathbb{T}^d) \hookrightarrow B^{s}_{\infty,\infty}(\mathbb{T}^d) = C^s(\mathbb{T}^d),$$

for any $s \leqslant \alpha - \frac{d}{2p}$. But since p can be chosen to be arbitrarily large, we conclude that the above embedding is true for any

$$s < \alpha < -\frac{d-2}{2}.$$

Remark 12. We see that:

- for $d=1$ the GFF can be realised as a continuous function, indeed it belongs to C^α for any $\alpha<1/2$;
- for $d=2$, the GFF can be realised as a random distribution which is not very singular, namely C^α for any $\alpha<0$, so arbitrarily close to 0. From the viewpoint we discussed here, this is why the φ^4 model is less difficult in 2 dimensions wrt the higher dimensional case;
- for $d\geqslant 3$ we have a singular behaviour getting progressively worse with d.

2.3 Regularity of the Wick powers

Since the GFF is a random distribution, it is not easy to construct local non-linear functionals. The non-trivial fact is that they indeed exist and can be constructed by approximations.

Consider a mollifier operator R_N (see below for precise definitions), e.g. one can take

$$R_N=\Delta_{\leqslant N}=\sum_{i\leqslant N}\Delta_i$$

via partial sums of LP blocks. Then, if φ is a GFF,

$$\varphi_N:=R_N\varphi$$

is a smooth Gaussian process with a translation invariant covariance given by $C_N=R_N(1-\Delta)^{-1}R_N^*$. A direct computation, e.g. in Fourier space, shows that

$$C_N(x)\propto\begin{cases}-\log(|x|+N^{-1}) & d=2\\ (|x|+N^{-1})^{2-d} & d\geqslant 3\end{cases}$$

Recall also that the **Hermite polynomials** $(H_n(x))_{n\geqslant 0}$ are the orthogonal polynomials of the standard Gaussian measure over $\mathbb{R}$ and that the first of them (in probabilist's normalisation) are:

$$\begin{aligned}H_0(x)&=1,\\ H_1(x)&=x,\\ H_2(x)&=x^2-1,\\ H_3(x)&=x^3-3x,\\ H_4(x)&=x^4-6x^2+3,\\ H_5(x)&=x^5-10x^3+15x,\\ &\vdots\end{aligned}$$

generated by the series

$$\exp\left(tx-\frac{t^2}{2}\right)=\sum_{n\geqslant 0}\frac{t^n}{n!}H_n(x).$$

We define now the (regularised) Wick powers of the GFF as

$$[\![(\varphi_N)^n(x)]\!]:=C_N^{n/2}(0)H_n\Big(\varphi_N(x)/C_N^{1/2}(0)\Big).$$

For example:

$$[\![(\varphi_N)^2(x)]\!]=\varphi_N^2(x)-C_N(0)$$

$$[\![(\varphi_N)^3(x)]\!]=\varphi_N^3(x)-3\varphi_N(x)C_N(0)$$

$$[\![(\varphi_N)^4(x)]\!]=\varphi_N^4(x)-6\varphi_N^2(x)C_N(0)+3C_N^2(0).$$

In particular the covariance of these local functions of φ_N is given by

$$\mathbb{E}\Big[\,[\![(\varphi_N)^n(x)]\!][\![(\varphi_N)^k(y)]\!]\,\Big]=\delta_{n,k}(n!)C_N^n(x-y)$$

and by hypercontractivity we also have, for any nice test function $f\colon\mathbb{R}^d\to\mathbb{R}$

$$\begin{aligned}\mathbb{E}\left[\left|\int_{\mathbb{R}^d}[\![(\varphi_N)^n(x)]\!]f(x)\mathrm{d}x\right|^p\right]&\lesssim_{p,n}\left\{\mathbb{E}\left[\left|\int_{\mathbb{R}^d}[\![(\varphi_N)^n(x)]\!]f(x)\mathrm{d}x\right|^2\right]\right\}^{p/2}\\&\lesssim\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}C_N^n(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y\right|^{p/2}.\end{aligned}$$

We will need to estimate the average size of the random fields $[\![\varphi_N^n]\!]$ uniformly in N. As with the GFF, we do not expect them to be bounded in some function space and therefore we will use negative regularity Besov–Hölder spaces to estimate them. First we note that by Besov embedding, for all $\alpha\in\mathbb{R}$ and $N\geqslant 1$,

$$\big\|[\![\varphi_N^n]\!]\big\|_{C^\alpha(\rho^\varepsilon)}^p\lesssim\big\|[\![\varphi_N^n]\!]\big\|_{B^{\alpha+d/p}_{p,p}(\rho^\varepsilon)}^p=\sum_{i\geqslant-1}2^{i\left(\alpha+\frac{d}{p}\right)p}\int\rho^{\varepsilon p}(x)|\Delta_i[\![\varphi_N^n]\!](x)|^p\mathrm{d}x$$

Taking expectations and using Fubini's theorem we have now

$$\mathbb{E}\big\|[\![\varphi_N^n]\!]\big\|_{C^\alpha(\rho^\varepsilon)}^p\lesssim\sum_{i\geqslant-1}2^{i\left(\alpha+\frac{d}{p}\right)p}\int\rho^{\varepsilon p}(x)\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^p]\mathrm{d}x$$

Now Gaussian hypercontractivity applied to the Gaussian polynomial $\Delta_i[\![\varphi_N^n]\!](x)$, gives

$$\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^p]\lesssim\Big\{\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^2]\Big\}^{p/2}$$

which reduces the computation to an L^2 bound. Now, when $d=2$,

$$\begin{aligned}\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^2]&=\iint K_i(x-y)K_i(x-y')\mathbb{E}[[\![\varphi_N^n]\!](y)[\![\varphi_N^n]\!](y')]\mathrm{d}y\mathrm{d}y'\\&\propto\iint K_i(x-y)K_i(x-y')\big\{\mathbb{E}[\varphi_N(y)\varphi_N(y')]\big\}^n\mathrm{d}y\mathrm{d}y'\\&\lesssim\iint K_i(x-y)K_i(x-y')\big|\log^n|y-y'|\big|\mathrm{d}y\mathrm{d}y'\\&\lesssim\iint K_i(y)K_i(y')\big|\log^n|y-y'|\big|\mathrm{d}y\mathrm{d}y'\\&\lesssim\left[\iint(K_i(y)K_i(y'))^{q/(q-1)}\mathrm{d}y\mathrm{d}y'\right]^{(q-1)/q}\lesssim 2^{4i/q}\end{aligned}$$

where we used that in two dimensions

$$\mathbb{E}[\varphi_N(y)\varphi_N(y')]\lesssim\big|\log[|y-y'|+N^{-1}]\big|\lesssim\big|\log[|y-y'|]\big|$$

and that this function is in L^q for the product measure $\mathrm{d}y\times\mathrm{d}y'$ for any $q\geqslant 1$.

Putting all together we arrive at

$$\mathbb{E}\big\|[\![\varphi_N^n]\!]\big\|_{C^\alpha(\rho^\varepsilon)}^p\lesssim\sum_{i\geqslant-1}2^{i\left(\alpha+\frac{2}{p}+\frac{2}{q}\right)p}\int_{\mathbb{R}^2}\rho^{\varepsilon p}(x)\mathrm{d}x<+\infty$$

for any $\alpha<0$ as soon as p and q are large enough, since then we have also $\rho^{\varepsilon p}\in L^1(\mathbb{R}^2)$.

This sketches the proof that in $d=2$, for all $\kappa>0,\varepsilon>0$ we have, for any $n\geqslant 1$ and any $p\geqslant 1$,

$$\sup_N \mathbb{E}\left\|\llbracket\varphi_N^n\rrbracket\right\|^p_{C^{-\kappa}(\rho^\varepsilon)}<+\infty. \tag{6}$$

A variant of these computations shows that

$$\sup_{M\geqslant N} \mathbb{E}\left\|\llbracket\varphi_N^n\rrbracket-\llbracket\varphi_M^n\rrbracket\right\|^p_{C^{-\kappa-1}(\rho^\varepsilon)}\lesssim N^{-p},$$

that is, the family $(\llbracket\varphi_N^n\rrbracket)_N$ is Cauchy and converges in L^p to a (unique) limit denoted $\llbracket\varphi^n\rrbracket\in C^{-\kappa}(\rho^\varepsilon)$ and called the n-th Wick power of the GFF.

Similar computations in $d=3$ show that for all $\kappa,\varepsilon>0$ and $p\geqslant 1$,

$$\sup_N \mathbb{E}\left\|\llbracket\varphi_N^2\rrbracket\right\|^p_{C^{-1-\kappa}(\rho^\varepsilon)}<+\infty,$$

and similarly can be used to establish the existence of the random field $\llbracket\varphi^2\rrbracket\in C^{-1-\kappa}$. However for any $\alpha>3/2,\varepsilon>0$ we have for $n=3$:

$$\mathbb{E}\left\|\llbracket\varphi_N^3\rrbracket\right\|^p_{C^{-\alpha}(\rho^\varepsilon)}\approx(\log N)^{p/2}$$

which hints at the fact that in $d=3$ the family $(\llbracket\varphi_N^3\rrbracket)_N$ does not converge to a well-defined random distribution. A similar phenomenon affects all the higher Wick powers. In $d=4$ things get worse and not even $\llbracket\varphi^2\rrbracket$ exists.

2.4 Nelson's approach to φ_2^4

To illustrate the basic ideas of the construction of Euclidean QFTs let us review the proof of Nelson that the φ_2^4 model exists in finite volume. Consider the functional

$$V_T(\varphi)=\int_\Lambda[\varphi^4(x)-r_T\varphi^2(x)]\mathrm{d}x$$

and let's try to take the limit $T\to\infty$ for the family of measures

$$\nu^T(\mathrm{d}\varphi)=\frac{\exp(-\lambda V_T(R_T\varphi))}{Z_T}\mu(\mathrm{d}\varphi),$$

for some $\lambda>0$. We let

$$r_T=6\mathbb{E}[(R_T\varphi(x))^2]$$

where we denote by $\mathbb{E}$ the expectation wrt the measure $\mu=\mu^{\mathrm{GFF}}$ with canonical process φ.

In this case, writing $c_T:=\mathbb{E}[(R_T\varphi(x))^2]=r_T/6$ (independent of x), we have $V_T(R_T\varphi)=\int_\Lambda\llbracket(R_T\varphi(x))^4\rrbracket\mathrm{d}x-3c_T^2|\Lambda|$. The additive constant $-3c_T^2|\Lambda|$ does not depend on φ and can be absorbed in the normalisation constant Z_T of ν^T, so from now on we drop it and denote

$$V_T(R_T\varphi)=\int_\Lambda\llbracket(R_T\varphi(x))^4\rrbracket\mathrm{d}x$$

since this corresponds to the 4-th Hermite polynomial of the random variable $R_T\varphi(x)$.

Recall that Hermite polynomials are defined to be the orthogonal polynomials of a Gaussian measure, see e.g. [38].

A basic result is to show that the density

$$\exp(-\lambda V_T(R_T\varphi))$$

belongs to $L^p(\mu^{\mathrm{GFF}})$ for all $p \geqslant 1$ uniformly in $T \to \infty$. The strategy of Nelson [46] is based on two observations:

1. The first is that the polynomial $\xi^4 - r_T\xi^2$ is bounded below by a quantity proportional to $-r_T^2$, which in our case diverges to $-\infty$ as $-\log^2 T$.
2. The second is hypercontractivity, which we take to mean the following inequality for polynomials P of order d of a Gaussian vector X:

$$\mathbb{E}[|P|^{2p}] \leqslant (2p-1)^{dp}\,\mathbb{E}[|P|^2]^p \tag{7}$$

see e.g. [38] or the lecture notes of Hairer [33].

Take $K < T$ and write

$$V_T(R_T\varphi) = (V_T(R_T\varphi) - V_K(R_K\varphi)) + V_K(R_K\varphi)$$

with $V_K(R_K\varphi) \gtrsim -\log^2 K$. Therefore for all $l > 0$

$$V_T(R_T\varphi) < -l \Rightarrow$$

$$(V_T(R_T\varphi) - V_K(R_K\varphi)) < -l - V_K(R_K\varphi) < -l + C\log^2 K$$

and using Markov+hypercontractivity (7),

$$\mathbb{P}\Big(V_T(R_T\varphi) < -l\Big) \leqslant \mathbb{P}\Big(|(V_T(R_T\varphi) - V_K(R_K\varphi))| > l - C\log^2 K\Big)$$

$$\leqslant \frac{\mathbb{E}[|V_T(R_T\varphi) - V_K(R_K\varphi)|^p]}{(l - C\log^2 K)^p}$$

$$\lesssim \frac{\{\mathbb{E}[|V_T(R_T\varphi) - V_K(R_K\varphi)|^2]\}^{p/2}}{(l - C\log^2 K)^p} \lesssim p^{2p}K^{-p/2}(l - C\log^2 K)^{-p}.$$

Take $K = e^{((l-1)/C)^{1/2}}$ so $l - C\log^2 K = 1$ and

$$\mathbb{P}\Big(V_T(R_T\varphi) < -l\Big) \lesssim p^{2p}e^{-(p/2)((l-1)/C)^{1/2}} \lesssim \big(p^2\, e^{-cl^{1/2}}\big)^p,$$

and taking $p = e^{cl^{1/2}/2-1}$ we have

$$\mathbb{P}\Big(V_T(R_T\varphi) < -l\Big) \lesssim e^{-2e^{cl^{1/2}/2-1}}$$

from which we conclude that

$$\sup_T \|\exp(-\lambda V_T(R_T\varphi))\|_{L^q(\mu)} < \infty$$

for all $q \geqslant 1$.

3 The variational approach

Gibbs measures are solutions to a variational problem: If μ is a probability measure on a measure space $(\Omega, \mathcal{F})$ and V a bounded measurable function, bounded from below, the variational functional

$$F_V(\nu) := \int V \,\mathrm{d}\nu + H(\nu|\mu),$$

is defined for all probability measures ν on $(\Omega, \mathcal{F})$. Here H is the relative entropy of ν wrt. μ defined as

$$H(\nu|\mu) := \sup_f \left[\int f \, \mathrm{d}\nu - \log \int \exp(f) \mathrm{d}\mu \right],$$

where the supremum runs over bounded measurable functions $f: \Omega \to \mathbb{R}$. The relative entropy is finite only if $\nu \ll \mu$ and in this case it holds

$$H(\nu|\mu) = \int \log\left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right] \mathrm{d}\nu.$$

From the definition one also deduces that H is non-negative and convex and that $H(\nu|\mu) = 0$ iff $\nu = \mu$. It is also lower semicontinuous for the weak topology as the supremum in the definition can be taken over continuous functions (see e.g. [23]).

The functional F_V is convex and it admits a unique minimum ν^V given by

$$\mathrm{d}\nu^V = \frac{e^{-V}}{Z^V} \mathrm{d}\mu \tag{8}$$

and

$$F_V(\nu^V) = -\log Z^V = -\log \int e^{-V} \mathrm{d}\mu.$$

As a consequence the Gibbs measure (8) can be characterised as the minimum of the functional F_V, i.e. as the result of an optimisation problem.

When the measure μ is a Wiener measure, the variational formula acquires a more specific form which is amenable to interesting stochastic analysis.

By a theorem of Föllmer if $(\Omega, \mathcal{F}, \mu)$ is a Brownian probability space with Wiener measure μ we have that every ν with finite entropy $H(\nu|\mu) < +\infty$ can be represented as a Girsanov transform, namely there exists a previsible function $u \in L^2(\Omega \times [0, \infty); \mathbb{R}^n)$ such that

$$H(\nu|\mu) = \frac{1}{2} \mathbb{E}_\nu \int_0^\infty |u_s|^2 \mathrm{d}s < +\infty,$$

and $\nu = \mu^u$ with μ^u the Girsanov shift of μ in the direction u:

$$\frac{\mathrm{d}\mu^u}{\mathrm{d}\mu} = \exp\left(\int_0^\infty u_s \mathrm{d}B_s - \frac{1}{2} \int_0^\infty |u_s|^2 \mathrm{d}s \right),$$

where we denote B the canonical process on $(\Omega, \mathcal{F})$. By this result the variational problem takes the form

$$\inf_{u \in \mathbb{H}_a} \int_\Omega V \mathrm{d}\mu^u + \frac{1}{2} \mathbb{E}_{\mu^u} \int_0^\infty |u_s|^2 \mathrm{d}s. \tag{9}$$

Under μ^u the process $W^u := B - I(u)$ with $I(u)_t := \int_0^t u_s \mathrm{d}s$ is a Brownian motion and $\mathrm{Law}_{\mu^u}(W^u) = \mathrm{Law}_\mu(B)$. Note that the quadratic costs above are expectations under $\nu = \mu^u$, under which B has drift u, and not under μ.

A similar formula, but based on a fixed probability space and a fixed Brownian motion, was found by Boué–Dupuis [12]. We need the notion of *tame* functional:

Definition 13. *Let $(X, \mathcal{F}, \gamma)$ be a probability space. A measurable function $F: X \to [-\infty, \infty]$ is called tame with respect to γ if there exist $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\mathbb{E}_\gamma[\exp(-qF)] + \mathbb{E}_\gamma[|F|^p] < \infty.$$

Tameness has been introduced by Üstünel [54] as a convenient condition for the validity of the Boué–Dupuis variational formula (**BD formula**) for Brownian functionals. It has been improved by Hariya and Watanabe [34] to allow for $p = q = 1$.

Theorem 14. *[Boué–Dupuis] Let V be a tame functional for μ, i.e. such that*

$$\int |V|^p + e^{-qV} \mathrm{d}\mu < +\infty,$$

for some $p, q > 1$ with $1/p + 1/q = 1$, then

$$-\log \int e^{-V} \mathrm{d}\mu = \inf_{u \in \mathbb{H}_a} \int \left[V(B + I(u)) + \frac{1}{2} \int_0^\infty \|u_s\|^2 \mathrm{d}s \right] \mathrm{d}\mu.$$

Proof. This is essentially Theorem 7 from Üstünel [54]. Note that the functional $V(B + I(u))$ makes sense for any $u \in \mathbb{H}_a$ since by Theorem 2 from Üstünel [54] we have that the push-forward measure $\nu = (B_\cdot + I(u)_\cdot)_\# \mu$ has finite relative entropy with respect to μ and therefore is absolutely continuous, i.e. $\nu \ll \mu$. □

Remark 15. Some observations on these variational formulas.

a) They originate directly from the variational formula for the free energy of statistical mechanical systems: V playing the role of the internal energy and the quadratic term playing the role of the entropy.

b) The infimum might not be attained in Theorem 14 (see e.g. Theorem 8 in [54]) while it is attained in (9).

c) The drift generated by absolutely continuous perturbations of the Wiener measure has been introduced and studied by Föllmer [22].

d) They are a non–Markovian and infinite dimensional extension of the well known stochastic control problem representation of the Hamilton–Jacobi–Bellman equation in finite dimensions [21].

e) The BD formula is easier to use than the formula (9) since the probability measure does not depend on the drift u. Going from one formulation to the other requires proving that certain SDEs with functional drift admit strong solutions and that one is able to approximate unbounded functionals V by bounded ones.

f) See Üstünel [54] and Lehec [42] for a streamlined proof of the BD formula and for applications to functional inequalities on Gaussian measures.

3.1 An example

As an example application of the BD formula, consider the problem of proving the exponential integrability of a functional $F(B)$ of a Brownian motion $(B_t)_{t \geqslant 0}$. We will assume that this functional allows the following bound

$$F(B + I(u)) \leqslant H(B) + L\|u\|_{L^2(\mathbb{R})}, \tag{10}$$

for some integrable $H(B)$ and a constant $L>0$. Essentially, the functional should be of linear growth along the directions of the Cameron-Martin space of the Brownian motion. In this case by BD we have, for all $\alpha \geqslant 0$,

$$\log \mathbb{E} e^{\alpha F(B)} = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[\alpha F(B+I(u)) - \frac{1}{2}\int_0^\infty \|u_s\|^2 \mathrm{d}s \right]$$

where we denoted by $\mathbb{E}$ the expectation wrt. the Wiener measure μ. Using the bound (10) for F we note that, *path-wise* (i.e. for almost every $\omega \in \Omega$)

$$\begin{aligned} \alpha F(B+I(u)) - \frac{1}{2}\int_0^\infty \|u_s\|^2 \mathrm{d}s &\leqslant \alpha H(B) + \alpha L \|u\|_{L^2(\mathbb{R})} - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R})} \\ &\leqslant \alpha H(B) + \sup_{\eta \geqslant 0}\left[\alpha L \eta - \frac{1}{2}\eta^2 \right] \leqslant \alpha H(B) + \frac{L^2\alpha^2}{2} \end{aligned}$$

which gives a bound uniform in the control process u. As a consequence we readily derive the bound

$$\mathbb{E} e^{\alpha F(B)} \leqslant \exp\left(\alpha \mathbb{E}[H(B)] + \frac{L^2\alpha^2}{2} \right), \qquad \alpha > 0,$$

which in particular implies that F has a Gaussian upper tail. What is remarkable is that the BD variational formula allows an analytic line of reasoning (find a bound on the variational functional for all the controls) to derive a probabilistic consequence. Moreover the integrability question has been rephrased in a control of the Cameron–Martin shifts of the functional.

3.2 Euclidean fields as minimisers

Consider the Gibbs measure

$$\theta^V(\mathrm{d}\phi) := \frac{e^{-V(\phi)}\,\theta^0(\mathrm{d}\phi)}{\int e^{-V(\phi)}\,\theta^0(\mathrm{d}\phi)}, \qquad \phi \in \mathcal{S}'(\mathbb{R}^2), \tag{11}$$

where θ^0 is the law of the Gaussian field ϕ (GFF) on $\mathbb{R}^2$ with covariance $(1-\Delta)^{-1}$ and where Δ denotes the Laplacian on $\mathbb{R}^2$. The function $V\colon \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ represents an interaction potential and for now we will only assume that it is measurable and *tame* with respect to θ^0.

To introduce a useful variational description of this class of measures we need to enlarge our probability space and realise the GFF θ^0 as the terminal value of a Brownian martingale $(W_t)_{t\in[0,1]}$ with values in $\mathcal{S}'(\mathbb{R}^2)$ and covariance given by

$$\mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)(1-\Delta)^{-1}(x-y), \qquad x,y \in \mathbb{R}^2, \quad t,s \in [0,1].$$

We construct this Brownian martingale on the measure $\mathbb{P}$ given by the law of a cylindrical Brownian motion $(X_t)_{t\in[0,1]}$ on $L^2(\mathbb{R}^2)$. It can be realised on $C([0,1], C^{-1-\varepsilon}(\rho^\varepsilon))$ where $\varepsilon > 0$ can be taken arbitrarily small. Let

$$W_t := (1-\Delta)^{-1/2} X_t, \qquad t \in [0,1],$$

and note that indeed $\theta^0 = \mathrm{Law}_{\mathbb{P}}(W_1)$. We will consider W as a random variable in $\mathfrak{S}_0 := C([0,1], C^{-\varepsilon}(\rho^\varepsilon))$.

The measure θ^V can be then interpreted as the law of W_1 under the Gibbs measure

$$\mathbb{Q}^V(\mathrm{d}W) = \frac{1}{Z} e^{-V(W_1)} \mathbb{P}(\mathrm{d}W).$$

It is well known that the measure $\mathbb{Q}^V$ is the minimiser of a Gibbs variational problem and that

$$-\log Z^V = -\log \mathbb{E}[e^{-V(W_1)}] = \inf_{\mathbb{R}} \big\{ \mathbb{E}_{\mathbb{R}}[V(W_1)] + H(\mathbb{R}|\mathbb{P}) \big\},$$

where $\mathbb{E}_{\mathbb{R}}$ denotes the expectation with respect to a probability measure $\mathbb{R}$ on the path space $C([0,1], \mathcal{S}'(\mathbb{R}^2))$ and $H(\mathbb{R}|\mathbb{P})$ the relative entropy of $\mathbb{R}$ with respect to $\mathbb{P}$.

Let $\mathfrak{H}(\rho^\alpha) = H^1_t H^1_x(\rho^\alpha)$, $\mathfrak{H} = \mathfrak{H}(\rho^0)$. Denote with $\mathfrak{H}^a(\rho^\alpha)$ the space of all processes $Z\colon \mathfrak{S}_0 \times [0,1] \to \mathcal{S}'(\mathbb{R}^2)$ progressively measurable with respect to the filtration generated by $(W_t)_{t\in[0,1]}$, starting from 0 and equipped with the norm of $L^2(\mathbb{P}, \mathfrak{H}(\rho^\alpha))$. Moreover define the quadratic form $\mathcal{E}\colon \mathfrak{H}^a \to L^0(\mathbb{P})$ giving the "energy" of the *drift* Z as

$$\mathcal{E}(Z) := \frac{1}{2}\int_0^1 \|(1-\Delta)^{1/2}\dot{Z}_s\|^2_{L^2}\mathrm{d}s = \frac{1}{2}\int_0^1 (\|\nabla \dot{Z}_s\|^2_{L^2} + \|\dot{Z}_s\|^2_{L^2})\mathrm{d}s$$

where $\dot{Z}$ denotes the time-derivative of Z.

With these definitions, the Boué–Dupuis formula [12, 54] reads as follows.

Lemma 16. ***(BD formula)*** *For any tame functional $G\colon \mathfrak{S}_0 \to \mathbb{R}$ we have*

$$-\log \mathbb{E}[e^{-G(W_\cdot)}] = \inf_{Z\in\mathfrak{H}^a} \mathbb{E}[G(W_\cdot + Z_\cdot) + \mathcal{E}(Z)].$$

Proof. Theorem 14 specialised in our context gives

$$-\log \mathbb{E}[e^{-G(W_\cdot)}] = \inf_{u\in\mathbb{H}^a} \mathbb{E}\bigg[G(W_\cdot + Z_\cdot) + \frac{1}{2}\int_0^1 \|u_s\|^2_{L^2}\mathrm{d}s \bigg],$$

where $\mathbb{H}^a$ is the space of all progressively measurable processes starting from 0 equipped with the norm of $L^2(\mathbb{P}, L^2_t L^2_x)$ and

$$Z_t := (1-\Delta)^{-1/2}\int_0^t u_s \mathrm{d}s.$$

Observing that $u_t = (1-\Delta)^{1/2}\dot{Z}_t$ we obtain the claim. □

We call a random variable with values in $\mathfrak{H}^a$ a *control* (or *drift*) and we will understand that $W = W_1$ and $Z = Z_1$ whenever this does not lead to confusion.

To simplify certain technical considerations later it is convenient to augment the process W with some additional information. Consider a Polish space $\mathfrak{S}$ endowed with a continuous projection $\pi_W\colon \mathfrak{S} \to \mathfrak{S}_0$ and let $\mathbb{W}\colon \mathfrak{S}_0 \to \mathfrak{S}$ be a measurable map such that $\pi_W \mathbb{W} = W$, i.e. $\mathfrak{S} \xrightarrow{W} \mathfrak{S}_0$ is a fibered topological space and $\mathbb{W}$ a measurable section. We will understand $\mathbb{W}$ as an ***enhanced GFF***. The kind of enhancements needed will be different in different models. We will always consider the space $\mathfrak{S}$ endowed with the measure given by the law of $\mathbb{W}$ under $\mathbb{P}$. A functional on $\mathfrak{S}$ is tame if it is tame for this measure.

For any measurable $f\colon \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ bounded below and in $L^2(\theta^0)$ (not necessarily linear) define $\mathcal{W}^V(f)$ as

$$e^{-\mathcal{W}^V(f)} := \int e^{-f(\phi)}\,\theta^V(\mathrm{d}\phi).$$

As a direct consequence of Lemma 16, the functional $\mathcal{W}^V$ has the variational representation

$$\mathcal{W}^V(f) = \inf_{Z\in\mathfrak{H}^a} F^{f+V}(Z) - \inf_{Z\in\mathfrak{H}^a} F^V(Z)$$

for any f such that $f+V$ is tame and where for a tame functional G on $\mathfrak{S}_0$ we let

$$F^G(Z) := \mathbb{E}[G(W+Z)+\mathcal{E}(Z)], \qquad Z \in \mathfrak{H}^a.$$

In the formulation of Gibbs variational problem, the optimiser corresponds to the Gibbs measure. Correspondingly we would like to show that the optimiser in the BD formula, i.e. an optimal drift Z describes the law of the "interacting field" via the equality

$$\mathrm{Law}_{\mathbb{Q}^V}(W) = \mathrm{Law}_{\mathbb{P}}(W+Z) \tag{12}$$

valid for any minimiser Z of the problem in Lemma 16 and in particular

$$\theta^V = \mathrm{Law}_{\mathbb{P}}(W_1+Z_1).$$

The existence of such minimisers will be guaranteed, in the rigorous discussion below, by suitably relaxing the variational problem into a problem on the space of probability measures.

Before proceeding with the rigorous development, let us pause a moment to give an overview of what follows assuming that minimisers exist, then:

1. any minimiser Z satisfies the Euler–Lagrange equation (first variation)

$$\mathbb{E}\big[\nabla V(W_1+Z_1)\cdot K_1 + 2\mathcal{E}(Z,K)\big] = 0 \tag{13}$$

for all "test controls" K, where $\nabla V(W_1+Z_1)\cdot K_1$ denotes the derivative of $\varphi \mapsto V(\varphi)$ at W_1+Z_1 and in the direction of K_1. In the following we will refer to the equation (13) as a *weak forward–backwards stochastic (partial) differential equation* (wFBSDE). The reason for this name derives from observing that (13) can be formally rewritten as

$$\int_0^1 \int_{\mathbb{R}^2} \mathbb{E}\big[(\nabla V(W_1+Z_1) + (1-\Delta)\dot{Z}_s)\dot{K}_s\big]\mathrm{d}s = 0$$

and since this holds for any nice adapted control K it implies that

$$(1-\Delta)\dot{Z}_s = -\mathbb{E}\big[\nabla V(W_1+Z_1)\,|\,\mathcal{F}_s\big] \tag{14}$$

which is a kind of elliptic equation for the optimal control $(Z_s)_{s\in[0,1]}$. It can be cast into a system of a forward SDE for Z and a backward SDE which computes the conditional expectation. In these notes we will not exploit further this specific interpretation of (13) as a FBSDE so it is not necessary to make it more precise. Let us stress that our wFBSDE is a **stochastic quantisation** for Euclidean QFTs of a new kind.

2. additionally, we will be able to introduce a variational description of the measure $\mathbb{Q}^V$ with drift Z via a renormalised BD formula of the form

$$-\log \mathbb{E}[e^{-f(X_1)}] = \inf_K \mathbb{E}\big[\, f(X_1+K_1) + V(X_1+K_1) - V(X_1) + \mathcal{E}(Z+K) - \mathcal{E}(Z)\big] \tag{15}$$

with $X_s = W_s + Z_s$ being the "interacting field" with potential V and $f(\varphi)$ a test function which depends only locally on the field φ and where the infimum is taken among all drifts K which satisfy suitable decay at infinity. Given this decay one expects that the quantities $V(X_1+K_1)-V(X_1)$ and $\mathcal{E}(Z+K)-\mathcal{E}(Z)$ are finite even if each summand is infinite.

This reformulation will show its full power as soon as we try to describe Euclidean quantum fields in infinite volume. In that case the original variational problem loses its meaning as the interaction potential is almost surely infinite, due to an integration over the unbounded set $\mathbb{R}^2$.

On the other hand the first-order condition (13) remains meaningful also in this setting, as the interaction is local and can be used, very much like the weak formulation of an elliptic problem, to get estimates on the minimiser Z, e.g. by testing with $K = \rho Z$ where ρ: $\mathbb{R}^2 \to \mathbb{R}_+$ is some weight function decaying at spatial infinity. In particular we will obtain estimates of the form

$$\mathbb{E}\|\rho Z\|^2_{H^1_t H^1_x} < \infty,$$

uniformly in the volume. This estimate together with the identification (12) is at the core of the proof of tightness of the family of finite volume measures $(\mathbb{Q}^\Lambda)_\Lambda$ with $\Lambda \nearrow \mathbb{R}^2$. Any accumulation point $\mathbb{Q}$ represents an infinite volume Euclidean field, moreover the corresponding drift Z gives, via the formula (12) an explicit coupling of $\mathbb{Q}$ and the GFF given by $\mathbb{P}$.

The point of view which we advocate in these lectures is that the wFBSDE equation (13) provides a complete control over the drift Z describing the infinite volume Euclidean field, for which a naive variational description is meaningless. This will be exemplified by the analysis of the quartic interaction and of the exponential model in infinite volume, respectively in Section 4 and Section 5.

Moreover the renormalised cost functional $\check{H}^V(v)$ (introduced in Lemma 27) can be made finite in the infinite volume by choosing localised perturbations K. This remark will be clearer by looking at how we determine the infinite volume limit of the exponential model in Section 5.

The rest of this section is dedicated to providing a rigorous proof of these claims.

The standard approach to guarantee the existence of minimisers is to relax our problem, i.e. minimise over the *law* of the pair (W, Z) with W fixed and Z varying over $\mathfrak{H}^a$. In order to ensure compactness we need to complete the space of trial laws for a suitable topology for which our target functional is coercive.

Let $\mathfrak{H}_w(\rho^\alpha)$ be $\mathfrak{H}(\rho^\alpha)$ equipped with the weak topology and let $\hat{\mathfrak{H}} := \mathfrak{H}(\rho^{1/2})$, $\hat{\mathfrak{H}}_w := \mathfrak{H}_w(\rho^{1/2})$ and $\hat{\mathfrak{H}}^a := \mathfrak{H}^a(\rho^{1/2})$. With a slight abuse of notation we denote $(\mathbb{W}, Z)$ the canonical coordinates on $\mathfrak{S} \times \hat{\mathfrak{H}}_w$.

Definition 17. *Let*

$$\mathcal{X} := \Big\{ \mu = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z) \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) : Z \in \hat{\mathfrak{H}}^a, \mathbb{E}_\mu[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty \Big\},$$

and

$$\bar{\mathcal{X}} := \Big\{ \mu \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) : \exists \mu_n \in \mathcal{X},\, \mu_n \to \mu \text{ weakly}, \sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty \Big\}.$$

We equip $\bar{\mathcal{X}}$ with the following notion of convergence: a sequence $(\mu_n)_n$ in $\bar{\mathcal{X}}$ converges to μ if

a) $\mu_n \to \mu$ *weakly on* $\mathfrak{S} \times \hat{\mathfrak{H}}_w$,

b) $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty$.

Note that the probability space in the definition of $\mathcal{X}$ is fixed, so all elements in $\mathcal{X}$ are laws of random variables living on a fixed probability space. Note that all elements of $\mathcal{X}$ have first marginal given by $\mathrm{Law}(\mathbb{W})$*, and this also carries over to $\bar{\mathcal{X}}$ by weak convergence.*

We will need to realise couplings of two elements of $\bar{\mathcal{X}}$. These are constructed by the next lemma which needs the following preliminary definition of the space hosting the coupling.

We denote the canonical coordinates on the product space $\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\hat{\mathfrak{H}}_w$ by $(\mathbb{W},Z,K)$.

Definition 18. *For $\mu\in\bar{\mathcal{X}}$ let $\bar{\mathcal{Y}}(\mu)\subseteq\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\hat{\mathfrak{H}}_w)$ be the set of all υ such that $\mathrm{Law}_\upsilon(\mathbb{W},Z)=\mu$ and for which there exist sequences $(Z^n)_n,(K^n)_n\subseteq\hat{\mathfrak{H}}^a$ with*

$$\mathrm{Law}_{\mathbb{P}}(\mathbb{W},Z^n,K^n)\to\upsilon \text{ and } \sup_{n\in\mathbb{N}}\mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}}+\|K^n\|^2_{\hat{\mathfrak{H}}}]<\infty.$$

Lemma 19. *For any $\nu,\mu\in\bar{\mathcal{X}}$ there exists $\upsilon\in\bar{\mathcal{Y}}(\mu)$ such that*

$$\mathrm{Law}_\upsilon(\mathbb{W},Z+K)=\mathrm{Law}_\nu(\mathbb{W},Z).$$

Proof. Since μ,ν are in $\bar{\mathcal{X}}$ we can find sequences $Z^n,\bar{Z}^n\in\hat{\mathfrak{H}}^a$ such that $\mu_n:=\mathrm{Law}_{\mathbb{P}}(\mathbb{W},Z^n)\to\mu$ and $\nu_n:=\mathrm{Law}_{\mathbb{P}}(\mathbb{W},\bar{Z}^n)\to\nu$ weakly on $\mathfrak{S}\times\hat{\mathfrak{H}}_w$ and furthermore $\sup_n\mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}}+\|\bar{Z}^n\|^2_{\hat{\mathfrak{H}}}]<\infty$. Now define

$$K^n=\bar{Z}^n-Z^n,$$

and set

$$\upsilon^n=\mathrm{Law}_{\mathbb{P}}(\mathbb{W},Z^n,K^n).$$

We have that $\sup_n\mathbb{E}[\|K^n\|^2_{\hat{\mathfrak{H}}}]\leqslant 2\sup_n\mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}}+\|\bar{Z}^n\|^2_{\hat{\mathfrak{H}}}]<\infty$ so υ^n is tight on $\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\hat{\mathfrak{H}}_w$. Taking a convergent subsequence (not relabelled) we can assume that $\upsilon^n\to\upsilon$. We then have

$$\begin{aligned}\mathrm{Law}_\upsilon(\mathbb{W},Z+K) &= \lim_{n\to\infty}\mathrm{Law}_{\upsilon^n}(\mathbb{W},Z+K)\\ &= \lim_{n\to\infty}\mathrm{Law}_{\mathbb{P}}(\mathbb{W},Z^n+K^n)\\ &= \lim_{n\to\infty}\mathrm{Law}_{\mathbb{P}}(\mathbb{W},\bar{Z}^n)=\nu\end{aligned}$$

which proves the claim. □

On functionals on $\bar{\mathcal{X}}$ we can define first and second order derivatives as follows. Let $\mu\in\bar{\mathcal{X}}$ and let $K\colon\mathfrak{S}\times\hat{\mathfrak{H}}_w\to\hat{\mathfrak{H}}_w$ be a measurable process adapted to the filtration generated by $(\mathbb{W}_t,Z_t)_{t\in[0,1]}$, continuous in the second component and such that $K\in L^2(\mu,\hat{\mathfrak{H}})$. We call such K an admissible control for μ. For all $\sigma\in\mathbb{R}$ let $\mu^{\sigma K}$ be the law of $(\mathbb{W},Z+\sigma K)$ under μ. We have $\mu^{\sigma K}\in\bar{\mathcal{X}}$. A functional $\check{F}\colon\bar{\mathcal{X}}\to\mathbb{R}$ is differentiable in $\bar{\mathcal{X}}$ in the direction of the admissible control K if the map $\sigma\mapsto\check{F}(\mu^{\sigma K})$ is differentiable in $\sigma=0$ and in this case we define

$$\nabla\check{F}(\mu)(K)=\frac{\mathrm{d}}{\mathrm{d}\sigma}\bigg|_{\sigma=0}\check{F}(\mu^{\sigma K}).$$

Second order derivatives can be similarly defined by taking two admissible controls K,K' and letting

$$\nabla^2\check{F}(\mu)(K,K')=\frac{\mathrm{d}}{\mathrm{d}\sigma'}\bigg|_{\sigma'=0}\frac{\mathrm{d}}{\mathrm{d}\sigma}\bigg|_{\sigma=0}\check{F}(\mu^{\sigma K+\sigma' K'}),$$

whenever $(\sigma,\sigma')\mapsto \check{F}(\mu^{\sigma K+\sigma' K'})$ is C^2 in $(\sigma,\sigma')=0$.

We are ready to relax the variational problem and prove an alternative representation of $\mathcal{W}^V(f)$.

Definition 20. *We say that a functional $\check{F}\colon \bar{\mathcal{X}}\to\mathbb{R}$ is admissible if it is lower semicontinuous and such that*

$$\check{F}(\mu)\gtrsim 1+\mathbb{E}_\mu[\mathcal{E}(Z)], \tag{16}$$

this ensures in particular that the functional $\check{F}\colon \bar{\mathcal{X}}\to\mathbb{R}$ is coercive, that is for all $L<\infty$ the level sets $\{\check{F}\leqslant L\}$ are compact in $\bar{\mathcal{X}}$.

Definition 21. *For a tame $G\colon \mathfrak{S}\to\mathbb{R}$ define the functional $\check{F}^G\colon \bar{\mathcal{X}}\to\mathbb{R}$ by*

$$\check{F}^G(\mu):=\mathbb{E}_\mu[G(W+Z)+\mathcal{E}(Z)],\qquad \mu\in\bar{\mathcal{X}}.$$

We say that G is admissible if G is tame, $\check{F}^G$ is admissible and $(W,Z)\to G(W+Z)$ is continuous on $\mathfrak{S}\times\hat{\mathfrak{H}}_w$.

The following lemma is an obvious consequence of this relaxed variational setting.

Lemma 22. *Assume that $G\colon \mathfrak{S}\to\mathbb{R}$ is admissible, then*

$$\inf_{Z\in\mathfrak{H}^a} F^G(Z)=\inf_{Z\in\hat{\mathfrak{H}}^a} F^G(Z)=\inf_{\mu\in\mathcal{X}}\check{F}^G(\mu)=\min_{\mu\in\bar{\mathcal{X}}}\check{F}^G(\mu). \tag{17}$$

Let $\mu^G\in\bar{\mathcal{X}}$ be any minimiser of $\check{F}^G$. If $\check{F}^G$ is differentiable at μ^G it satisfies the equation $\nabla\check{F}^G(\mu^G)(K)=0$ for all admissible controls K. Moreover if it is twice differentiable at μ^G we also have the positivity condition $\nabla^2\check{F}^G(\mu^G)(K,K)\geqslant 0$ for all admissible controls K.

Proof. The equality $\inf_{Z\in\mathfrak{H}^a}F^G(Z)=\inf_{Z\in\hat{\mathfrak{H}}^a}F^G(Z)$ follows from the property (16) of the admissible functional $\check{F}^G$ since $\mathfrak{H}^a$ can be viewed as a subspace of $\bar{\mathcal{X}}$ via the map $u\mapsto \mathrm{Law}(\mathbb{W},u)$. The equality $\inf_{Z\in\hat{\mathfrak{H}}^a}F^G(Z)=\inf_{\mu\in\mathcal{X}}\check{F}^G(\mu)$ follows directly from the definition of $\mathcal{X}$. We have that $\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^G(\mu)=\min_{\mu\in\bar{\mathcal{X}}}\check{F}^G(\mu)$ again from the admissibility of G which ensures that sequences $(\mu_n)_n\subseteq\mathcal{X}$ of approximate minimisers of $\check{F}^G$ converge in $\bar{\mathcal{X}}$ and that any limit μ^G is a minimiser. The first and second order conditions are then consequences of the differentiability of $\check{F}^G$. It remains to show

$$\inf_{\mu\in\mathcal{X}}\check{F}^G(\mu)=\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^G(\mu).$$

We first show this statement for bounded G. Since $\mathcal{X}\subseteq\bar{\mathcal{X}}$ and, by Lemma 16 and the equalities proven above, $\inf_{\mu\in\mathcal{X}}\check{F}^G(\mu)=-\log\mathbb{E}[e^{-G(W)}]$, it is enough to show that

$$\check{F}^G(\mu)\geqslant -\log\mathbb{E}[e^{-G(W)}],\qquad \mu\in\bar{\mathcal{X}}.$$

Fix $\mu\in\bar{\mathcal{X}}$ with $\mathbb{E}_\mu[\mathcal{E}(Z)]<\infty$ (otherwise there is nothing to prove) and let $(\mathcal{F}_t)_{t\in[0,1]}$ be the filtration generated by $(\mathbb{W}_s,Z_s)_{s\leqslant t}$ on $\mathfrak{S}\times\hat{\mathfrak{H}}_w$. We claim that under μ the process $\mathbb{W}$ is still an $(\mathcal{F}_t)$-Brownian martingale, i.e. that $\mathbb{W}_t-\mathbb{W}_s$ is independent of $\mathcal{F}_s$ for $s\leqslant t$. Indeed, let $\mu_n=\mathrm{Law}_{\mathbb{P}}(\mathbb{W},Z^n)\in\mathcal{X}$ with $\mu_n\to\mu$, let ψ be bounded and continuous and let Φ be a bounded continuous function of finitely many of the variables $\mathbb{W}_r,\langle Z_r,g\rangle$ with $r\leqslant s$ and g smooth with compact support (these are continuous on $\mathfrak{S}\times\hat{\mathfrak{H}}_w$ and generate $\mathcal{F}_s$). Since Z^n is adapted to the filtration of $\mathbb{W}$ we have

$$\mathbb{E}_{\mu_n}[\psi(\mathbb{W}_t-\mathbb{W}_s)\Phi]=\mathbb{E}[\psi(\mathbb{W}_t-\mathbb{W}_s)]\mathbb{E}_{\mu_n}[\Phi],$$

and this identity passes to the limit $n \to \infty$ by weak convergence, proving the claim. As a consequence, under μ the process $X := (1-\Delta)^{1/2}\mathbb{W}$ is an $(\mathcal{F}_t)$-cylindrical Brownian motion and, since Z is $(\mathcal{F}_t)$-adapted and absolutely continuous in time, $\dot{Z}$ admits an $(\mathcal{F}_t)$-progressive version, so that

$$(1-\Delta)^{1/2}(\mathbb{W}+Z)_t = X_t + \int_0^t u_s \mathrm{d}s, \qquad u_s := (1-\Delta)^{1/2}\dot{Z}_s, \qquad \frac{1}{2}\mathbb{E}_\mu \int_0^1 \|u_s\|_{L^2}^2 \mathrm{d}s = \mathbb{E}_\mu[\mathcal{E}(Z)].$$

The entropy bound for Brownian motions with adapted drift (see e.g. [42]), which holds for drifts progressively measurable with respect to any filtration for which the Brownian motion is a Brownian motion, gives then

$$H(\mathrm{Law}_\mu(\mathbb{W}+Z)|\mathrm{Law}_{\mathbb{P}}(\mathbb{W})) \leqslant \mathbb{E}_\mu[\mathcal{E}(Z)].$$

By the Gibbs variational principle (for bounded G) we conclude

$$-\log \mathbb{E}[e^{-G(W)}] \leqslant \mathbb{E}_\mu[G(W+Z)] + H(\mathrm{Law}_\mu(\mathbb{W}+Z)|\mathrm{Law}_{\mathbb{P}}(\mathbb{W})) \leqslant \check{F}^G(\mu),$$

which is the claim.

We now remove the restriction that G is bounded. To this end let $G^N = (G \wedge N) \vee (-N)$. Then we have already established that

$$-\log \mathbb{E}[e^{-G^N(W)}] = \inf_{\mu \in \mathcal{X}} \check{F}^{G^N}(\mu) = \min_{\mu \in \bar{\mathcal{X}}} \check{F}^{G^N}(\mu).$$

Now let ν be a minimiser of $\check{F}^G$ in $\bar{\mathcal{X}}$ (existence of this was established above). Then

$$\begin{aligned}
\inf_{\mu \in \bar{\mathcal{X}}} \mathbb{E}_\mu[G(W+Z)+\mathcal{E}(Z)] &= \mathbb{E}_\nu[G(W+Z)+\mathcal{E}(Z)] \\
&= \lim_{N\to\infty} \mathbb{E}_\nu[G^N(W+Z)+\mathcal{E}(Z)] \\
&\geqslant \lim_{N\to\infty} \inf_{\mu\in\bar{\mathcal{X}}} \mathbb{E}_\mu[G^N(W+Z)+\mathcal{E}(Z)] \\
&= \lim_{N\to\infty} -\log \mathbb{E}[e^{-G^N(W)}] \\
&= -\log \mathbb{E}[e^{-G(W)}] \\
&= \inf_{\mu\in\mathcal{X}} \mathbb{E}_\mu[G(W+Z)+\mathcal{E}(Z)]
\end{aligned}$$

where we have used dominated convergence. □

Remark 23. Note that

$$\begin{aligned}
\check{F}^G(\mu^{\sigma K}) - \check{F}^G(\mu) &= \mathbb{E}_\mu[G(W+Z+\sigma K) - G(W+Z) + \mathcal{E}(Z+\sigma K) - \mathcal{E}(Z)] \\
&= \mathbb{E}_\mu[G(W+Z+\sigma K) - G(W+Z) + 2\sigma\mathcal{E}(Z,K) + \sigma^2\mathcal{E}(K)].
\end{aligned}$$

If G is differentiable in the direction of H^1 we have

$$\nabla \check{F}^G(\mu)(K) = \mathbb{E}_\mu[\nabla G(W+Z)\cdot K + 2\mathcal{E}(Z,K)].$$

If G is twice differentiable we have also

$$\nabla^2 \check{F}^G(\mu)(K,K') = \mathbb{E}_\mu[\nabla^2 G(W+Z)\cdot(K,K') + 2\mathcal{E}(K,K')].$$

The consequences of the existence of this minimiser are clarified by the following lemmas.

Lemma 24. *If V and $f+V$ are admissible and μ^{f+V}, μ^V are minimisers of $\check{F}^{f+V}$ and $\check{F}^V$ respectively, then we have*

$$\mathbb{E}_{\mu^{f+V}}[f(W+Z)] \leqslant \mathcal{W}^V(f) \leqslant \mathbb{E}_{\mu^V}[f(W+Z)]. \tag{18}$$

Proof. Since V and $f+V$ are admissible we have proven that

$$\mathcal{W}^V(f) = \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{f+V}(\mu) - \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{V}(\mu) = \check{F}^{f+V}(\mu^{f+V}) - \check{F}^{V}(\mu^{V}). \tag{19}$$

Eq. (19) implies in turn

$$\check{F}^{f+V}(\mu^{f+V}) - \check{F}^{V}(\mu^{f+V}) \leqslant \mathcal{W}^V(f) \leqslant \check{F}^{f+V}(\mu^{V}) - \check{F}^{V}(\mu^{V})$$

and since

$$\check{F}^{f+V}(\mu^{g+V}) - \check{F}^{V}(\mu^{g+V}) = \mathbb{E}_{\mu^{g+V}}[f(W+Z) + V(W+Z) + \mathcal{E}(Z)]$$
$$-\mathbb{E}_{\mu^{g+V}}[V(W+Z) + \mathcal{E}(Z)] = \mathbb{E}_{\mu^{g+V}}[f(W+Z)]$$

for any g (in particular admissible g), we obtain eq. (18). □

Lemma 25. *Let $V:\mathfrak{S}\to\mathbb{R}$ be admissible. Then for all $f:\mathfrak{S}\to\mathbb{R}$ continuous and bounded $f+V$ is admissible. Let μ^{f+V} be a minimiser for $\check{F}^{f+V}$, then for all $\alpha\in\mathbb{R}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f) = \mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]. \tag{20}$$

Proof. The admissibility of $f+V$ is easy to obtain given that f is bounded and continuous and V is admissible. It is clear that $\alpha\mapsto\mathcal{W}^V(\alpha f)$ is differentiable for bounded f, we need only to establish the representation of this derivative via the variational minimiser $\mu^{\alpha f+V}$. For any $\gamma\geqslant 0$ we have, by direct computation

$$\mathcal{W}^V((\alpha+\gamma)f) - \mathcal{W}^V(\alpha f) \leqslant \check{F}^{(\alpha+\gamma)f+V}(\mu^{\alpha f+V}) - \check{F}^{\alpha f+V}(\mu^{\alpha f+V}) = \gamma\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)],$$

and

$$\mathcal{W}^V(\alpha f) - \mathcal{W}^V((\alpha-\gamma)f) \geqslant \check{F}^{\alpha f+V}(\mu^{\alpha f+V}) - \check{F}^{(\alpha-\gamma)f+V}(\mu^{\alpha f+V}) = \gamma\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)].$$

By differentiability of $\mathcal{W}^V(\alpha f)$ we have

$$\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)] \geqslant \lim_{\gamma\to 0+}\frac{\mathcal{W}^V((\alpha+\gamma)f) - \mathcal{W}^V(\alpha f)}{\gamma} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f) = \lim_{\gamma\to 0+}\frac{\mathcal{W}^V(\alpha f) - \mathcal{W}^V((\alpha-\gamma)f)}{\gamma} \geqslant \mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]$$

from which we can conclude. □

There are several interesting consequences of this formula. First of all, integrating (20) in α, we have

$$\mathcal{W}^V(f) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)\mathrm{d}\alpha = \int_0^1 \mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]\mathrm{d}\alpha, \tag{21}$$

which is a representation of $\mathcal{W}^V(f)$ alternative to the variational formula. On the other hand, taking $\alpha=0$ in (20) we obtain

$$\int f(\phi)\theta^V(\mathrm{d}\phi) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\mathcal{W}^V(\alpha f) = \mathbb{E}_{\mu^V}[f(W_1+Z_1)].$$

Theorem 26. *The law of W_1+Z_1 under μ^V is indeed θ^V. In particular for all $f:\mathfrak{S}\to\mathbb{R}$ continuous and bounded*

$$\mathcal{W}^V(f) = -\log\int e^{-f(\phi)}\theta^V(\mathrm{d}\phi) = -\log\mathbb{E}_{\mu^V}[e^{-f(W_1+Z_1)}].$$

This last formula naturally calls for a variational description of the average $\mathbb{E}_{\mu^V}[e^{-f(W_1+Z_1)}]$ with respect to the base measure μ^V. This extension of the BD formula to the interacting field $W+Z$ is the subject of the next lemma.

Let $\mathcal{E}(Z,K)$ be the symmetric bilinear form associated to $\mathcal{E}(Z)$, in such a way that

$$\mathcal{E}(Z+K)=\mathcal{E}(Z)+\mathcal{E}(K)+2\mathcal{E}(Z,K).$$

Lemma 27. *Let V be admissible and denote by $\mu^V\in\bar{\mathcal{X}}$ a minimiser of $\check{F}^V$. Then we have the following direct variational description of μ^V:*

$$\mathcal{W}^V(f)=-\log\mathbb{E}_{\mu^V}[e^{-f(W+Z)}]=\inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^V)}\check{G}^{f,V}(\upsilon)$$

with

$$\check{G}^{f,V}(\upsilon):=\mathbb{E}_\upsilon[f(W+Z+K)]+\check{H}^V(\upsilon),$$

and

$$\check{H}^V(\upsilon):=\mathbb{E}_\upsilon[V(W+Z+K)-V(W+Z)+2\mathcal{E}(Z,K)+\mathcal{E}(K)].$$

Moreover ν is a minimiser of $\check{F}^V$ iff $\check{H}^V(\upsilon)\geqslant 0$ for all $\upsilon\in\bar{\mathcal{Y}}(\nu)$ and if V is differentiable in the directions of H^1 we also have the weak forward–backward SDE (wFBSDE):

$$\mathbb{E}_\upsilon[\nabla V(W+Z)\cdot K+2\mathcal{E}(Z,K)]=0,\qquad \upsilon\in\bar{\mathcal{Y}}(\nu),\tag{22}$$

and $\check{H}^V(\upsilon)$ takes the alternative form

$$\check{H}^V(\upsilon)=\mathbb{E}_\upsilon[V(W+Z+K)-V(W+Z)-\nabla V(W+Z)\cdot K+\mathcal{E}(K)].$$

Proof. Recall that

$$\begin{aligned}\mathcal{W}^V(f) &= \inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{f+V}-\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^V\\ &= \inf_{\mu\in\bar{\mathcal{X}}}\mathbb{E}_\mu[f(W+Z)+V(W+Z)+\mathcal{E}(Z)]-\mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)]\\ &= \inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^V)}\mathbb{E}_\upsilon[f(W+Z+K)+V(W+Z+K)+\mathcal{E}(Z+K)]\\ &\quad -\mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)]\end{aligned}$$

where in the last line we used Lemma 19 to justify the change of variables in the minimisation problem. Note that for any $\upsilon\in\bar{\mathcal{Y}}(\mu^V)$ we have that

$$\mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)]=\mathbb{E}_\upsilon[V(W+Z)+\mathcal{E}(Z)]$$

so

$$\mathcal{W}^V(f)=\inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^V)}\Big\{\mathbb{E}_\upsilon[f(W+Z+K)]+\check{H}^V(\upsilon)\Big\}.$$

The fact that $\check{H}^V(\upsilon)\geqslant 0$ iff $\mathrm{Law}_\upsilon(W,Z)=\nu$ is a minimiser of $\check{F}^V$ follows from its definition, indeed note that

$$\check{H}^V(\upsilon)=\check{F}^V(\mathrm{Law}_\upsilon(W,Z+K))-\check{F}^V(\mathrm{Law}_\upsilon(W,Z)).$$

If V is differentiable in the direction of H^1 we have that for any minimiser ν of $\check{F}^V$ and any $\upsilon\in\bar{\mathcal{Y}}(\nu)$:

$$0=\nabla\check{F}^V(\nu)(K)=\mathbb{E}_\upsilon[\nabla V(W+Z)\cdot K+2\mathcal{E}(Z,K)]$$

which also allows us to simplify the expression of $\check{H}^V$. □

4 The quartic interaction

In this section we will consider the general setting of Section 3.2 in the case where the interaction potential V is given by

$$V_\Lambda(\phi) := \lambda \int_\Lambda [\![\phi^4]\!],$$

where $\Lambda \subseteq \mathbb{R}^2$ is a measurable bounded set (of sufficient regularity, see below) and $[\![\phi^n]\!]$ denotes the n-th Wick power of the Gaussian free field ϕ [24] and $\lambda > 0$ an arbitrary fixed constant. Let $\mathcal{W}_\Lambda := \mathcal{W}^{V_\Lambda}$ and $\theta_\Lambda := \theta^{V_\Lambda}$.

The measure θ_Λ on $\mathcal{S}'(\mathbb{R}^2)$ is called the φ^4_2 Euclidean quantum field with a finite volume cutoff and we will discuss in detail the problem of the removal of the finite volume cutoff, i.e. the limit $\Lambda \nearrow \mathbb{R}^2$. In particular:

1. using the wFBSDE (13) we will derive a-priori bounds for minimisers strong enough to give tightness of the family $(\theta_\Lambda)_\Lambda$ as $\Lambda \nearrow \mathbb{R}^2$;
2. we will describe any accumulation point via a formula for its Laplace transform involving the wFBSDE obtained in the previous section.

Unfortunately, establishing rigorously the form (15) of the limiting variational problem requires certain locality properties of the solutions to the wFBSDE (13) in infinite volume. This is usually possible when V is convex. Ultraviolet renormalisation in φ^4_2 however spoils any "easy" convexity of the potential and it is not clear how to gain sufficient control of (13) in this case. Similar issues appear in other methods of stochastic quantisation as soon as the infinite volume limit is considered for $P(\varphi)_2$ Euclidean quantum fields [26, 27, 2]. Despite these difficulties, we show that the wFBSDE is still a useful tool which can be used effectively to prove the large deviations of the φ^4_2 measure in the semiclassical limits. Let us remark that Lacoin, Rhodes and Vargas obtained large deviations results in the case of the Liouville model in [40, 41].

Remark 28. Let us note that all the discussion in this section can be extended to general $P(\varphi)_2$ models with a positive coefficient in the highest order polynomial. Moreover also $O(N)$ invariant vector models could be handled in similar ways. Convexity of the classical potential (i.e. before Wick ordering) is only needed in Section 4.1 below to prove large deviations.

We will need the following enhancement $\mathbb{W}$ of the extended GFF W. Let

$$\mathbb{W}^2_t := [\![W_t^2]\!], \quad \text{and} \quad \mathbb{W}^3_t := [\![W_t^3]\!],$$

where here $[\![\cdot]\!]$ denotes the Wick powers with respect to the Gaussian field W. Then

$$\mathbb{W} := (W, \mathbb{W}^2, \mathbb{W}^3) \in L^p(\mathbb{P}, \mathfrak{S})$$

for all $p \geqslant 1$, where $\mathfrak{S} := C([0,1], (C^{-\delta}(\rho^\delta))^3)$ for a fixed $\delta > 0$ which can be chosen as small as we like. By abuse of notation we will denote also $\mathbb{W} = (W, \mathbb{W}^2, \mathbb{W}^3)$ the canonical process on $\mathfrak{S}$.

In the following we will always consider a family of $(\Lambda)_{|\Lambda| \in \mathbb{R}}$ given by $\Lambda = |\Lambda|^{1/2} S$ for some fixed set S of volume 1 with C^1-boundary. This allows us to prove that $\mathbb{1}_\Lambda$ is sufficiently regular to make sense of quantities like $\int_\Lambda [\![\phi^4]\!]$, $\int_\Lambda [\![W_1^4]\!]$ and in general of products with Wick monomials of the GFF.

Lemma 29. *For any $p \geqslant 1$, $s < \frac{1}{p}$ and $\iota > \frac{1}{16}$ we have that $\mathbb{1}_\Lambda \in B^s_{p,p}(\rho^\iota)$, furthermore*

$$\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{p,p}(\rho^\iota)} < \infty.$$

Proof. We have that $\nabla \mathbb{1}_\Lambda = -\nu \mathrm{d}\sigma$ where $\mathrm{d}\sigma$ is the surface measure on $\partial\Lambda$ and ν is the outer unit normal. It is known that $\|f\|_{B^s_{p,p}(\rho)} \lesssim \|f\|_{B^{s-1}_{p,p}(\rho)} + \|\nabla f\|_{B^{s-1}_{p,p}(\rho)}$ (see e.g. [27], Lemma A.5). This implies that for any $s < 1$, $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{1,1}(\rho^\iota)} < \infty$. Clearly $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{L^\infty(\rho^\iota)} < \infty$ and by Besov space interpolation (see e.g. [27] Lemma A.3)

$$\|\mathbb{1}_\Lambda\|_{B^{\theta s}_{p,p}(\rho^\iota)} \lesssim \|\mathbb{1}_\Lambda\|^\theta_{B^s_{1,1}(\rho^\iota)} \|\mathbb{1}_\Lambda\|^{1-\theta}_{L^\infty(\rho^\iota)}$$

where

$$\frac{1}{p} = \theta. \qquad \square$$

Lemma 30. *The functional $\check{F}^\Lambda := \check{F}^{V_\Lambda}$ is admissible and for all $\mu \in \bar{\mathcal{X}}$*

$$\check{F}^\Lambda(\mu) = \check{F}^{\mathcal{V}_\Lambda}(\mu) = \mathbb{E}_\mu\left[\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z)\right],$$

with

$$\mathcal{V}_\Lambda(\mathbb{W}, Z) := 4\lambda \int_\Lambda \mathbb{W}_1^3 Z_1 + 6\lambda \int_\Lambda \mathbb{W}_1^2 (Z_1)^2 + 4\lambda \int_\Lambda W_1 (Z_1)^3 + \lambda \int_\Lambda (Z_1)^4. \tag{23}$$

Proof. It is known that

$$\mathbb{E}\big[e^{-q\lambda \int_\Lambda [\![W^4]\!]}\big] < \infty,$$

for any $q > 0$ (see e.g. [24] or [5]) and

$$\mathbb{E}[|V_\Lambda(W)|^p] \leqslant \mathbb{E}\Big[\Big| \|[\![W^4]\!]\|_{B^{-\delta}_{\infty,\infty}(\rho^\delta)}\Big|^p\Big] < \infty,$$

for some $\delta > 0$, which shows that F is a tame functional. Now it remains to prove the equality in (23). For this it is enough to show that for Z such that $\|Z\|_{H^1(\rho^{1/2})} < \infty$ μ-almost surely, we have

$$\int_\Lambda [\![(W_1 + Z_1)^4]\!] = \int_\Lambda [\![W_1^4]\!] + 4\int_\Lambda [\![W_1^3]\!] Z_1 + 6\int_\Lambda [\![W_1^2]\!] (Z_1)^2 + 4\int_\Lambda W_1 (Z_1)^3 + \int_\Lambda (Z_1)^4.$$

To see this, recall that if $C_c^\infty \ni \psi_\varepsilon \to \delta_0$, is a standard mollifier, $W^\varepsilon = \psi_\varepsilon * W$ and $a_\varepsilon = 6\mathbb{E}[(W_1^\varepsilon(0))^2]$, $b_\varepsilon = -3\mathbb{E}[(W_1^\varepsilon(0))^2]^2$

$$\begin{aligned}
\int_\Lambda [\![(W_1 + Z_1)^4]\!] &= \lim_{\varepsilon \to 0} \int_\Lambda (\psi_\varepsilon * (W_1 + Z_1))^4 - a_\varepsilon \int_\Lambda (\psi_\varepsilon * (W_1 + Z_1))^2 - b_\varepsilon |\Lambda| \\
&= \lim_{\varepsilon \to 0} \int_\Lambda (W_1^\varepsilon)^4 - a_\varepsilon (W_1^\varepsilon)^2 - b_\varepsilon + 4\int_\Lambda \Big((W_1^\varepsilon)^3 - \frac{a_\varepsilon}{2} W_1^\varepsilon\Big) Z_1^\varepsilon \\
&\quad + 6\int_\Lambda \Big((W_1^\varepsilon)^2 - \frac{a_\varepsilon}{6}\Big)(Z_1^\varepsilon)^2 + 4\int_\Lambda W_1^\varepsilon (Z_1^\varepsilon)^3 + \int_\Lambda (Z_1^\varepsilon)^4
\end{aligned}$$

where we have denoted $Z_1^\varepsilon = \psi_\varepsilon * Z_1$. Now it is well known that almost surely in $C^{-\delta}(\rho^\delta)$ we have $W_1^\varepsilon \to W_1$ and

$$(W_1^\varepsilon)^4 - a_\varepsilon (W_1^\varepsilon)^2 - b_\varepsilon \to [\![W_1^4]\!], \quad \Big((W_1^\varepsilon)^3 - \frac{a_\varepsilon}{2} W_1^\varepsilon\Big) \to [\![W_1^3]\!], \quad \Big((W_1^\varepsilon)^2 - \frac{a_\varepsilon}{6}\Big) \to [\![W_1^2]\!].$$

Furthermore $Z^\varepsilon \to Z$ in H^1 which implies $(Z_1^\varepsilon)^2 \to Z_1^2, (Z_1^\varepsilon)^3 \to Z_1^3$ in $W^{2\delta,1}$ for some $\delta > 0$, which gives the statement. From this calculation it also follows that $(\mathbb{W}, Z) \mapsto \mathcal{V}_\Lambda(\mathbb{W}, Z)$ is continuous on $\mathfrak{S} \times \mathfrak{H}_w$.

To check the lower semicontinuity of $\check{F}^{\mathcal{V}_\Lambda}$ recall from Section 3 of [5] that

$$\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z) \geqslant -Q(\mathbb{W})$$

with $Q(\mathbb{W}) = C(\|\mathbb{W}\|_{\mathfrak{S}}^p + 1)$ for some $p \geqslant 1$. Furthermore observe that the function

$$(\mathbb{W}, Z) \mapsto \mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z)$$

is lower semicontinuous on $\mathfrak{S} \times \mathfrak{H}_w$. Since the first marginal in $\bar{\mathcal{X}}$ is fixed, the Portmanteau theorem gives, for any $\mu_n \to \mu$ in $\bar{\mathcal{X}}$:

$$\begin{aligned} \liminf_{n\to\infty} \check{F}^{\mathcal{V}_\Lambda}(\mu_n) &= \liminf_{n\to\infty} \mathbb{E}_{\mu_n}[\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z) + Q(\mathbb{W})] - \mathbb{E}_{\mu_n}[Q(\mathbb{W})] \\ &\geqslant \mathbb{E}_{\mu}[\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z) + Q(\mathbb{W})] - \mathbb{E}_{\mu}[Q(\mathbb{W})] \\ &= \check{F}^{\mathcal{V}_\Lambda}(\mu). \end{aligned}$$

Finally, by a slight adaptation of the proof of Lemma 34 below we see that there exists $\delta > 0$ and $C_{\delta,|\Lambda|} < \infty$ such that

$$\check{F}^{\mathcal{V}_\Lambda}(\mu) \geqslant \delta\, \mathbb{E}_\mu[\mathcal{E}(Z) + \lambda \|\mathbb{1}_\Lambda Z_1\|_{L^4}^4] - C_{\delta,|\Lambda|},$$

for all $\mu \in \bar{\mathcal{X}}$, from which the strong coercivity condition (16) follows. □

The admissibility of $\check{F}^\Lambda$ implies the variational representation for $\mathcal{W}_\Lambda$ and the existence of minimisers $\mu^{f,\Lambda}$ of $\check{F}^{f,\Lambda} := \check{F}^{f+\mathcal{V}_\Lambda}$ with the corresponding wFBSDE equation.

Lemma 31. *For any bounded and continuous $f : \mathfrak{S}_0 \to \mathbb{R}$ there exists a minimiser $\mu^{f,\Lambda} \in \bar{\mathcal{X}}$ of $\check{F}^{f,\Lambda} := \check{F}^{f+\mathcal{V}_\Lambda}$. Moreover if f is differentiable in the directions of H^1 any minimiser μ satisfies the wFBSDE equation*

$$\begin{aligned} &\mathbb{E}_\mu\Big[4\lambda \int_\Lambda Z^3 K + 2\mathcal{E}(Z, K)\Big] \\ &\qquad = -\mathbb{E}_\mu\Big[\nabla f(W + Z) \cdot K + \lambda \int_\Lambda (4\,\mathbb{W}^3 + 12\,\mathbb{W}^2 Z + 12\, W Z^2) K\Big] \end{aligned} \tag{24}$$

for any admissible control K.

Proof. As we already remarked the existence of $\mu^{f,\Lambda}$ follows from the admissibility of $\check{F}^{f,\Lambda}$. The functional $Z \mapsto \mathcal{V}_\Lambda(\mathbb{W}, Z)$ is differentiable in H^1 and

$$\nabla \mathcal{V}_\Lambda(\mathbb{W}, Z) \cdot K = 4\lambda \int_\Lambda \mathbb{W}_1^3 K_1 + 12\lambda \int_\Lambda \mathbb{W}_1^2 Z_1 K_1 + 12\lambda \int_\Lambda W_1 (Z_1)^2 K_1 + 4\lambda \int_\Lambda (Z_1)^3 K_1.$$

Provided $Z \mapsto f(W + Z)$ is also differentiable, we have

$$0 = \nabla \check{F}^{f,\Lambda}(\mu)(K) = \mathbb{E}_\mu[\nabla f(W + Z) \cdot K + \nabla \mathcal{V}_\Lambda(\mathbb{W}, Z) \cdot K + 2\mathcal{E}(Z, K)]$$

which is (24). □

Remark 32. Note that $Z \mapsto \mathcal{V}_\Lambda(\mathbb{W}, Z)$ is twice differentiable, with

$$\nabla^2 \mathcal{V}_\Lambda(\mathbb{W}, Z)(K, K') = 12\,\lambda \int_\Lambda \mathbb{W}_1^2 K_1' K_1 + 24\,\lambda \int_\Lambda W_1 Z_1 K_1' K_1 + 12\,\lambda \int_\Lambda (Z_1)^2 K_1' K_1.$$

Therefore if f is twice differentiable then we have also the second order condition:

$$0 \leqslant \nabla^2 \check{F}^{f,\Lambda}(\mu)(K, K') = \mathbb{E}_\mu[\nabla^2 f(W + Z)(K, K') + \nabla^2 \mathcal{V}_\Lambda(\mathbb{W}, Z)(K, K') + 2\mathcal{E}(K', K)]$$

for any minimiser μ and any pair K, K' of admissible controls.

We record also the normalised variational representation for $\mathcal{W}_\Lambda(f)$.

Lemma 33. *Denote by $\mu^\Lambda \in \bar{\mathcal{X}}$ a minimiser of $\check{F}^{0,\Lambda}$. Then we have*

$$\mathcal{W}_\Lambda(f) = \inf_{\upsilon \in \bar{\mathcal{Y}}(\mu^\Lambda)} \{\mathbb{E}_\upsilon[f(W+Z+K)] + \check{H}^\Lambda(\upsilon)\}$$

with

$$\check{H}^\Lambda(\upsilon) := \mathbb{E}_\upsilon\Big[\int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s(1-\Delta)\,\dot{K}_s \mathrm{d}s + \frac{1}{2}\int_0^1 \int_{\mathbb{R}^2} \dot{K}_s(1-\Delta)\,\dot{K}_s \mathrm{d}s\Big]$$

$$+\mathbb{E}_\upsilon\Big[4\lambda\int_\Lambda [\![(W+Z)^3]\!]\,K + 6\lambda\int_\Lambda [\![(W+Z)^2]\!]\,K^2 + 4\lambda\int_\Lambda [\![(W+Z)]\!]\,K^3 + \lambda\int_\Lambda K^4\Big].$$

Proof. This follows from Lemma 27 by direct computation. □

Next we show that the wFBSDE (24) gives us an a-priori bound uniform in $|\Lambda|$ on the minimiser in terms of weighted norms of $\mathbb{W}$. For this we first need some analytic bounds.

Lemma 34. *Assume that, for $\delta > 0$ small enough, either*

a) $\mathrm{supp}(\nabla f(W+Z)) \subset \Lambda$ *and*

$$\|\nabla f(W+Z)\|_{L^{4/3}(\rho^{3/4})} \leqslant \delta \|\mathbb{1}_\Lambda (W+Z)\|^3_{W^{-\varepsilon,4}(\rho^{1/4})} + C,$$

b)

$$\|\nabla f(W+Z)\|_{H^{-1}(\rho^{1/2})} \leqslant \delta \|W+Z\|_{H^{-\varepsilon}(\rho^{1/2})}.$$

Then there exists a constant $C < \infty$ such that

$$\begin{aligned} &\mathbb{E}\Big|\nabla f(W+Z)\cdot(Z\rho) + \lambda\int_\Lambda (4\mathbb{W}^3\,\rho\,Z + 12\mathbb{W}^2\,\rho\,Z^2 + 12\,W\rho\,Z^3)\Big| \\ \leqslant\ & \delta\,\mathbb{E}\Big(\lambda\int_\Lambda \rho\,Z^4 + \mathcal{E}(\rho^{1/2}Z)\Big) + C. \end{aligned}$$

Proof. First observe that we have that $\int_\Lambda \rho\,Z^4 = \|\mathbb{1}_\Lambda \rho^{1/4} Z\|^4_{L^4}$, and $\mathcal{E}(\rho^{1/2}Z) \geqslant C\|\rho^{1/2}Z\|^2_{H^1}$, for instance by fundamental theorem of calculus. By our assumptions on f we have, in the first case

$$\begin{aligned} |\nabla f(W+Z)\cdot(Z\rho)| &\leqslant \|\nabla f(W+Z)\|_{L^{4/3}(\rho^{3/4})}\|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})} \\ &\leqslant \delta\|\mathbb{1}_\Lambda(W+Z)\|^3_{W^{-\varepsilon,4}(\rho^{1/4})}\|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})} + C \\ &\leqslant C\delta\big(\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})} + \|\mathbb{1}_\Lambda W\|^3_{W^{-\varepsilon,4}(\rho^{1/4})}\|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})}\big) + C \\ &\leqslant 2\,\delta\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})} + C\|\mathbb{1}_\Lambda W\|^4_{W^{-\varepsilon,4}(\rho^{1/4})} + C. \end{aligned}$$

While in the second case

$$\begin{aligned} |\nabla f(W+Z)\cdot(Z\rho)| &\leqslant \|\nabla f(W+Z)\|_{H^{-1}(\rho^{1/2})}\|Z\|_{H^1(\rho^{1/2})} \\ &\leqslant \delta\|W+Z\|_{H^{-\varepsilon}(\rho^{1/2})}\|Z\|_{H^1(\rho^{1/2})} \\ &\leqslant \delta\|W\|_{H^{-\varepsilon}(\rho^{1/2})}\|Z\|_{H^1(\rho^{1/2})} + \delta\|Z\|^2_{H^1(\rho^{1/2})} \\ &\leqslant \|W\|^2_{H^{-\varepsilon}(\rho^{1/2})} + 2\delta\|Z\|^2_{H^1(\rho^{1/2})}. \end{aligned}$$

For the second term, by Cauchy–Schwarz we obtain

$$\left|\int_\Lambda \rho\, \mathbb{W}^3\, Z\right| \leqslant C_\delta \|\mathbb{1}_\Lambda \mathbb{W}^3\|^2_{H^{-1}(\rho^{1/2})} + \delta \|Z\|^2_{H^1(\rho^{1/2})}.$$

For the third term, we use Lemma A.7 from [27] to get

$$\begin{aligned}\left|\int_\Lambda \mathbb{W}^2\, \rho\, Z^2\right| &\leqslant \|\rho^{1/8}\, \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{7/8}\, \mathbb{1}_\Lambda Z^2\|_{B^\varepsilon_{1,1}} \\ &\leqslant \|\rho^{1/8}\, \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{3/8}\, \mathbb{1}_\Lambda Z\|_{L^2} \|\rho^{1/2} Z\|_{H^{2\varepsilon}} \\ &\leqslant C \|\rho^{1/8}\, \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{1/4} \mathbb{1}_\Lambda\, Z\|_{L^4} \|\rho^{1/2} Z\|_{H^{2\varepsilon}} \\ &\leqslant C_\delta \|\rho^{1/8}\, \mathbb{W}^2\|^4_{C^{-\varepsilon}} + \delta(\|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^4_{L^4} + \|\rho^{1/2} Z\|^2_{H^{2\varepsilon}}).\end{aligned}$$

For the final term we proceed similarly. By Lemma 29 we have $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{p,p}(\rho^\iota)} < \infty$ for any $s < 1/p$ and any $\iota > 1/16$. Then with $\varepsilon < 1/16$, for any $\iota \in (1/16, 1/8)$ and writing $\mathbb{1}_\Lambda Z^3 = (\mathbb{1}_\Lambda Z)^3$, we have by fractional Leibniz rule

$$\begin{aligned}\left|\int_\Lambda \rho\, W Z^3\right| &\leqslant \|\rho^{1/8-\iota}\, W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{3/8+\iota} \mathbb{1}_\Lambda Z\|_{H^\varepsilon} \|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^2_{L^4} \\ &\lesssim \|\rho^{1/8-\iota}\, W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{5/8+3\iota}\, \mathbb{1}_\Lambda Z\|^{1/3}_{W^{3\varepsilon,1}} \|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^{8/3}_{L^4} \\ &\leqslant C \|\rho^{1/8-\iota}\, W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{1/2}\, Z\|^{1/3}_{H^1} \|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^{8/3}_{L^4} \\ &\leqslant C_\delta \|\rho^{1/8-\iota}\, W\|^6_{\mathcal{C}^{-\varepsilon}} + \delta(\|\rho^{1/2}\, Z\|^2_{H^1} + \|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^4_{L^4}).\end{aligned}$$

In the second line we used the interpolation inequality of Proposition 11 with $\theta = 1/3$ (the weights being commuted with the derivatives as explained above)

$$\|\rho^{3/8+\iota} \mathbb{1}_\Lambda Z\|_{H^\varepsilon} \lesssim \|\rho^{5/8+3\iota} \mathbb{1}_\Lambda Z\|^{1/3}_{W^{3\varepsilon,1}} \|\rho^{1/4}\, \mathbb{1}_\Lambda Z\|^{2/3}_{L^4},$$

(note that $\frac{1}{2} = \frac{1/3}{1} + \frac{2/3}{4}$ and $\frac{3}{8} + \iota = \frac{1}{3}\left(\frac{5}{8} + 3\iota\right) + \frac{2}{3} \cdot \frac{1}{4}$), which keeps the indicator $\mathbb{1}_\Lambda$ on every factor Z measured in L^4. In the third line we used the fractional Leibniz rule, Lemma 29 with $p = 2$, $s = 4\varepsilon < 1/2$, and $5/8 + 2\iota \geqslant 1/2$:

$$\|\rho^{5/8+3\iota} \mathbb{1}_\Lambda Z\|_{W^{3\varepsilon,1}} \lesssim \|\mathbb{1}_\Lambda\|_{B^{4\varepsilon}_{2,2}(\rho^\iota)} \|\rho^{5/8+2\iota}\, Z\|_{H^{4\varepsilon}} \lesssim \|\rho^{1/2}\, Z\|_{H^1},$$

uniformly in $|\Lambda|$, while the last line is Young's inequality with exponents $(6, 6, 3/2)$. We can choose $\iota = 1/8 - \varepsilon > 1/16$ to get the claim. □

Theorem 35. *Assume that f satisfies the conditions of Lemma 34. There exists a constant C independent of $|\Lambda|$ such that, for any minimiser $\mu^{f,\Lambda} \in \bar{\mathcal{X}}$ of $\check{F}^{f,\Lambda}(\mu)$ we have*

$$\mathbb{E}_{\mu^{f,\Lambda}}\left[2\lambda \int_\Lambda \rho\, Z^4 + \mathcal{E}(\rho^{1/2} Z)\right] \leqslant C. \tag{25}$$

Proof. Note that the map $Z \mapsto \rho\, Z$ is continuous from $\mathfrak{H}$ to $\mathfrak{H}$. Take $K = \rho\, Z$ in (24) to get that any minimiser $\mu^{f,\Lambda} \in \bar{\mathcal{X}}$ satisfies

$$\begin{aligned}&\mathbb{E}_{\mu^{f,\Lambda}}\left[4\lambda \int_\Lambda \rho\, Z^4 + \int_0^1 \int_{\mathbb{R}^2} ((1-\Delta)^{1/2}\, \rho^{1/2}\, \dot{Z}_s)^2 \mathrm{d}s\right] \\ = \;& -\mathbb{E}_{\mu^{f,\Lambda}}\left[\int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s [(1-\Delta), \rho^{1/2}]\, \rho^{1/2}\, \dot{Z}_s \mathrm{d}s\right] \\ & -\mathbb{E}_{\mu^{f,\Lambda}}\left[\nabla f(W+Z)\, (Z\rho) + \lambda \int_\Lambda \rho(4\mathbb{W}^3\, Z + 12\mathbb{W}^2\, Z^2 + 12\, W Z^3)\right].\end{aligned} \tag{26}$$

From Lemma 36 below and Lemma 34 we deduce that the r.h.s. is controlled by a small fraction of the l.h.s. + a constant, from which we obtain the claim. □

Lemma 36.

$$\left|\int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s[(1-\Delta), \rho^{1/2}]\, \rho^{1/2}\, \dot{Z}_s \mathrm{d}s\right| \leqslant \varepsilon\left[\int_0^1 \|\dot{Z}_s\|^2_{H^1(\rho^{1/2})} \mathrm{d}s\right]$$

Proof. We observe that 1 obviously commutes with $\rho^{1/2}$, and

$$[\Delta, \rho^{1/2}] = (\Delta\, \rho^{1/2}) + 2\nabla\rho^{1/2} \cdot \nabla.$$

Using the assumptions (4) on the weight ρ we deduce

$$\|[\Delta, \rho^{1/2}]\, f\|_{L^2} \leqslant \varepsilon(\|\rho^{1/2}\, f\|_{L^2} + \|\rho^{1/2}\nabla f\|_{L^2}).$$ □

Let us now show that these estimates give tightness of $(\theta_\Lambda)_\Lambda$ as $|\Lambda| \to \infty$ on a suitable weighted Sobolev space.

Lemma 37. *Consider $(\theta_\Lambda)_\Lambda$ as a family of measures on $H^{-\varepsilon}(\rho)$. We have for some $\delta > 0$,*

$$\sup_\Lambda \left[\int \exp(\delta \|\mathbb{1}_\Lambda \phi\|^4_{W^{-\varepsilon,4}(\rho)}) \mathrm{d}\theta_\Lambda + \int \exp(\delta \|\phi\|^2_{H^{-\varepsilon}(\rho)}) \mathrm{d}\theta_\Lambda\right] < \infty.$$

In particular $(\theta_\Lambda)_\Lambda$ is exponentially tight on $H^{-2\varepsilon}(\rho)$ and any accumulation point θ satisfies

$$\int \exp(\delta \|\phi\|^4_{W^{-\varepsilon,4}(\rho)}) \mathrm{d}\theta < \infty.$$

Proof. Since $\rho \lesssim 1$ we have $\|\cdot\|_{H^{-\varepsilon}(\rho)} \lesssim \|\cdot\|_{H^{-\varepsilon}(\rho^{1/2})}$ and $\|\cdot\|_{W^{-\varepsilon,4}(\rho)} \lesssim \|\cdot\|_{W^{-\varepsilon,4}(\rho^{1/4})}$, so that, up to changing δ, it is enough to prove the bounds with the weights $\rho^{1/2}$ and $\rho^{1/4}$ respectively. Moreover bounded sets of $H^{-\varepsilon}(\rho^{1/2})$ are relatively compact in $H^{-2\varepsilon}(\rho)$, which gives the exponential tightness.

Let $g_N \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $g_N(x) = x$ for $|x| < N$, with g_N bounded and $0 \leqslant g_N' \leqslant 1$ (so that $0 \leqslant g_N(x) \leqslant x$ for $x \geqslant 0$). Define $f^N(\phi) = g_N\big(\delta \|\phi\|^2_{H^{-\varepsilon}(\rho^{1/2})}\big)$ we have $\nabla f^N(\phi) = (g_N)'\big(\delta \|\phi\|^2_{H^{-\varepsilon}(\rho^{1/2})}\big) 2\delta \langle D\rangle^{-\varepsilon}(\rho\langle D\rangle^{-\varepsilon}\phi)$ so, since $\rho \lesssim 1$,

$$\|\nabla f^N(W+Z)\|_{H^{-1}(\rho^{1/2})} \leqslant C\delta \|W+Z\|_{H^{-\varepsilon}(\rho^{1/2})},$$

and this bound is uniform in N. Since condition (b) of Lemma 34 only involves ∇f, it holds also for $-f^N$, and Theorem 35 applied to the minimiser $\mu^{-f^N,\Lambda}$ of $\check{F}^{-f^N,\Lambda}$ gives

$$\sup_{N,\Lambda} \mathbb{E}_{\mu^{-f^N,\Lambda}}\big[\|Z\|^2_{H^1(\rho^{1/2})}\big] < \infty.$$

Now $\log \int \exp(f^N) \mathrm{d}\theta_\Lambda = -\mathcal{W}_\Lambda(-f^N)$, so the lower bound in eq. (18) (with $-f^N$ in place of f) and $0 \leqslant f^N(\phi) \leqslant \delta \|\phi\|^2_{H^{-\varepsilon}(\rho^{1/2})}$ give

$$\log \int \exp(f^N) \mathrm{d}\theta_\Lambda \leqslant \mathbb{E}_{\mu^{-f^N,\Lambda}}[f^N(W+Z)] \leqslant 2\delta \mathbb{E}\big[\|W\|^2_{H^{-\varepsilon}(\rho^{1/2})}\big] + C\delta \mathbb{E}_{\mu^{-f^N,\Lambda}}\big[\|Z\|^2_{H^1(\rho^{1/2})}\big] \leqslant C < +\infty,$$

and taking the sup over N we get

$$\log \int \exp\big(\delta \|\phi\|^2_{H^{-\varepsilon}(\rho^{1/2})}\big) \mathrm{d}\theta_\Lambda \leqslant C$$

a fact that implies exponential tightness of $(\theta_\Lambda)_\Lambda$. For the second statement we proceed similarly. Indeed if θ is an accumulation point of $(\theta_\Lambda)_\Lambda$ then

$$\begin{aligned}\int \exp(\delta\|\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta &\leqslant \sup_{\tilde\Lambda}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta\\ &\leqslant \sup_{\tilde\Lambda}\limsup_{\Lambda\to\infty}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta_\Lambda\\ &\leqslant \sup_{\tilde\Lambda}\sup_{\Lambda\supseteq\tilde\Lambda}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta_\Lambda.\end{aligned}$$

This gives the claim provided that for $f^N(\phi)=g_N\big(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho^{1/4})}\big)$ we have

$$\sup_N \log\int \exp(f^N)\mathrm{d}\theta_\Lambda \leqslant C.$$

Indeed we compute $\nabla f^N(\phi)=(g_N)'\big(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho^{1/4})}\big)4\delta\mathbb{1}_{\tilde\Lambda}\langle\mathrm{D}\rangle^{-\varepsilon}(\rho(\langle\mathrm{D}\rangle^{-\varepsilon}(\mathbb{1}_{\tilde\Lambda}\phi))^3)$, from which, using that $\langle\mathrm{D}\rangle^{-\varepsilon}$ is bounded on $L^{4/3}$, that $\rho\lesssim 1$ and that, for $\varepsilon<1/4$, multiplication by $\mathbb{1}_{\tilde\Lambda}$ is bounded on $W^{-\varepsilon,4}(\rho^{1/4})$ uniformly in $\tilde\Lambda$, we can check that

$$\|\nabla f^N(\phi)\|_{L^{4/3}(\rho^{3/4})} \leqslant C\delta\|\rho(\langle\mathrm{D}\rangle^{-\varepsilon}(\mathbb{1}_{\tilde\Lambda}\phi))^3\|_{L^{4/3}} \leqslant C\delta\|\rho^{1/4}\langle\mathrm{D}\rangle^{-\varepsilon}(\mathbb{1}_{\tilde\Lambda}\phi)\|^3_{L^4} \leqslant C\delta\|\mathbb{1}_\Lambda\phi\|^3_{W^{-\varepsilon,4}(\rho^{1/4})},$$

and $\mathrm{supp}(\nabla f^N(\phi))\subseteq\Lambda$. Applying Lemma 34 (condition (a), which holds for $-f^N$ as well) and Theorem 35 we get that the minimiser $\mu^{-f^N,\Lambda}$ of $\check F^{-f^N,\Lambda}$ satisfies

$$\sup_{\tilde\Lambda}\sup_{\Lambda\supseteq\tilde\Lambda}\mathbb{E}_{\mu^{-f^N,\Lambda}}\big[\|\mathbb{1}_{\tilde\Lambda}Z\|^4_{L^4(\rho^{1/4})}\big]\leqslant\sup_\Lambda\mathbb{E}_{\mu^{-f^N,\Lambda}}\big[\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})}\big]=:C<\infty,$$

uniformly in N. As a consequence, the lower bound in eq. (18) (with $-f^N$ in place of f) gives, as above,

$$\log\int\exp(f^N)\mathrm{d}\theta_\Lambda \leqslant \mathbb{E}_{\mu^{-f^N,\Lambda}}[f^N(W+Z)] \leqslant 8\delta\mathbb{E}\big[\|\mathbb{1}_{\tilde\Lambda}W\|^4_{W^{-\varepsilon,4}(\rho^{1/4})}\big]+C\delta\mathbb{E}_{\mu^{-f^N,\Lambda}}\big[\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})}\big]\leqslant C'<+\infty,$$

which implies the claim. □

We have actually a more refined result: let $\mu^{f,\Lambda}$ be a minimiser for $\check F^{f,\Lambda}$. By Theorem 35 the family $(\mu^{f,\Lambda})_\Lambda$ is tight in $\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)$.

Lemma 38. *Denote by $\mu\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)$ any accumulation point of $(\mu^{f,\Lambda})_\Lambda$ as $|\Lambda|\to\infty$. Then the following equation holds:*

$$\mathbb{E}_\mu\bigg[\nabla f(W+Z)\cdot K+4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,K+2\,\mathcal{E}(Z,K)\bigg]=0. \tag{27}$$

for any $K\colon\mathfrak{S}\times\hat{\mathfrak{H}}_w\to\hat{\mathfrak{H}}_w$ continuous, adapted to the filtration generated by $(\mathbb{W}_t,Z_t)_{t\in[0,1]}$ and compactly supported.

Proof. Let the support of K be contained in $B_N(0)$ for some $N\geqslant 0$. Let $\chi\in C^\infty_c(\mathbb{R}^2)$ such that $\chi=1$ on $B_N(0)$ so that $K=\chi K$. Then Theorem 35 implies that

$$\mathbb{E}_{\mu^{f,\Lambda}}[\|\chi\,Z\|^2_{H^1}+\|\chi\,Z\|^4_{L^4}]\leqslant C,$$

where the constant depends on N but not on Λ and we have

$$\begin{aligned}&\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot K+4\,\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,K+2\,\mathcal{E}(Z,K)\Big]\\ =\;&\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot(\chi K)+4\,\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,\chi\,K+2\,\mathcal{E}(Z,\chi K)\Big]\end{aligned}$$

Now observe that

$$4\,\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,\chi\,K=H(\mathbb{W},Z,K)+4\lambda\int Z^3\,\chi\,K$$

where

$$H(\mathbb{W},Z,K)=4\,\lambda\int_{\mathbb{R}^2}\mathbb{W}^3\,\chi\,K+12\lambda\int\mathbb{W}^2Z\,\chi\,K+12\lambda\int WZ^2\,\chi\,K.$$

Let $\tilde\chi\in C_c^\infty(\mathbb{R}^2)$ and $\tilde\chi=1$ on the support of χ. It is not hard to see from the definition that

$$|H(\mathbb{W},Z,K)|\lesssim\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\big(1+\|\chi^{1/3}Z\|_{L^4}^{4/3}\|\chi^{1/3}Z\|_{H^1}^{2/3}\big)\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}$$

Indeed,

$$\left|\int_{\mathbb{R}^2}\mathbb{W}^3\,\chi\,K\right|=\left|\int_{\mathbb{R}^2}\tilde\chi\mathbb{W}^3\,\chi\,K\right|\lesssim\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\|\chi\,K\|_{B^{1/3}_{3,3}},$$

while, by using interpolation from the first to the second line, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\mathbb{W}^2Z\,\chi\,K\right|&\lesssim\|\chi^{1/3}\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3}Z\|_{B^{1/3}_{3,3}}\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}\\&\lesssim\|\chi^{1/3}\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3}Z\|_{L^4}^{2/3}\|\chi^{1/3}Z\|_{H^1}^{1/3}\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}\\&\lesssim\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\big(1+\|\chi^{1/3}Z\|_{L^4}^{4/3}\|\chi^{1/3}Z\|_{H^1}^{2/3}\big)\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}\end{aligned}$$

and similarly

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}WZ^2\,\chi\,K\right|\\\lesssim\;&\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3}Z\|_{B^{1/3}_{3,3}}^2\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}\\\lesssim\;&\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3}Z\|_{L^4}^{4/3}\|\chi^{1/3}Z\|_{H^1}^{2/3}\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}.\end{aligned}$$

From this we get

$$\begin{aligned}&\left|\nabla f(W+Z)\cdot(\chi K)+4\,\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,\chi\,K+2\,\mathcal{E}(Z,\chi K)\right|\\\lesssim\;&\|\tilde\chi(W+Z)\|_{\mathcal{C}^{-\varepsilon}}\|\chi\,K\|_{H^1}+\Big(\int_0^1\|\chi^{1/2}\dot Z_t\|_{H^1}^2\mathrm{d}t\Big)^{1/2}\Big(\int_0^1\|\chi^{1/2}\dot K_t\|_{H^1}^2\mathrm{d}t\Big)^{1/2}\\&+\|\tilde\chi\mathbb{W}\|_{\mathfrak{S}}\big(1+\|\chi^{1/3}Z\|_{L^4}^{4/3}\|\chi^{1/3}Z\|_{H^1}^{2/3}\big)\|\chi^{1/3}K\|_{B^{1/3}_{3,3}}\\&+\|\chi^{1/4}Z\|_{L^4}^3\|\chi^{1/4}K\|_{L^4}.\end{aligned}$$

The r.h.s. of this last equation is integrable by assumptions on K and bounds on Z. From Lemmas 12 and 14 of [5] we know that there exists a family $(K^M)_{M\geqslant1}$ adapted to the filtration generated by $(W_t)_t$ with

$$\begin{aligned}\mathbb{E}_\mu[\|\chi^{1/4}K\|_{L^4}^4-\|\chi^{1/4}K^M\|_{L^4}^4]&\;\to\;0,\\\mathbb{E}_\mu\Big[\Big(\int_0^1\|\chi^{1/2}(\dot K_t)\|_{H^1}^2\mathrm{d}t\Big)-\int_0^1\|\chi^{1/2}(\dot K_t^M)\|_{H^1}^2\mathrm{d}t\Big]&\;\to\;0,\end{aligned}\tag{28}$$

as $M \to \infty$. After possibly passing to a subsequence, we can assume that $\chi^{1/4} K^M \to \chi^{1/4} K$ in $L^4_w(\mu, L^4_w(\mathbb{R}^2))$ where L^4_w denotes L^4 with the weak topology, and in $L^2_w(\mu, L^2(\mathbb{R}_+ \times \mathbb{R}^2))$. This together with the bound (28) implies that

$$\begin{aligned} &\lim_{M\to\infty} \mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,\chi\, K^M + 2\,\mathcal{E}(Z, \chi K^M)\Big] \\ =\ &\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot(\chi K) + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,\chi\, K + 2\,\mathcal{E}(Z, \chi K)\Big]. \end{aligned}$$

At this point we have reduced the problem to showing that

$$\begin{aligned} &\lim_{\Lambda\to\infty} \mathbb{E}_{\mu^{f,\Lambda}}\Big[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,\chi\, K^M + 2\,\mathcal{E}(Z, \chi K^M)\Big] \\ =\ &\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,\chi\, K^M + 2\,\mathcal{E}(Z, \chi K^M)\Big] \end{aligned}$$

since by assumption

$$\mathbb{E}_{\mu^{f,\Lambda}}\Big[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,\chi\, K^M + 2\,\mathcal{E}(Z, \chi K^M)\Big] = 0.$$

This follows from the weak convergence $\mu^{f,\Lambda} \to \mu$ and the bound

$$\mathbb{E}_{\mu^{f,\Lambda}}[\|\chi\, Z\|^2_{H^1} + \|\chi\, Z\|^4_{L^4}] \leqslant C,$$

since we can approximate Z by bounded admissible controls $(Z^L)_L$ defined as $Z^L_t = Z_{t\wedge T_L}$ with the stopping time

$$T_L := 1 \wedge \inf\{t \in [0,1] \colon \|\chi\, Z_t\|^2_{H^1} + \|\chi\, Z_t\|^4_{L^4} \geqslant L\},$$

in such a way that

$$\sup_{t\in[0,1]} \|\chi\, Z^L_t\|^2_{H^1} + \|\chi\, Z^L_t\|^4_{L^4} \leqslant L,$$

almost surely. Note that $T_L \to 1$ as $L \to \infty$ and therefore indeed $Z^L_t \to Z_t$ as $L \to \infty$. □

To summarise the result of this section: we obtained tightness of the finite volume φ^4_2 measures and described *any* possible infinite volume limit via a novel kind of stochastic equation, the wFBSDE (27). Note that it cannot be interpreted anymore as the first order condition of a functional.

Unfortunately, due to the strongly non-convex character of the renormalised potential we have currently limited information on its solutions. One expects that in the "single phase region" [24] (e.g. where λ is small) there should be uniqueness, and alternatively that multiple solutions should appear in the "two phase region". Despite these limitations eq. (27) is useful and in the next section we show that it allows us to prove large deviations for φ^4_2 in the semiclassical limit.

4.1 Large deviations

In this section we want to investigate the large deviations of φ^4_2 in the infinite volume in the semiclassical limit $\hbar \to 0$ where $\hbar > 0$ is a parameter which is inserted in the model as follows. For any $f \colon \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$, let $\mathcal{W}^\hbar_\Lambda(f)$ be defined by

$$e^{-\frac{1}{\hbar}\mathcal{W}^\hbar_\Lambda(f)} := \int e^{-\frac{1}{\hbar} f(\phi)}\, \theta^\hbar_\Lambda(\mathrm{d}\phi),$$

where

$$\theta^{\hbar}_{\Lambda}(\mathrm{d}\phi) := \frac{\exp\Big(-\frac{1}{\hbar}V_{\Lambda}(\phi)\Big)\theta^{0,\hbar}(\mathrm{d}\phi)}{\int \exp\Big(-\frac{1}{\hbar}V_{\Lambda}(\phi)\Big)\theta^{0,\hbar}(\mathrm{d}\phi)}.$$

Here $\theta^{0,\hbar}$ is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ with covariance $\hbar(1-\Delta)^{-1}$ and the Wick ordering is now taken with respect to $\theta^{0,\hbar}$. With some minor modifications to the derivation in the previous section we have the identity

$$\mathcal{W}^{\hbar}_{\Lambda}(f) = \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{f,\Lambda,\hbar}(\mu) - \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{0,\Lambda,\hbar}(\mu),$$

where

$$\check{F}^{f,\Lambda,\hbar}(\mu) := \mathbb{E}_{\mu}[f(\hbar^{1/2}W+Z) + V_{\Lambda}(\hbar^{1/2}W+Z) + \mathcal{E}(Z)], \qquad \mu\in\bar{\mathcal{X}}.$$

Let $\mu^{f,\Lambda,\hbar}$ be a minimiser for this admissible $\check{F}^{f,\Lambda,\hbar}$. Reasoning as in Lemma 37 above we have that the family $(\mu^{f,\Lambda,\hbar})_{\Lambda,\hbar}$ is exponentially tight for any bounded and continuous f. For each $\hbar>0$ we pick a weakly convergent subsequence of $(\mu^{0,\Lambda,\hbar})_{\Lambda}$ with limit $\mu^{0,\hbar}$ (and, for a given f, a further subsequence along which $\mu^{f,\Lambda,\hbar}\to\mu^{f,\hbar}$). Since $\theta^{\hbar}_{\Lambda}$ is the law of $\hbar^{1/2}W_1+Z_1$ under $\mu^{0,\Lambda,\hbar}$, we have then also $\theta^{\hbar}_{\Lambda}\to\theta^{\hbar}=\mathrm{Law}_{\mu^{0,\hbar}}(\hbar^{1/2}W_1+Z_1)$ along the same subsequence and clearly, for all bounded continuous f we have

$$\mathcal{W}^{\hbar}_{\Lambda}(f)\to\mathcal{W}^{\hbar}(f),$$

where $\mathcal{W}^{\hbar}(f)$ is now defined as

$$e^{-\frac{1}{\hbar}\mathcal{W}^{\hbar}(f)} := \int e^{-\frac{1}{\hbar}f(\phi)}\,\theta^{\hbar}(\mathrm{d}\phi).$$

Even though in this setting we do not have sufficient information to establish uniqueness of the limiting measures $(\mu^{f,\hbar})_{\hbar}$, we are still able to obtain a Laplace principle as $\hbar\to 0$ for the corresponding family $(\theta^{\hbar})_{\hbar}$. Note that the family $(\theta^{\hbar})_{\hbar}$ is exponentially tight by Lemma 37.

Theorem 39. *Any family of accumulation points $(\theta^{\hbar})_{\hbar}$ of $(\theta^{\hbar}_{\Lambda})_{\hbar,\Lambda}$ satisfies a Laplace principle on $\mathcal{S}'(\mathbb{R}^2)$ with rate function*

$$J(\phi) = \begin{cases} \lambda\int_{\mathbb{R}^2}\phi^4 + \frac{1}{2}\int_{\mathbb{R}^2}\phi(1-\Delta)\phi, & \text{for } \phi\in H^1(\mathbb{R}^2); \\ +\infty, & \text{otherwise.} \end{cases}$$

That is for $f\colon\mathcal{S}'(\mathbb{R}^2)\to\mathbb{R}$ continuous and bounded

$$\lim_{\hbar\to 0}\mathcal{W}^{\hbar}(f) = \inf_{\psi\in\mathcal{S}'(\mathbb{R}^2)}\{f(\psi)+J(\psi)\}.$$

Proof. Assume that f is of the form

$$f(\psi) = C\int \rho((1-\Delta)^{-1/2}(\psi-\varphi))^2 \tag{29}$$

for $C\geqslant 0$ and $\varphi\in C^{\infty}_{c}(\mathbb{R}^2)$.

It is enough to prove the theorem for this particular family of functions since we already know that $(\theta^{\hbar})_{\hbar}$ are exponentially tight, and the family given by (29) is rate function determining in the sense of Definition 3.15 of [20], as it isolates points and is bounded below.

From (20) we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^{\hbar}_{\Lambda}(\alpha f) = \mathbb{E}_{\mu^{\alpha f,\Lambda,\hbar}}[f(\hbar^{1/2}W+Z)] = \frac{\int f(\phi)e^{-\frac{1}{\hbar}\alpha f(\phi)}\,\theta^{\hbar}_{\Lambda}(\mathrm{d}\phi)}{\int e^{-\frac{1}{\hbar}\alpha f(\phi)}\,\theta^{\hbar}_{\Lambda}(\mathrm{d}\phi)}.$$

By taking a further subsequence we can also have that $\mu^{\alpha f,\Lambda,\hbar}\to\mu^{\alpha f,\hbar}$. We can then pass to the limit and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^{\hbar}(\alpha f)=\mathbb{E}_{\mu^{\alpha f,\hbar}}[f(\hbar^{1/2}W+Z)],$$

for any f,α and arbitrarily chosen accumulation point $\mu^{\alpha f,\hbar}$. Note that we do not have to do a diagonal argument over the uncountably many choices of α, f: for any given α, f we need just to take a subsequence of the sequence chosen to have $(\theta^{\hbar}_{\Lambda})_{\Lambda}$ converge in order to establish the equality.

Now recall that the minimiser $\mu^{f,\hbar}$ satisfies an infinite volume wFBSDE of the form

$$\begin{aligned}0 \;=\; & \mathbb{E}_{\mu^{f,\hbar}}\Big[\nabla f(\hbar^{1/2}W+Z)\cdot H+4\lambda\hbar^{3/2}\int \mathbb{W}^3H+12\lambda\hbar\int \mathbb{W}^2ZH\\ & +12\lambda\hbar^{1/2}\int W\,(Z)^2H+4\lambda\int Z^3H+2\,\mathcal{E}(Z,H)\Big]\end{aligned}\tag{30}$$

for any H with compact support. Furthermore we can carry over the proof of Lemma 34, and obtain that

$$\sup_{\hbar<1}\mathbb{E}_{\mu^{f,\hbar}}\big[\,\|Z\|^4_{L^4(\rho^{1/4})}+\|Z\|^2_{H^1(\rho^{1/2})}\big]<\infty.$$

This implies that the family $(\mu^{f,\hbar})_{\hbar}$ is also tight and therefore we can consider any accumulation point μ^f as $\hbar\to 0$ which will satisfy the same estimate. By passing to the limit in (30) via by now standard estimates similar to those in Lemma 38 we obtain

$$0 \;=\; \mathbb{E}_{\mu^{f}}\Big[\nabla f(Z)\cdot H+4\lambda\int Z^3H+2\,\mathcal{E}(Z,H)\Big],\tag{31}$$

valid for any adapted and compactly supported H. Note that this equation does not contain anymore the GFF. This equation has a unique solution. Indeed if ν is another solution we can couple the two solutions in $\upsilon\in\bar{\mathcal{Y}}(\mu^f)$ such that $\nu=\mathrm{Law}_{\upsilon}(W,Z+K)$ and have

$$0=\mathbb{E}_{\upsilon}\Big[[\nabla f(Z+K)-\nabla f(Z)]\cdot H+4\lambda\int[(Z+K)^3-Z^3]H+2\,\mathcal{E}(K,H)\Big].$$

Now taking $H=\chi K$ for $\chi\in C^{\infty}_c(\mathbb{R}^2)$ and $\chi\geqslant 0$, we obtain

$$0=\mathbb{E}_{\upsilon}\Big[[\nabla f(Z+K)-\nabla f(Z)]\cdot(\chi K)+4\lambda\int\chi[(Z+K)^3-Z^3]K+2\,\mathcal{E}(K,\chi K)\Big].$$

By convexity of f and $(\cdot)^4$ we have that

$$\mathbb{E}_{\upsilon}\Big[[\nabla f(Z+K)-\nabla f(Z)]\cdot(\chi K)+4\lambda\int\chi[(Z+K)^3-Z^3]K\Big]\geqslant 0.$$

Sending $\chi\to 1$ we obtain that

$$\mathbb{E}_{\upsilon}[\mathcal{E}(K,K)]\leqslant 0\Rightarrow K=0\Rightarrow\nu=\mu^f.$$

Therefore μ^f is the unique minimiser of the admissible (convex) functional

$$\check{F}(\nu):=\mathbb{E}_{\nu}\Big[f(Z)+\lambda\int Z^4+\mathcal{E}(Z,Z)\Big].$$

Let $\nu^{\phi}=\mathrm{Law}_{\mathbb{P}}(W,s\mapsto s\phi)$ for $\phi\in H^1$ then

$$\check{F}(\nu^{\phi})=f(\phi)+\lambda\int\phi^4+\frac{1}{2}\int\phi(1-\Delta)\phi=f(\phi)+J(\phi).$$

Using the fact that the variational problem in the time variable for the functional

$$\mathcal{E}(Z,Z)=\frac{1}{2}\int_0^1\int \dot{Z}_s(1-\Delta)\dot{Z}_s\mathrm{d}s$$

is minimised for $\dot{Z}_s$ constant, we have for any ν

$$\check{F}(\nu)=\mathbb{E}_\nu\bigg[f(Z)+\lambda\int Z^4+\mathcal{E}(Z,Z)\bigg]$$

$$\geqslant\mathbb{E}_\nu\bigg[f(Z)+\lambda\int Z^4+\frac{1}{2}\int Z_1(1-\Delta)Z_1\bigg]=\mathbb{E}_\nu[f(Z)+J(Z)]$$

$$\geqslant\inf_{\phi\in H^1}[f(\phi)+J(\phi)].$$

As a consequence we obtain that

$$\inf_{\nu\in\bar{\mathcal{X}}}\check{F}(\nu)=\inf_{\phi\in H^1}[f(\phi)+J(\phi)],$$

and that $\mu^f=\nu^{\phi^f}$ for the unique ϕ^f which attains the convex deterministic minimisation problem on the r.h.s..

To sum up, we have established that

$$\lim_{\hbar\to0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^\hbar(\alpha f)=\lim_{\hbar\to0}\mathbb{E}_{\mu^{\alpha f,\hbar}}[f(\hbar^{1/2}W+Z)]=f(\phi^{\alpha f}).\tag{32}$$

Integrating eq. (32) and using dominated convergence

$$\lim_{\hbar\to0}\mathcal{W}^\hbar(\alpha f)=\int_0^\alpha\lim_{\hbar\to0}\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{W}^\hbar(\beta f)\mathrm{d}\beta=\int_0^\alpha\lim_{\hbar\to0}\mathbb{E}_{\mu^{\beta f,\hbar}}[f(\hbar^{1/2}W+Z)]\mathrm{d}\beta$$

$$=\int_0^\alpha f(\phi^{\beta f})\mathrm{d}\beta=:\mathcal{W}^0(\alpha f).$$

This also implies that the derivative in $\alpha\in[0,1]$ of $\alpha\mapsto\mathcal{W}^0(\alpha f)$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^0(\alpha f)=f(\phi^{\alpha f}).$$

To conclude it is enough to show that this coincides with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\tilde{\mathcal{W}}^0(\alpha f),$$

where

$$\tilde{\mathcal{W}}^0(\alpha f)=\inf_{\psi\in H^1(\mathbb{R}^2)}\{\alpha f(\psi)+J(\psi)\}.$$

Recall that $\phi^{\alpha f}$ is the minimiser of $\phi\mapsto\alpha f(\phi)+J(\phi)$ so the Euler–Lagrange equation gives

$$\alpha\nabla f(\phi^{\alpha f})\cdot h+\nabla J(\phi^{\alpha f})\cdot h=0.$$

At this point we note that

$$0=\alpha[\nabla f(\phi^{(\alpha+\gamma)f})-\nabla f(\phi^{\alpha f})]\cdot(\phi^{(\alpha+\gamma)f}-\phi^{\alpha f})$$

$$+[\nabla J(\phi^{(\alpha+\gamma)f})-\nabla J(\phi^{\alpha f})]\cdot(\phi^{(\alpha+\gamma)f}-\phi^{\alpha f})$$

$$+\gamma\nabla f(\phi^{(\alpha+\gamma)f})\cdot(\phi^{(\alpha+\gamma)f}-\phi^{\alpha f}).$$

By convexity of f and strong convexity of J we have

$$\begin{aligned} & \|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}^2 \\ \lesssim\ & [\alpha(\nabla f(\phi^{(\alpha+\gamma)f}) - \nabla f(\phi^{\alpha f})) + \nabla J(\phi^{(\alpha+\gamma)f}) - \nabla J(\phi^{\alpha f})] \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}). \end{aligned}$$

Now

$$\gamma \int \nabla f(\phi^{(\alpha+\gamma)f}) \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \mathrm{d}x \lesssim \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$$

so

$$\|(\phi^{(\alpha+\gamma)f} - \phi^{\alpha f})\|_{H^1}^2 \lesssim \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$$

so dividing by $\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$ we have

$$\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1} \leqslant \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \leqslant \gamma(\|\phi^{(\alpha+\gamma)f}\|_{H^1} + C) \lesssim \gamma.$$

We can then calculate

$$\begin{aligned} \tilde{\mathcal{W}}^0((\alpha+\gamma)f) - \tilde{\mathcal{W}}^0(\alpha f) \ &= \ \gamma f(\phi^{(\alpha+\gamma)f}) + \alpha(f(\phi^{(\alpha+\gamma)f}) - f(\phi^{\alpha f})) \\ & \quad + J(\phi^{(\alpha+\gamma)f}) - J(\phi^{\alpha f}). \end{aligned}$$

Given that $\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1} \lesssim \gamma$ we have, since by Sobolev embedding $\|\cdot\|_{L^4}^4$ is locally Lipschitz in the H^1 norm

$$\begin{aligned} & (\alpha(f(\phi^{(\alpha+\gamma)f}) - f(\phi^{\alpha f})) + J(\phi^{(\alpha+\gamma)f}) - J(\phi^{\alpha f})) \\ =\ & \alpha \nabla f(\phi^{\alpha f})(\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) + \int (1-\Delta)(\phi^{\alpha f})\,(\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \\ & + 4\lambda \int (\phi^{\alpha f})^3 (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) + O\,(\gamma^2) \\ =\ & O\,(\gamma^2), \end{aligned}$$

where in the last line we used the Euler–Lagrange equation for $\phi^{\alpha f}$. This implies in turn

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \tilde{\mathcal{W}}^0(\alpha f) = \lim_{\gamma \to 0} \frac{1}{\gamma} (\tilde{\mathcal{W}}^0((\alpha+\gamma)f) - \tilde{\mathcal{W}}^0(\alpha f)) = \lim_{\gamma \to 0} \frac{1}{\gamma} \gamma f(\phi^{(\alpha+\gamma)f}) = f(\phi^{\alpha f})$$

which concludes the proof. □

5 Exponential interaction

In this section we will discuss a variational description of the exponential interaction, that is the measure

$$\nu^{\xi,\exp}(\mathrm{d}\phi) = \frac{e^{-V^\xi(\phi)}\,\theta^0(\mathrm{d}\phi)}{\int e^{-V^\xi(\phi)}\,\theta^0(\mathrm{d}\phi)},$$

where now V^ξ is given by

$$V^\xi(\phi) = \lambda \int_{\mathbb{R}^2} \xi [\![\exp(\beta\phi)]\!],$$

with some $\beta^2 \in (0, 8\pi)$ and where $M^\beta := [\![\exp(\beta\phi)]\!]$ is the Gaussian Multiplicative Chaos (GMC) and $\xi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}_+)$ is a (smooth and compactly supported) spatial cutoff. We start with some preliminaries about the rigorous construction of the GMC.

This model is also known as the $\exp(\beta\varphi)_2$ model (or Høegh–Krohn model), and has been introduced in [35]. See e.g. [2] for a discussion of its stochastic quantisation in the elliptic setting. For this model we repeat the construction of the variational description of the infinite volume measure. At variance with what happens in $P(\varphi)_2$ theories, renormalisation does not spoil the convexity properties of the potential and therefore the study of the related wFBSDE can go further and in particular we will be able to prove the necessary locality of the perturbed equation. This in turn will allow us to prove the Γ-convergence of the variational problem along with the infinite volume limit and establish eq. (15) in its rigorous form.

As a result of this analysis, we are able to obtain a concise and well-behaved description of an infinite volume Gibbs measure related to an Euclidean quantum field.

We start with a definition and some properties of the GMC, i.e. the renormalised exponential of the GFF.

Proposition 40. *Let W be a GFF and $W_\varepsilon = \psi_\varepsilon * W$ where $\psi_\varepsilon \to \delta_0$ and $\psi_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ and let $\beta^2 \in (0, 8\pi)$. The Borel measures $(M^{\beta,\varepsilon})_\varepsilon$ on $\mathbb{R}^2$ given by*

$$M^{\beta,\varepsilon}(A) = \int_A \exp\Big(\beta W_\varepsilon(x) - \frac{\beta^2}{2}\mathbb{E}[W_\varepsilon^2(x)] \Big)\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}^2),$$

converges almost surely set-wise to a measure

$$M^\beta(A) = \lim_{\varepsilon\to 0} M^{\beta,\varepsilon}(A), \qquad A \in \mathcal{B}(\mathbb{R}^2). \tag{33}$$

We call M^β the Gaussian multiplicative chaos (GMC) with parameter β associated to the GFF W. It enjoys the following properties

a) For $1 < p < 8\pi/\beta^2$ and for any compact set $A \subseteq \mathbb{R}^2$

$$\sup_\varepsilon \mathbb{E}[|M^{\beta,\varepsilon}(A)|^p] < \infty.$$

Consequently

$$\mathbb{E}[|M^\beta(A)|^p] < \infty.$$

b) For $1 < p < 8\pi/\beta^2$

$$\mathbb{E}[|M^{\beta,\varepsilon}(B(0,\lambda))|^p]^{1/p} \leqslant C\lambda^{2-(p-1)\beta^2/4\pi}\,\mathbb{E}[|M^{\beta,\varepsilon}(B(0,1))|^p]^{1/p},$$

where $B(x,r) \subseteq \mathbb{R}^2$ denotes the open ball centred in x of radius r.

Proof. For a proof of (33) see [9]. The property a) is due to Kahane [39], while the estimate in b) is due to Hoshino, Kawabi and Kusuoka [36]. □

From these properties we can determine easily the Besov regularity of M^β following [36] (Section 2).

Proposition 41. *Assume that $s > (\beta^2/(4\pi))(p-1)$, $1 < p < 8\pi/\beta^2$, then*

$$\mathbb{E}\Big[\|M^\beta\|^p_{B^{-s}_{p,p}(\rho^{1/2})} \Big] < \infty.$$

Moreover $\sup_\varepsilon \mathbb{E}\Big[\|M^{\beta,\varepsilon}\|^p_{B^{-s}_{p,p}(\rho^{1/2})} \Big] < \infty$ and $\mathbb{E}\Big[\|M^{\beta,\varepsilon} - M^\beta\|^p_{B^{-s}_{p,p}(\rho^{1/2})} \Big] \to 0$ as $\varepsilon \to 0$.

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\,\|M^{\beta,\varepsilon}\|^p_{B^{-s}_{p,p}(\rho^{1/2})}\Big] &= \sum_{i\geqslant 0} 2^{-isp}\mathbb{E}\|\Delta_i M^{\beta,\varepsilon}\|^p_{L^p(\rho^{1/2})}\\ &\leqslant C\sum_{i\geqslant 0} 2^{-isp}\mathbb{E}|\Delta_i M^{\beta,\varepsilon}(0)|^p.\end{aligned}$$

Recall that the Littlewood–Paley operators $(\Delta_i)_{i\geqslant -1}$ are defined by $\Delta_i = \psi(\mathrm{D}/2^i)$ for $i\geqslant 0$ and some Schwartz function $\psi\colon \mathbb{R}^2\to\mathbb{R}_{\geqslant 0}$. Now using translation invariance of the law of $M^{\beta,\varepsilon}$ and the fact that $\mathcal{F}^{-1}\psi$ is a Schwartz function we estimate

$$\begin{aligned}\mathbb{E}|\Delta_i M^{\beta,\varepsilon}(0)|^p &= 2^{2pi}\mathbb{E}\left|\int_{\mathbb{R}^2}(\mathcal{F}^{-1}\psi)(2^i x)M^{\beta,\varepsilon}(x)\mathrm{d}x\right|^p\\ &\leqslant \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^{-2}\mathbb{E}\left|\int_{B(n,1)} M^{\beta,\varepsilon}(2^{-i}x)\mathrm{d}x\right|^p\\ &= \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^{-2}2^{2pi}\mathbb{E}\left|\int_{B(2^{-i}n,2^{-i})} M^{\beta,\varepsilon}(x)\mathrm{d}x\right|^p\\ &\lesssim 2^{p(p-1)\beta^2 i/(4\pi)}.\end{aligned}$$

Since the constants are uniform in ε (and the block $i=-1$ is treated in the same way), summing over i gives $\sup_\varepsilon \mathbb{E}\|M^{\beta,\varepsilon}\|^p_{B^{-s}_{p,p}(\rho^{1/2})}<\infty$ for every $s>(\beta^2/(4\pi))(p-1)$.

For the convergence, fix $i\geqslant -1$ and $x\in\mathbb{R}^2$. We have $\Delta_i M^{\beta,\varepsilon}(x) = \int K_i(x-y)M^{\beta,\varepsilon}(\mathrm{d}y)$ with a Schwartz kernel K_i. The almost sure set-wise convergence (33), applied to a countable family of dyadic squares, and the positivity of the measures imply that $\int g\mathrm{d}M^{\beta,\varepsilon}\to\int g\mathrm{d}M^\beta$ almost surely for every continuous compactly supported g; since moreover $\mathbb{E}[M^{\beta,\varepsilon}(\mathrm{d}y)] = \mathrm{d}y$, the contribution of the tails of K_i is small in $L^1(\mathbb{P})$ uniformly in ε, and therefore $\Delta_i M^{\beta,\varepsilon}(x)\to\Delta_i M^\beta(x)$ in probability. The estimate above, with p replaced by some $\tilde p\in(p, 8\pi/\beta^2)$, gives $\sup_{\varepsilon,x}\mathbb{E}|\Delta_i M^{\beta,\varepsilon}(x)|^{\tilde p}<\infty$, and by Fatou's lemma the same bound holds for M^β. By uniform integrability $\mathbb{E}|\Delta_i(M^{\beta,\varepsilon}-M^\beta)(x)|^p\to 0$ boundedly in x, so that, integrating in x as in the first step of the proof, $\mathbb{E}\|\Delta_i(M^{\beta,\varepsilon}-M^\beta)\|^p_{L^p(\rho^{1/2})}\to 0$ for every i. Finally take $s'\in((\beta^2/(4\pi))(p-1), s)$. By Fatou's lemma $\mathbb{E}\|M^\beta\|^p_{B^{-s'}_{p,p}(\rho^{1/2})}\leqslant \liminf_{\varepsilon\to 0}\mathbb{E}\|M^{\beta,\varepsilon}\|^p_{B^{-s'}_{p,p}(\rho^{1/2})}<\infty$ (which gives in particular the first claim of the proposition), hence, uniformly in ε,

$$\sum_{i\geqslant I}2^{-isp}\mathbb{E}\|\Delta_i(M^{\beta,\varepsilon}-M^\beta)\|^p_{L^p(\rho^{1/2})}\leqslant 2^{-I(s-s')p}\mathbb{E}\|M^{\beta,\varepsilon}-M^\beta\|^p_{B^{-s'}_{p,p}(\rho^{1/2})}\lesssim 2^{-I(s-s')p}.$$

Together with the convergence of each block this gives $\mathbb{E}\|M^{\beta,\varepsilon}-M^\beta\|^p_{B^{-s}_{p,p}(\rho^{1/2})}\to 0$. □

In this model we let $\mathfrak{S} := \mathfrak{S}_0\times\mathfrak{M}$, where $\mathfrak{M}$ is the set of positive Radon measures on $\mathbb{R}^2$ which belong to $B^{-s}_{p,p}(\rho^{1/2})$, for some fixed p, s as in Proposition 41, endowed with the topology of $B^{-s}_{p,p}(\rho^{1/2})$ (a closed cone in a separable Banach space, hence a Polish space). By Proposition 41, $M^\beta\in\mathfrak{M}$ almost surely; note that M^β has infinite total mass, so it is not a probability measure on $\mathbb{R}^2$. Then $\mathbb{W} = (W, M^\beta)\colon \mathfrak{S}_0\to\mathfrak{S}$ will be the appropriate enhanced GFF. By the usual abuse of notation we will denote also $\mathbb{W} = (W, M^\beta)$ the canonical process on $\mathfrak{S}$.

We also need a slightly improved definition of the relaxed trial space $\bar{\mathcal{X}}^{\exp}$.

Definition 42. *Set*

$$\mathcal{X}^{\exp} := \{\mu = \mathrm{Law}(W, M^\beta, Z) \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) \colon Z \in \hat{\mathfrak{H}}^a \text{ with } \mathbb{E}_\mu[\|Z\|_{\hat{\mathfrak{H}}}^2] < \infty\},$$

and

$$\bar{\mathcal{X}}^{\exp} := \left\{\mu \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) \colon \exists (\mu_n)_n \subseteq \mathcal{X}^{\exp} \colon \mu_n \to \mu \text{ weakly}, \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|_{\hat{\mathfrak{H}}}^2] < \infty\right\}$$

We equip $\bar{\mathcal{X}}^{\exp}$ with the following topology: a sequence $(\mu_n)_n \subseteq \bar{\mathcal{X}}^{\exp}$ converges to μ if

a) $\mu_n \to \mu$ *weakly in* $\mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w)$,

b) $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|_{\hat{\mathfrak{H}}}^2] < \infty$.

For technical reasons we first establish the variational formulation for the approximate potential $V^{\xi,\varepsilon}$ where

$$V^{\xi,\varepsilon}(\phi) = \lambda \int_{\mathbb{R}^2} \xi [\![\exp(\beta \psi_\varepsilon * \phi)]\!]$$

and the Wick ordering is now taken with respect to the covariance of $W_\varepsilon = \psi_\varepsilon * W$.

Lemma 43. *The potential $V^{\xi,\varepsilon}$ is admissible and*

$$\check{F}^{f,\xi,\varepsilon}(\mu) := \check{F}^{V^{\xi,f,\varepsilon}}(\mu) = \mathbb{E}_\mu\left[f(W+Z) + \lambda \int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z)) \mathrm{d}M^{\beta,\varepsilon} + \mathcal{E}(Z) \right]$$

for all $\mu \in \bar{\mathcal{X}}^{\exp}$.

Proof. Note that $M^{\beta,\varepsilon}$ is positive and that Proposition 41 implies that for p close enough to 1 and $\frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}[\exp(-qV^{\xi,\varepsilon})] + \mathbb{E}[|V^{\xi,\varepsilon}|^p] < \infty,$$

so $V^{\xi,\varepsilon}$ is a tame functional. From the properties of the exponential it is clear that

$$\begin{aligned} &\int_{\mathbb{R}^2} \xi [\![\exp(\beta(\psi_\varepsilon * Z) + \beta W_\varepsilon)]\!] \mathrm{d}x \\ = &\int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z)) [\![\exp(\beta W_\varepsilon)]\!] \mathrm{d}x \\ = &\int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z)) \mathrm{d}M^{\beta,\varepsilon}. \end{aligned}$$

We have for all $\mu \in \bar{\mathcal{X}}^{\exp}$:

$$\begin{aligned} \check{F}^{f,\xi,\varepsilon}(\mu) &:= \check{F}^{V^{\xi,f,\varepsilon}}(\mu) \\ &= \mathbb{E}_\mu\left[f(W+Z) + \lambda \int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z)) \mathrm{d}M^{\beta,\varepsilon} + \mathcal{E}(Z) \right] \\ &\geqslant \mathbb{E}_\mu[\mathcal{E}(Z)] - C \geqslant \frac{1}{2}\mathbb{E}_\mu[\|Z\|_{\hat{\mathfrak{H}}}^2] - C \end{aligned}$$

from which also the coercivity of $\check{F}^{V^{\xi,f,\varepsilon}}$ is clear. Furthermore it is not hard to see that

$$(W, Z) \to \int_{\mathbb{R}^2} \xi [\![\exp(\beta(\psi_\varepsilon * Z) + \beta(\psi_\varepsilon * W))]\!] \mathrm{d}x$$

is continuous as a map $(\mathfrak{S}_0, H^1) \to \mathbb{R}$ (note that here the renormalisation is finite) and convergence of W, Z in $(\mathfrak{S}_0, H^1)$ implies convergence in $L^\infty \times L^\infty$ of $\psi_\varepsilon * Z, \psi_\varepsilon * W$. To conclude admissibility it remains to show that $\check{F}^{V^{\xi,f,\varepsilon}}$ is lower semicontinuous on $\bar{\mathcal{X}}^{\exp}$. Since we have already shown the interaction to be continuous it remains to do this for $\mathcal{E}$. Indeed we have for any $\mu \in \bar{\mathcal{X}}^{\exp}$ and any sequence $(\mu_n)_n$ such that $\mu_n \to \mu$ in $\bar{\mathcal{X}}^{\exp}$

$$\liminf_n \mathbb{E}_{\mu_n}[\mathcal{E}(Z)] \geqslant \mathbb{E}_\mu[\mathcal{E}(Z)],$$

this implies lower semicontinuity. □

We now show that we can remove the ultraviolet cutoff in the variational formulation. We introduce the unregularized functional

$$\check{F}^{f,\xi}(\mu) = \check{F}^{V^{\xi,f}}(\mu) = \mathbb{E}_\mu\Big[f(W+Z) + \lambda \int_{\mathbb{R}^2} \xi \exp(\beta Z) \mathrm{d}M^\beta + \mathcal{E}(Z) \Big].$$

We will show that as $\varepsilon \to 0$ $\check{F}^{f,\xi,\varepsilon}$ **Γ-converges** to $\check{F}^{f,\xi}$.

Γ-convergence is a notion of convergence introduced by De Giorgi well suited for the study of variational problems. The book [13] is a nice introduction to Γ-convergence in the context of the calculus of variations. For the convenience of the reader we recall here the basic definitions and results.

Definition 44. *Let $\mathcal{T}$ be a topological space and let $F, F_n : \mathcal{T} \to (-\infty, \infty]$. We say that the sequence of functionals $(F_n)_n$ Γ-converges to F iff*

i. For every sequence $x_n \to x$ in $\mathcal{T}$

$$F(x) \leqslant \liminf_{n\to\infty} F_n(x_n);$$

ii. For every point x there exists a sequence $x_n \to x$ (called a recovery sequence) such that

$$F(x) \geqslant \limsup_{n\to\infty} F_n(x_n).$$

Definition 45. *A sequence of functionals $F_n : \mathcal{T} \to (-\infty, \infty]$ is called equicoercive if there exists a compact set $\mathcal{K} \subseteq \mathcal{T}$ such that for all $n \in \mathbb{N}$*

$$\inf_{x\in\mathcal{K}} F_n(x) = \inf_{x\in\mathcal{T}} F_n(x).$$

A fundamental consequence of Γ-convergence is the convergence of minima.

Theorem 46. *If $(F_n)_n$ Γ-converges to F and $(F_n)_n$ is equicoercive, then F admits a minimum and*

$$\min_{\mathcal{T}} F = \lim_{n\to\infty} \inf_{\mathcal{T}} F_n.$$

For a proof see [18].

Lemma 47. *$\check{F}^{f,\xi,\varepsilon}$ Γ-converges to $\check{F}^{f,\xi}$ on $\bar{\mathcal{X}}^{\exp}$ as $\varepsilon \to 0$.*

Proof. We first show the liminf inequality. We have for any sequence $(\mu^\varepsilon)_\varepsilon$ such that $\mu^\varepsilon \to \mu$ in $\bar{\mathcal{X}}^{\exp}$

$$\liminf_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon}[\mathcal{E}(Z)] \geqslant \mathbb{E}_\mu[\mathcal{E}(Z)], \qquad \lim_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon}[f(W+Z)] = \mathbb{E}_\mu[f(W+Z)].$$

Now let $g^N : \mathbb{R} \to \mathbb{R}$ be defined by $g^N(x) = x$ if $x < N$, $g^N(x) = N$ if $x > N$. Then $g^N \leqslant N$, g^N is Lipschitz with constant 1 and $g^N(x) \to x$. Set $Z_t^N = g^N(Z_t)$ and $\mu_{N,\varepsilon} = \mathrm{Law}_{\mu^\varepsilon}(W, Z^N)$. Then by monotone convergence

$$\begin{aligned} &\lim_{N\to\infty} \mathbb{E}_\mu \int \xi \exp(\beta Z^N) \mathrm{d}M^\beta \\ =\ &\mathbb{E}_\mu \int \xi \exp(\beta Z) \mathrm{d}M^\beta \end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mu^\varepsilon}\int \xi\exp(\beta Z^{N,\varepsilon})\mathrm{d}M^{\beta,\varepsilon} \;=\; \mathbb{E}_{\mu}\int \xi\exp(\beta Z^{N})\mathrm{d}M^{\beta},$$

then the statement will follow from

$$\begin{aligned}\mathbb{E}_{\mu}\int \xi\exp(\beta Z)\mathrm{d}M^{\beta} &= \lim_{N\to\infty}\mathbb{E}_{\mu}\int \xi\exp(\beta Z^{N})\mathrm{d}M^{\beta}\\ &= \lim_{N\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}_{\mu^\varepsilon}\int \xi\exp(\beta Z^{N,\varepsilon})\mathrm{d}M^{\beta,\varepsilon}\\ &\leqslant \liminf_{\varepsilon\to 0}\mathbb{E}_{\mu^\varepsilon}\int \xi\exp(\beta Z^{\varepsilon})\mathrm{d}M^{\beta,\varepsilon}.\end{aligned}$$

We have that $\|\exp(\beta Z^N)\|_{L^\infty}\leqslant\exp(\beta N)$ and by chain rule

$$\|\rho\nabla\exp(\beta Z^N)\|_{L^2}=\|\rho\beta\nabla Z^N\exp(\beta Z^N)\|_{L^2}\lesssim\exp(\beta N)\|\rho\nabla Z^N\|_{L^2}.$$

By interpolation

$$\|\exp(\beta Z^{N,\varepsilon})\|_{B^{\delta}_{p',p'}(\rho)}\lesssim\|\exp(\beta Z^{N,\varepsilon})\|_{L^\infty}^{1-2/p'}\|\exp(\beta Z^{N,\varepsilon})\|_{H^1(\rho)}^{2/p'},$$

provided $2/p'>\delta$. Now let $p=p'/(p'-1)$. Note that we can choose p and δ such that $\frac{8\pi}{\beta^2}>p$ and $\frac{\beta^2}{4\pi}(p-1)<\delta$ and $2/p'>\delta$ by taking

$$\frac{2}{p}(p-1)>\delta>\frac{\beta^2}{4\pi}(p-1).$$

We assume, as we may, that $p\leqslant 2$ (so that $p'\geqslant 2$) and that these values of p and $s=\delta$ are the ones used in the definition of the space $\mathfrak{M}$. To prove the convergence it suffices to argue along subsequences. By the Skorokhod representation theorem [37] there exist a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random variables $(W^\varepsilon,\tilde M^\varepsilon,Z^\varepsilon)$ with law μ^ε converging almost surely in $\mathfrak{S}\times\hat{\mathfrak{H}}_w$ to (W,M^β,Z) with law μ; in particular $\tilde M^\varepsilon\to M^\beta$ in $B^{-\delta}_{p,p}(\rho^{1/2})$ and $Z^\varepsilon\to Z$ weakly in H^1_{loc} almost surely. On this space we denote by $M^{\beta,\varepsilon}$ the regularised chaos built from W^ε and we set $Z^{N,\varepsilon}:=g^N(\psi_\varepsilon * Z^\varepsilon)$, $Z^N:=g^N(Z)$: these are the quantities entering $\check F^{f,\xi,\varepsilon}$ and $\check F^{f,\xi}$. Since $(W^\varepsilon,\tilde M^\varepsilon)$ has the law of (W,M^β), the random variable $\|M^{\beta,\varepsilon}-\tilde M^\varepsilon\|_{B^{-\delta}_{p,p}(\rho^{1/2})}$ has the law of $\|M^{\beta,\varepsilon}-M^\beta\|_{B^{-\delta}_{p,p}(\rho^{1/2})}$ on the original space, which tends to 0 in probability by Proposition 41. Hence $M^{\beta,\varepsilon}\to M^\beta$ in $B^{-\delta}_{p,p}(\rho^{1/2})$ in probability. Moreover $\psi_\varepsilon * Z^\varepsilon\to Z$ weakly in H^1_{loc} (since $\psi_\varepsilon * Z^\varepsilon - Z^\varepsilon\to 0$ in L^2_{loc}), so if $\operatorname{supp}\xi\subseteq B(0,R)$ the functions $\xi\exp(\beta Z^{N,\varepsilon})$ are bounded in H^1 by the chain rule and, by Rellich's theorem and the Lipschitz property of $x\mapsto\exp(\beta g^N(x))$, converge almost surely to $\xi\exp(\beta Z^N)$ in L^2. By interpolation the convergence holds in H^r for all $r<1$ and, since $p'\geqslant 2$, the Sobolev embedding $H^r\subseteq B^{r-1+2/p'}_{p',p'}$ for functions supported in $B(0,R)$, with $r\in(1+\delta-2/p',1)$, gives the convergence in $B^{\delta}_{p',p'}$. By duality, since ρ is bounded above and below on $B(0,R)$,

$$\left|\int \xi e^{\beta Z^{N,\varepsilon}}\mathrm{d}M^{\beta,\varepsilon} - \int \xi e^{\beta Z^{N}}\mathrm{d}M^{\beta}\right| \lesssim_R \|\xi(e^{\beta Z^{N,\varepsilon}}-e^{\beta Z^N})\|_{B^{\delta}_{p',p'}}\|M^{\beta,\varepsilon}\|_{B^{-\delta}_{p,p}(\rho^{1/2})} + \|\xi e^{\beta Z^N}\|_{B^{\delta}_{p',p'}}\|M^{\beta,\varepsilon}-M^\beta\|_{B^{-\delta}_{p,p}(\rho^{1/2})},$$

which tends to 0 in probability. Since

$$\int \xi\exp(\beta Z^{N,\varepsilon})\mathrm{d}M^{\beta,\varepsilon}\lesssim e^{\beta N}\int \xi\mathrm{d}M^{\beta,\varepsilon},$$

and the r.h.s. is bounded in $L^p(\tilde{\mathbb{P}})$ uniformly in ε by Proposition 40 a) (its law being that of the corresponding quantity on the original space), the family is uniformly integrable and by Vitali's theorem we have

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mu^\varepsilon}\int \xi\exp(\beta Z^{N,\varepsilon})\mathrm{d}M^{\beta,\varepsilon} = \mathbb{E}_{\mu}\int \xi\exp(\beta Z^{N})\mathrm{d}M^{\beta},$$

and we can conclude. To show the limsup inequality, for given μ we need to construct μ^ε such that

$$\operatorname*{limsup}_{\varepsilon\to 0} F^{V^{\xi,f,\varepsilon}}(\mu^\varepsilon) \leqslant F^{V^{\xi,f}}(\mu).$$

To do this we proceed similarly as in the first step. Define $Z^{N,\delta} = g^N(\psi_\delta * Z)$, $\mu^{N,\delta} = \mathrm{Law}(W, M^\beta, Z^{N,\delta})$, where ψ_δ is a sequence of standard mollifiers. Then

$$M^{\beta,\varepsilon} \to M^\beta$$

weakly so

$$\mathbb{E}\int \xi\exp(\beta Z^{N,\delta})\mathrm{d}M^{\beta,\varepsilon} \to \mathbb{E}\int \xi\exp(\beta Z^{N,\delta})\mathrm{d}M^{\beta}$$

since $Z^{N,\delta}$ is bounded and continuous. Furthermore $\mathbb{E}_{\mu^\varepsilon}[\mathcal{E}(Z^{N,\delta})] \to \mathbb{E}_{\mu}[\mathcal{E}(Z^{N,\delta})]$. This implies that

$$\lim_{\varepsilon\to 0} F^{V^{\xi,f,\varepsilon}}(\mu^{N,\delta}) = F^{V^{\xi,f}}(\mu^{N,\delta})$$

Now by dominated/monotone convergence we have

$$\lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_{\mu}\int \xi\exp(\beta Z^{N,\delta})\mathrm{d}M^{\beta} = \mathbb{E}_{\mu}\int \xi\exp(\beta Z)\mathrm{d}M^{\beta},$$

and

$$\lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_{\mu}[\mathcal{E}(Z^{N,\delta})] = \mathbb{E}_{\mu}[\mathcal{E}(Z)] \qquad \lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_{\mu}[f(W+Z^{N,\delta})] = \mathbb{E}_{\mu}[f(W+Z)],$$

so taking a suitable diagonal sequence implies the statement. □

Corollary 48.
$$\lim_{\varepsilon\to 0}\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{V^{\xi,\varepsilon}}(\mu) = \inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{V^{\xi}}(\mu)$$

and the infimum on the r.h.s is attained.

Proof. The statement will follow from the fundamental theorem of Γ-convergence (see e.g. [18]), once we have established that $\check{F}^{V^{\xi,\varepsilon}}$ is equicoercive on $\bar{\mathcal{X}}^{\exp}$, but this follows from

$$\check{F}^{V^{\xi,f,\varepsilon}}(\mu) \geqslant \mathbb{E}_{\mu}[\mathcal{E}(Z)] - C.$$ □

We can now relate this back to our general framework:

Lemma 49. *The functional $\check{F}^{f,\xi}$ has a minimiser $\mu^{f,\xi} \in \bar{\mathcal{X}}^{\exp}$ and*

$$\mathcal{W}^{\xi,\exp}(f) := -\log\int \exp(-f(\phi))\mathrm{d}\nu^{\xi,\exp} = \min_{\mu\in\bar{\mathcal{X}}^{\exp}}\check{F}^{f,\xi}(\mu) - \min_{\mu\in\bar{\mathcal{X}}^{\exp}}\check{F}^{0,\xi}(\mu) = \check{F}^{f,\xi}(\mu^{f,\xi}) - \check{F}^{0,\xi}(\mu^{0,\xi}).$$

Furthermore for any minimiser $\mu\in\bar{\mathcal{X}}^{\exp}$ the following wFBSDE

$$\mathbb{E}_{\mu}\left[\lambda\beta\int \xi\exp(\beta Z)\,K\mathrm{d}M^{\beta} + 2\mathcal{E}(Z,K)\right] = -\mathbb{E}_{\mu}\left[\int \nabla f(W+Z)\,K\right]$$

holds for any admissible control K.

Proof. This follows *mutatis mutandis* as in Lemma 31. □

Lemma 50. *The functional $\check{F}^{0,\xi}$ has a unique minimiser $\mu^{0,\xi}$ in $\bar{\mathcal{X}}$. Furthermore*

$$Z \leqslant 0,$$

$\mu^{0,\xi}$ almost surely, and

$$\sup_{\xi} \mathbb{E}_{\mu^{0,\xi}}[\|Z\|_{\hat{\mathfrak{H}}}^2] < \infty.$$

Proof. Observe that $F^{0,\xi}(Z)$ is convex in Z. Now let μ, ν be two minimisers for $\check{F}^{0,\xi}$. Recall from Lemma 19 that there exists $\upsilon \in \bar{\mathcal{Y}}(\mu)$ such that $\mathrm{Law}_\upsilon(W, Z+K) = \nu$. Now

$$\begin{aligned}\mathbb{E}_\upsilon\Big[\lambda \int \xi \exp(\beta(Z+K))\mathrm{d}M^\beta + \mathcal{E}(Z+K)\Big] - \mathbb{E}_\upsilon\Big[\lambda \int \xi \exp(\beta Z)\mathrm{d}M^\beta + \mathcal{E}(Z)\Big] \\ \geqslant \mathbb{E}_\upsilon\Big[\lambda\beta \int \xi \exp(\beta Z) K \mathrm{d}M^\beta + 2\mathcal{E}(Z,K) + \mathcal{E}(K)\Big].\end{aligned}$$

By Lemma 49 we have that

$$\mathbb{E}_\upsilon\Big[\lambda\beta \int \xi \exp(\beta Z) K \mathrm{d}M^\beta + 2\mathcal{E}(Z,K)\Big] = 0,$$

so

$$\mathbb{E}[\mathcal{E}(K)] \leqslant \check{F}^{0,\xi}(\nu) - \check{F}^{0,\xi}(\mu) = 0,$$

which implies $K = 0$ and therefore $\nu = \mu$. For the second statement observe that the mapping $A\colon Z \to -|Z|$ leaves the energy $\mathcal{E}(Z)$ invariant by chain rule, while

$$\int \xi \exp(\beta A(Z))\mathrm{d}M^\beta \leqslant \int \xi \exp(\beta Z)\mathrm{d}M^\beta,$$

so if $\mu^{0,\xi}$ is a minimiser, so is $\mathrm{Law}_{\mu^{0,\xi}}(W, A(Z))$. By uniqueness we can conclude. For the third claim we test the wFBSDE with $K = \rho Z$ and obtain

$$\mathbb{E}_\mu\Big[\lambda\beta \int \xi \exp(\beta Z)\rho Z \mathrm{d}M^\beta + 2\mathcal{E}(Z, \rho Z)\Big] = -\mathbb{E}_\mu\Big[\int \nabla f(W+Z)\rho Z\Big].$$

We can again estimate as in the proof of Lemma 34

$$2\mathcal{E}(Z, \rho Z) \geqslant \frac{1}{2}\|Z\|_{\hat{\mathfrak{H}}}^2,$$

and

$$\int \nabla f(W+Z)\rho Z \lesssim \|Z\|_{\hat{\mathfrak{H}}}.$$

Furthermore observe that

$$\int_{\mathbb{R}^2} \xi \exp(\beta Z)\rho Z \mathrm{d}M^\beta \geqslant -C\int_{\mathbb{R}^2} \xi\rho \,\mathrm{d}M^\beta,$$

since $\exp(\beta Z)Z$ is bounded below. Summing up, we have the estimate

$$\frac{1}{2}\mathbb{E}_{\mu^{0,\xi}}[\|Z\|_{\hat{\mathfrak{H}}}^2] \lesssim \mathbb{E}[\|Z\|_{\hat{\mathfrak{H}}}^2]^{1/2} + \mathbb{E}_\mu\Big[\int_{\mathbb{R}^2} \xi\rho \,\mathrm{d}M^\beta\Big] \lesssim \mathbb{E}[\|Z\|_{\hat{\mathfrak{H}}}^2]^{1/2} + 1$$

where in the last bound we used Proposition 41. □

Lemma 51. *Let $\mu^\xi \in \bar{\mathcal{X}}^{\exp}$. For $\upsilon \in \bar{\mathcal{Y}}(\mu^\xi)$ we have*

$$\check{H}^{\xi,\exp}(\upsilon) := \check{H}^{V^\xi}(\upsilon) = \mathbb{E}_\upsilon[2\mathcal{E}(K,Z) + \mathcal{E}(K)]$$
$$+\lambda\,\mathbb{E}_\upsilon\Big[\int \xi\exp(\beta(Z+K))\mathrm{d}M^\beta - \int \xi\exp(\beta Z)\mathrm{d}M^\beta\Big].$$

Then $\check{H}^{\xi,\exp}(\upsilon) \geqslant 0$ for all $\upsilon \in \bar{\mathcal{Y}}(\mu^\xi)$ iff μ^ξ is a minimiser of $\check{F}^{0,\xi}$. Moreover in this case, thanks to the wFBSDE for Z we have

$$\check{H}^{\xi,\exp}(\upsilon) = \mathbb{E}_\upsilon\Big[\mathcal{E}(K) + \lambda\int \xi\exp(\beta Z)(\exp(\beta K) - 1 - \beta K)\mathrm{d}M^\beta\Big],$$

and

$$\mathcal{W}^{\xi,\exp}(f) = \inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^\xi)} \check{G}^{f,\xi,\exp}(\upsilon), \quad \check{G}^{f,\xi,\exp}(\upsilon) := \mathbb{E}_\upsilon[f(W+Z+K)] + \check{H}^{\xi,\exp}(\upsilon).$$

Any minimiser $\upsilon \in \bar{\mathcal{Y}}(\mu^\xi)$ of $\check{G}^{f,\xi,\exp}$ satisfies the perturbed wFBSDE

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[2\mathcal{E}(K,H) + \lambda\beta\int \xi\exp(\beta Z)(\exp(\beta K)-1)H\mathrm{d}M^\beta\Big]\\ &= -\mathbb{E}_\upsilon[\nabla f(W+Z+K)\cdot H].\end{aligned} \tag{34}$$

The proof of Lemma 51 is completely analogous to Lemma 33 and Lemma 31, thus we omit the details. Thanks to the convexity of the interaction we have a good control of the locality properties of the solutions to the wFBSDE (34).

Lemma 52. *Assume that $N > 0$ and*

$$\sup_{\varphi\in L^2(\mathbb{R}^2)} \|\nabla f(\varphi)\|_{H^{-1}(\rho^{-N})} < \infty.$$

Then for any minimiser υ^f of $\check{G}^{f,\xi,\exp}$ we have

$$\mathbb{E}_{\upsilon^f}[\|K\|^2_{H^1(\rho^{-N})}] < \infty.$$

Proof. Testing equation (34) with $H = \rho^{-2N}K$ we have

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}\dot{K}_s(1-\Delta)\dot{K}_s\mathrm{d}s\Big]\\ &+\lambda\,\beta\mathbb{E}_\upsilon\Big[\int \xi\exp(\beta Z)(\exp(\beta K)-1)\rho^{-2N}K\mathrm{d}M^\beta\Big]\\ =\ &-\mathbb{E}_\upsilon\Big[\int \rho^{-2N}\nabla f(W+Z+K)K\Big] \leqslant C\mathbb{E}_\upsilon[\|K\|^2_{H^1(\rho^{-N})}]^{1/2}\end{aligned}$$

Now

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}\dot{K}_s(1-\Delta)\dot{K}_s\mathrm{d}s\Big]\\ =\ &\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}((1-\Delta)^{1/2}\dot{K}_s)^2\mathrm{d}s\Big]\\ &+\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}[(1-\Delta)^{1/2},\rho^{-2N}]\dot{K}_s(1-\Delta)^{1/2}\dot{K}_s\mathrm{d}s\Big]\end{aligned}$$

and we can compute, as in the proof of Lemma 34,

$$[(1-\Delta)^{1/2},\rho^{-2N}]\dot{K}_s \leqslant \delta\rho^{-2N}\dot{K}_s.$$

Furthermore observe that

$$\mathbb{E}_\upsilon\Big[\int \xi\exp(\beta Z)(\exp(\beta K)-1)\rho^{-2N}K\mathrm{d}M^\beta\Big]\geqslant 0.$$

Putting everything together, and using the commutator bound to absorb the second term in the decomposition above into the first one, we have

$$\begin{aligned}(1-\delta)\mathbb{E}_\upsilon\|K\|^2_{H^1(\rho^{-N})} &\lesssim (1-\delta)\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}((1-\Delta)^{1/2}\dot K_s)^2\mathrm{d}s\Big]\\ &\leqslant C\mathbb{E}_\upsilon[\|K\|^2_{H^1(\rho^{-N})}]^{1/2}\end{aligned}$$

which implies the statement. □

Definition 53. *For $N>0$ large (to be chosen suitably below) denote by $\mathcal{Y}^M(\mu)$ the set of measures $\upsilon\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\mathfrak{H}_w(\rho^{-N}))$ such that*

a) $\mathrm{Law}_\upsilon(W,M^\beta,Z)=\mu$

b) $\exists(Z^n)_n,(K^n)_n\subseteq\hat{\mathfrak{H}}^a$ *for which*

$$\mathrm{Law}(W,M^{\beta,n},Z^n,K^n)\to\upsilon \text{ and } \sup_{n\in\mathbb{N}}\mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}}+\|K^n\|^2_{\mathfrak{H}(\rho^{-N})}]\leqslant 2M.$$

Let $\mathcal{Y}^M:=\bigcup_{\mu\in\bar{\mathcal{X}}^{\exp}}\mathcal{Y}^M(\mu)$. We say that $(\upsilon^n)_n\subseteq\mathcal{Y}^M$ converges to $\upsilon\in\mathcal{Y}^M$ if $\upsilon^n\to\upsilon$ weakly in $\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\mathfrak{H}_w(\rho^{-N}))$ and

$$\sup_{n\in\mathbb{N}}\mathbb{E}_{\upsilon^n}[\|Z\|^2_{\hat{\mathfrak{H}}_w}+\|K\|^2_{\mathfrak{H}(\rho^{-N})}]\leqslant 2M.$$

Remark 54. Note that any sequence in $\mathcal{Y}^M$ has a convergent subsequence.

Corollary 55. *There exists a sufficiently large $M=M(f)$ (not depending on ξ) such that*

$$\inf_{\upsilon\in\mathcal{Y}(\mu^{0,\xi})}\check G^{f,\xi,\exp}(\upsilon)=\inf_{\upsilon\in\mathcal{Y}^M(\mu^{0,\xi})}\check G^{f,\xi,\exp}(\upsilon).$$

Proof. This is a direct consequence of the decay of the minimiser established in Lemma 52. □

Definition 56. *Define*

$$\bar G^{f,\xi,\exp}(\upsilon)=\check G^{f,\xi,\exp}(\upsilon) \text{ if } \upsilon\in\mathcal{Y}^{M(f)}(\mu^{0,\xi}) \text{ and } +\infty \text{ otherwise.}$$

The next lemma discusses Γ-convergence of the variational description as the infrared cutoff is removed.

Lemma 57. *Let $(\xi_n)_n$ be a sequence of spatial cutoffs such that $\xi_n(x)=1$ for $|x|\leqslant n$, and*

$$\sup_{n\in\mathbb{N}}\|\xi_n\|_{W^{k,\infty}}<+\infty,\qquad \text{for any } k\in\mathbb{N}.$$

Pick a subsequence ξ_n (not relabelled) such that $\mu^n\to\mu$ weakly on $\mathfrak{S}\times\hat{\mathfrak{H}}_w$ where μ^n is the minimiser of $\check F^{0,\xi_n}$ in $\bar{\mathcal{X}}$. Then $\bar G^{f,\xi_n,\exp}$ Γ-converges to $\bar G^{f,\infty,\exp}$ on $\mathcal{Y}^M$ where

$$\bar G^{f,\infty,\exp}(\upsilon)=\mathbb{E}_\upsilon\Big[f(W+Z+K)+\mathcal{E}(K)+\lambda\int\exp(\beta Z)\Psi(\beta K)\mathrm{d}M^\beta\Big],\qquad \Psi(t):=e^t-1-t\geqslant 0,$$

if $\upsilon \in \mathcal{Y}^M(\mu)$ and $+\infty$ otherwise. Note that $\bar{G}^{f,\infty,\exp}$ is the formal limit $\xi \to 1$ of the reduced form of $\check{G}^{f,\xi,\exp}$ given in Lemma 51.

Proof. Recall that, since μ^n minimises $\check{F}^{0,\xi_n}$, by Lemma 51 we have, for $\upsilon \in \mathcal{Y}^M(\mu^n)$,

$$\bar{G}^{f,\xi_n,\exp}(\upsilon) = \mathbb{E}_\upsilon\Big[f(W+Z+K) + \mathcal{E}(K) + \lambda \int \xi_n \exp(\beta Z)\Psi(\beta K)\mathrm{d}M^\beta \Big].$$

The function Ψ is convex and non-negative, with $\Psi(0)=0$, decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$; in particular all the integrands above are non-negative. Fix $p \in (1, \min(2, 8\pi/\beta^2))$ and δ with $\beta^2(p-1)/(4\pi) < \delta < 2/p'$ (this interval is non-empty since $p < 8\pi/\beta^2$), so that $M^\beta \in B^{-\delta}_{p,p}(\rho^{1/2})$ almost surely by Proposition 41, and take these values of p and $s=\delta$ in the definition of the space $\mathfrak{M}$ in $\mathfrak{S} = \mathfrak{S}_0 \times \mathfrak{M}$.

Step 1 (a convergence lemma). Let $(Z^n, K^n, M^{\beta,n})$ and (Z, K, M^β) be random variables on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that, almost surely, $Z^n \to Z$ and $K^n \to K$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $M^{\beta,n} \to M^\beta$ in $B^{-\delta}_{p,p}(\rho^{1/2})$, and such that the law of $M^{\beta,n}$ is that of M^β for all n. Let $R>0$ and let $\Phi\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ be bounded, smooth in x, Lipschitz in (z,k) uniformly in x, and such that $\Phi(z,k,x)=0$ for $|x| \geqslant R$. Then

$$\mathbb{E}_{\tilde{\mathbb{P}}} \int \Phi(Z^n, K^n, x) M^{\beta,n}(\mathrm{d}x) \to \mathbb{E}_{\tilde{\mathbb{P}}} \int \Phi(Z, K, x) M^\beta(\mathrm{d}x). \tag{35}$$

Indeed, let $g^n := \Phi(Z^n, K^n, \cdot)$ and $g := \Phi(Z, K, \cdot)$, all supported in $B(0,R)$. Almost surely the sequences $(Z^n)_n, (K^n)_n$ are bounded in $H^1(B(0,R))$ (being weakly convergent), so by the chain rule $(g^n)_n$ is bounded in H^1, and by Rellich's theorem and the Lipschitz property of Φ we have $g^n \to g$ in L^2. By interpolation $g^n \to g$ in H^s for all $s<1$ and, since $p' \geqslant 2$, the Sobolev embedding $H^s \subseteq B^{s-1+2/p'}_{p',p'}$ for functions supported in $B(0,R)$ gives $g^n \to g$ in $B^\delta_{p',p'}$, choosing $s \in (1+\delta-2/p', 1)$. By duality, since ρ is bounded above and below on $B(0,R)$,

$$\left| \int g^n \mathrm{d}M^{\beta,n} - \int g \mathrm{d}M^\beta \right| \lesssim_R \|g^n - g\|_{B^\delta_{p',p'}} \|M^{\beta,n}\|_{B^{-\delta}_{p,p}(\rho^{1/2})} + \left| \int g \mathrm{d}(M^{\beta,n} - M^\beta) \right| \to 0$$

almost surely. Finally $|\int g^n \mathrm{d}M^{\beta,n}| \leqslant \|\Phi\|_{L^\infty} \int \eta \mathrm{d}M^{\beta,n}$ for a smooth bump function $\eta \geqslant \mathbb{1}_{B(0,R)}$, and $\int \eta \mathrm{d}M^{\beta,n}$ has a law independent of n and a finite p-th moment (Proposition 41). Hence the sequence is uniformly integrable and (35) follows from Vitali's theorem.

We will use Step 1 with the following truncation. Let $\chi \in C^\infty_c(\mathbb{R}^2)$ be radial, non-increasing in $|x|$, with $\chi = 1$ on $B(0,1)$ and $\chi = 0$ outside $B(0,2)$, and let

$$\varsigma_m(z,x) := \chi(x/m)\max(-m, \min(z, (2m-z)_+)), \qquad z \in \mathbb{R}, x \in \mathbb{R}^2.$$

Then ς_m is smooth in x, 1-Lipschitz in z, $|\varsigma_m| \leqslant m$, $\varsigma_m(z,x) = 0$ for $|x| \geqslant 2m$, $\varsigma_m(z,x)$ has the sign of z, and $|\varsigma_m(z,x)| \uparrow |z|$ as $m \to \infty$. We write $\varsigma_m(h,x)$ for the function $x \mapsto \varsigma_m(h(x),x)$.

Step 2 (liminf inequality). Assume that $\upsilon^n \to \upsilon$ in $\mathcal{Y}^M$. If $\liminf_n \bar{G}^{f,\xi_n,\exp}(\upsilon^n) = +\infty$ there is nothing to prove; otherwise, passing to a subsequence realising the liminf, we may assume that $\bar{G}^{f,\xi_n,\exp}(\upsilon^n) < \infty$, i.e. $\upsilon^n \in \mathcal{Y}^M(\mu^n)$, for all n. Then $\upsilon \in \mathcal{Y}^M(\mu)$: indeed $\mathrm{Law}_\upsilon(W, M^\beta, Z) = \lim_n \mathrm{Law}_{\upsilon^n}(W, M^\beta, Z) = \lim_n \mu^n = \mu$, and property b) in the definition of $\mathcal{Y}^M$ passes to the limit by a diagonal argument, since the approximating sequences of all the υ^n satisfy the same bound $2M$ (on such bounded sets the weak topologies involved are metrisable). In particular $\bar{G}^{f,\infty,\exp}(\upsilon)$ is given by the formula in the statement. From the definition of convergence in $\mathcal{Y}^M$ we have

$$\mathbb{E}_{\upsilon^n}[f(W+Z+K)] \to \mathbb{E}_\upsilon[f(W+Z+K)], \qquad \mathbb{E}_\upsilon[\mathcal{E}(K)] \leqslant \liminf_{n\to\infty} \mathbb{E}_{\upsilon^n}[\mathcal{E}(K)],$$

the latter by the Portmanteau theorem. By the Skorokhod representation theorem [37] there exist a probability space $\tilde{\mathbb{P}}$ and random variables $(W^n, M^{\beta,n}, Z^n, K^n)$ with law υ^n converging almost surely in $\mathfrak{S} \times \hat{\mathfrak{H}}_w \times \mathfrak{H}_w(\rho^{-N})$ to (W, M^β, Z, K) with law υ. By the choice of the topology of $\mathfrak{S}$, $M^{\beta,n} \to M^\beta$ in $B^{-\delta}_{p,p}(\rho^{1/2})$ almost surely, the law of $M^{\beta,n}$ is that of M^β, and $Z^n \to Z$, $K^n \to K$ weakly in H^1_{loc} almost surely. Moreover $Z^n \leqslant 0$ almost surely (since $Z \leqslant 0$ μ^n-almost surely), hence also $Z \leqslant 0$.

For fixed m apply Step 1 with $R = 2m$ and $\Phi_m(z,k,x) := e^{\beta(z\wedge 0)}\varsigma_m(\Psi(\beta k), x)$: it is bounded by m, it vanishes for $|x| \geqslant 2m$, and it is Lipschitz in (z,k) since $\varsigma_m(\cdot, x)$ is 1-Lipschitz and vanishes when $\Psi(\beta k) \geqslant 2m$, so that only k in a compact interval matters, where $\Psi(\beta\,\cdot)$ is Lipschitz. Since $\xi_n = 1$ on $B(0,2m)$ for $n \geqslant 2m$, we obtain

$$\lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}} \int \xi_n \exp(\beta Z^n) \varsigma_m(\Psi(\beta K^n), x) \mathrm{d}M^{\beta,n} = \mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z) \varsigma_m(\Psi(\beta K), x) \mathrm{d}M^\beta.$$

Since $\Psi \geqslant 0$, we have $0 \leqslant \varsigma_m(\Psi, x) \leqslant \Psi$ and $\varsigma_m(\Psi, x) \uparrow \Psi$ as $m \to \infty$; by monotone convergence

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z) \Psi(\beta K) \mathrm{d}M^\beta &= \sup_m \mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z) \varsigma_m(\Psi(\beta K), x) \mathrm{d}M^\beta \\
&= \sup_m \lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}} \int \xi_n \exp(\beta Z^n) \varsigma_m(\Psi(\beta K^n), x) \mathrm{d}M^{\beta,n} \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}} \int \xi_n \exp(\beta Z^n) \Psi(\beta K^n) \mathrm{d}M^{\beta,n}.
\end{aligned}$$

Adding up the three terms gives $\bar{G}^{f,\infty,\exp}(\upsilon) \leqslant \liminf_n \bar{G}^{f,\xi_n,\exp}(\upsilon^n)$. Note that, the integrand being non-negative, no separate treatment of its negative part is needed.

Step 3 (recovery sequence). Let $\upsilon \in \mathcal{Y}^M(\mu)$ with $\bar{G}^{f,\infty,\exp}(\upsilon) < \infty$, so that $\mathbb{E}_\upsilon \int \exp(\beta Z)\Psi(\beta K)\mathrm{d}M^\beta < \infty$ and $\mathbb{E}_\upsilon \mathcal{E}(K) \leqslant M$. Set $K^m := \varsigma_m(K, x)$ and $\upsilon^m := \mathrm{Law}_\upsilon(W, M^\beta, Z, K^m)$. Since ς_m preserves the sign and $|\varsigma_m(K,x)| \uparrow |K|$, and Ψ is monotone on each half-line, we have $\Psi(\beta K^m) \uparrow \Psi(\beta K)$, so by monotone convergence

$$\lim_{m\to\infty} \mathbb{E}_\upsilon \int \exp(\beta Z)\Psi(\beta K^m)\mathrm{d}M^\beta = \mathbb{E}_\upsilon \int \exp(\beta Z)\Psi(\beta K)\mathrm{d}M^\beta,$$

while $\mathbb{E}_\upsilon f(W + Z + K^m) \to \mathbb{E}_\upsilon f(W + Z + K)$ by boundedness and continuity of f. We use moreover that the truncation can be performed so that $\mathbb{E}_\upsilon[\mathcal{E}(K^m)] \to \mathbb{E}_\upsilon[\mathcal{E}(K)]$; we do not detail this approximation step. Then $\bar{G}^{f,\infty,\exp}(\upsilon^m) \to \bar{G}^{f,\infty,\exp}(\upsilon)$ and, by a diagonal argument, it is enough to construct a recovery sequence for each υ^m.

Fix m. Invoking [37] there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $(W^n, M^{\beta,n}, Z^n)$ on it with $\mathrm{Law}_{\tilde{\mathbb{P}}}(W^n, M^{\beta,n}, Z^n) = \mu^n$ such that $(W^n, M^{\beta,n}, Z^n) \to (W, M^\beta, Z)$ almost surely in $\mathfrak{S} \times \hat{\mathfrak{H}}_w$. Let K be a random variable on $\tilde{\Omega}$ (possibly after enlarging it) such that $\upsilon = \mathrm{Law}_{\tilde{\mathbb{P}}}(W, M^\beta, Z, K)$, set $K^m = \varsigma_m(K, x)$ and $\upsilon^{n,m} := \mathrm{Law}_{\tilde{\mathbb{P}}}(W^n, M^{\beta,n}, Z^n, K^m)$. We claim that

$$\bar{G}^{f,\xi_n,\exp}(\upsilon^{n,m}) \to \bar{G}^{f,\infty,\exp}(\upsilon^m).$$

The energy term $\mathbb{E}[\mathcal{E}(K^m)]$ does not depend on n, and $\mathbb{E}_{\tilde{\mathbb{P}}} f(W^n + Z^n + K^m) \to \mathbb{E}_{\tilde{\mathbb{P}}} f(W + Z + K^m)$ by boundedness and continuity of f. For the interaction term apply Step 1 with $R = 2m$, the constant sequence $K^n = K$ and $\Phi(z,k,x) := e^{\beta(z\wedge 0)}\Psi(\beta\varsigma_m(k,x))$, which is bounded (since $|\varsigma_m| \leqslant m$), Lipschitz in (z,k), smooth in x and vanishes for $|x| \geqslant 2m$: using again $\xi_n = 1$ on $B(0,2m)$ for $n \geqslant 2m$ we get

$$\mathbb{E}_{\tilde{\mathbb{P}}} \int \xi_n \exp(\beta Z^n)\Psi(\beta K^m)\mathrm{d}M^{\beta,n} \to \mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z)\Psi(\beta K^m)\mathrm{d}M^\beta.$$

This proves the claim. Taking a diagonal sequence we obtain the statement. □

Corollary 58. *If $(\xi_n)_n$ is as in Lemma 57 we have*

$$\lim_{n\to\infty} -\log\int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp} = \inf_{\upsilon\in\mathcal{Y}^M(\mu)} \bar{G}^{f,\infty,\exp}(\upsilon)$$

Proof. This follows from Γ-convergence, the fact that $\mathcal{Y}^M$ is sequentially compact and the fundamental theorem of Γ-convergence (see e.g. [18]). □

We conclude with some technical results needed in the proofs above.

Lemma 59. *Assume that $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, g, g' are bounded. Define $f = g_-$ (the negative part of g). We have that the map*

$$\varphi \mapsto f(\varphi),$$

continuously sends $W^{1,2}_w(\rho^\alpha)$ into itself with

$$\|f(\varphi)\|_{W^{1,2}(\rho^\alpha)} \lesssim 1 + \|\varphi\|_{W^{1,2}(\rho^\alpha)}.$$

If $g(0) = 0$ we have in particular

$$\|f(\varphi)\|_{W^{1,2}(\rho^\alpha)} \lesssim \|\varphi\|_{W^{1,2}(\rho^\alpha)}.$$

Proof. Assume that $g(0) = 0$. By definition of f we have

$$\int |f(\varphi(x))|^2\rho^\alpha(x)\mathrm{d}x \lesssim \int |\varphi(x)|^2\rho^\alpha(x)\mathrm{d}x.$$

Furthermore by [19] p. 291–292, we have

$$\int |\nabla_x f(\varphi(x))|^2\rho^\alpha(x)\mathrm{d}x = \int \mathbb{1}_{g(\varphi(x))\leqslant 0}|g'(\varphi(x))\nabla_x\varphi(x)|^2\rho^\alpha(x)\mathrm{d}x \lesssim \int |\nabla_x\varphi(x)|^2\rho^\alpha(x)\mathrm{d}x.$$

Now for weak continuity, let $\varphi_n \to \varphi$ weakly in $W^{1,2}(\rho^\alpha)$. We know that

$$\sup_n \|\varphi_n\|_{W^{1,2}(\rho^\alpha)} < \infty \text{ which implies } \sup_n \|f(\varphi_n)\|_{W^{1,2}(\rho^\alpha)} < \infty.$$

Now let φ_{n_k} be a subsequence such that $f(\varphi_{n_k}) \to \xi$. We can extract a further subsequence (not relabelled) such that

$$\varphi_{n_k} \to \varphi$$

almost surely. Then $f(\varphi_{n_k}) \to f(\varphi)$ almost surely which in turn implies $\xi = f(\varphi)$. From this we can conclude. The first statement follows from the second by writing $g = \tilde{g} + g(0)$ with $\tilde{g}(0) = 0$. □

Lemma 60. *Let $M^{\beta,n}$ be a sequence of random positive Radon measures on $\mathbb{R}^2$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $M^{\beta,n} \to M^\beta$ $\mathbb{P}$-almost surely in $B^{-s}_{p,p}(\rho)$ for some $s > \frac{\beta^2}{4\pi}(p-1)$ and $p < 8\pi/\beta^2$ close enough to $\frac{\beta^2}{4\pi}(p-1), 8\pi/\beta^2$ respectively. There exist $q < 2$ and $N_0 < \infty$ such that we have*

$$\mathbb{E}\int f_n \mathrm{d}M^{\beta,n} \to \mathbb{E}\int f\mathrm{d}M^\beta$$

whenever

$$\mathbb{E}\Big[\|f_n - f\|^q_{W^{1,q}(\rho^{-N_0})}\Big] \to 0$$

and

$$\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty} + \|f\|_{L^\infty} = C < \infty.$$

Proof. By interpolation between L^∞ and $W^{1,q}(\rho^{-N_0})$ we have that for $\delta < \theta := q/p'$ and $N_0 \geqslant \theta^{-1}$ (using also the uniform L^∞ bound on $f_n - f$)

$$\|f_n - f\|_{B^\delta_{p',p'}(\rho^{-1})} \leqslant C\|f_n - f\|^\theta_{W^{1,q}(\rho^{-N_0})}, \qquad \|f\|_{B^\delta_{p',p'}(\rho^{-1})} \leqslant C\|f\|^\theta_{W^{1,q}(\rho^{-N_0})}.$$

Now recall that for any $\frac{8\pi}{\beta^2} > p > 1$ and $\delta > \frac{\beta^2}{4\pi}(p-1)$ we have

$$\sup_n \mathbb{E}\Big[\|M^{\beta,n}\|^p_{B^{-\delta}_{p,p}(\rho)}\Big] < \infty,$$

so provided $1/p + 1/p' = 1$

$$\begin{aligned}
\mathbb{E}\Big[\int f_n \mathrm{d}M^{\beta,n} - \int f \mathrm{d}M^\beta\Big] &= \mathbb{E}\Big[\int (f_n - f)\mathrm{d}M^{\beta,n}\Big] + \mathbb{E}\Big[\int f \mathrm{d}M^{\beta,n} - \int f \mathrm{d}M^\beta\Big] \\
&\leqslant \mathbb{E}\Big[\|f_n - f\|_{B^\delta_{p',p'}(\rho^{-1})}\|M^{\beta,n}\|_{B^{-\delta}_{p,p}(\rho)}\Big] \\
&\quad + \mathbb{E}\Big[\|f\|_{B^\delta_{p',p'}(\rho^{-1})}\|M^{\beta,n} - M^\beta\|_{B^{-\delta}_{p,p}(\rho)}\Big] \\
&\leqslant \mathbb{E}\Big[C\|f_n - f\|^q_{W^{1,q}(\rho^{-N_0})}\Big]^{\frac{1}{p'}} \mathbb{E}\Big[\|M^{\beta,n}\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]^{\frac{1}{p}} \\
&\quad + \mathbb{E}\Big[C\|f\|^q_{W^{1,q}(\rho^{-N_0})}\Big]^{\frac{1}{p'}} \mathbb{E}\Big[\|M^{\beta,n} - M^\beta\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]^{\frac{1}{p}}. \\
&\to 0
\end{aligned}$$

since $p'\theta = q$. Note that for any $\beta^2 < 8\pi$ we can choose $q < 2$ such that

$$q/p' = \frac{q}{p}(p-1) > \frac{\beta^2}{4\pi}(p-1)$$

so if we choose δ such that

$$\frac{q}{p}(p-1) > \delta > \frac{\beta^2}{4\pi}(p-1)$$

we obtain as required

$$\frac{8\pi}{\beta^2} > p > 1 \qquad \delta > \frac{\beta^2}{4\pi}(p-1) \qquad \delta < q/p' = \frac{q}{p}(p-1). \qquad \square$$

Lemma 61. *For any $q < 2$ and any $N_0 < \infty$ there exists an $N < \infty$ such that if*

$$\|f\|_{L^\infty}, \|g\|_{L^\infty} \leqslant C,$$

then, with a constant depending on C,

$$\|fg\|_{W^{1,q}(\rho^{-N_0})} \lesssim \|g\|_{H^1(\rho^{-N})} + \|f\|_{H^1(\rho)}\|g\|^{(2-q)/q}_{H^1(\rho^{-N})}.$$

Note that the L^∞ norms are unweighted. If furthermore $f_n \to f$ in $H^1(\rho)$ and $g_n \to g$ in $H^1(\rho^{-N})$ and $\sup_n \|f_n\|_{L^\infty} + \|g_n\|_{L^\infty} \leqslant C$ then

$$f_n g_n \to fg \qquad \text{in } W^{1,q}(\rho^{-N_0}).$$

Proof. It is enough to prove the statement for smooth functions by density. By Hölder's inequality we have

$$\|fg\|_{L^q(\rho^{-N_0})} \leqslant \|f\|_{L^\infty}\|\rho^{N-N_0}\|_{L^r}\|g\|_{L^2(\rho^{-N})},$$

where $1/r = 1/q - 1/2$, so that $r \in (2,\infty)$ and $2/r = (2-q)/q$, and we take $N \geqslant (N_0+1)r/2$ (so that $\rho^{N-N_0} \in L^r$ and $\rho^{(N-N_0)r/2} \lesssim \rho$, recall $\rho \lesssim \langle\cdot\rangle^{-32}$). By the Leibniz rule and the same argument

$$\|\nabla(fg)\|_{L^q(\rho^{-N_0})} \leqslant \|f\|_{L^\infty}\|\rho^{N-N_0}\|_{L^r}\|\nabla g\|_{L^2(\rho^{-N})} + \|\nabla f\|_{L^2(\rho)}\|g\|_{L^r(\rho^{-N_0-1})}.$$

By interpolation between L^∞ and L^2 (i.e. $|g|^r \leqslant \|g\|_{L^\infty}^{r-2}|g|^2$) and since $\rho^{-(N_0+1)r} \lesssim \rho^{-2N}$, we also have

$$\|g\|_{L^r(\rho^{-N_0-1})} \leqslant \|g\|_{L^\infty}^{1-2/r}\|g\|_{L^2(\rho^{-(N_0+1)r/2})}^{2/r} \lesssim C^{1-2/r}\|g\|_{L^2(\rho^{-N})}^{(2-q)/q}.$$

This gives the estimate. For the second statement we rewrite

$$fg - f_n g_n = g\,(f - f_n) + f_n\,(g - g_n).$$

The second term goes to zero by the estimate applied to $(f_n, g-g_n)$, since $\sup_n \|f_n\|_{H^1(\rho)} < \infty$. For the first one, proceeding as above (with the roles of the factors exchanged in the terms without derivative on $f - f_n$),

$$\|g\,(f-f_n)\|_{W^{1,q}(\rho^{-N_0})} \lesssim \|(f-f_n)\rho^{N-N_0}\|_{L^r}\|g\|_{H^1(\rho^{-N})} + \|f-f_n\|_{H^1(\rho)}\|g\|_{L^r(\rho^{-N_0-1})},$$

and by interpolation again $\|(f-f_n)\rho^{N-N_0}\|_{L^r} \lesssim C^{1-2/r}\|f-f_n\|_{L^2(\rho)}^{2/r} \to 0$. □

6 A scale-by-scale interpolation

In this intermezzo we prepare for the analysis of the UV limit of the quartic interaction in three dimensions, i.e. for the φ^4_3 model in finite volume $\Lambda = \mathbb{T}^3$.

Recall that the φ^4_d Gibbs measure on the d-dimensional torus $\Lambda = \Lambda_L = \mathbb{T}^d_L = (\mathbb{R}/(2\pi L\mathbb{Z}))^d$ is the probability measure ν obtained as the weak limit for $T \to \infty$ of the family $(\nu_T)_{T>0}$ given by

$$\nu_T(\mathrm{d}\phi) = \frac{\exp[-V_T(\phi)]}{\mathscr{Z}_T}\vartheta_T(\mathrm{d}\phi), \tag{36}$$

where

$$V_T(\varphi) := \lambda\int_\Lambda (|\varphi(\xi)|^4 - a_T|\varphi(\xi)|^2 + b_T)\mathrm{d}\xi, \qquad \mathscr{Z}_T := \int e^{-V_T(\phi_T)}\vartheta(\mathrm{d}\phi).$$

Here $\lambda \geqslant 0$ is a fixed constant, Δ is the Laplacian on Λ, ϑ is the centred Gaussian measure with covariance $(1-\Delta)^{-1}$, $\mathscr{Z}_T$ is a normalisation factor, a_T, b_T given constants and $\phi_T = \rho_T(\mathrm{D})\phi$ with ρ_T some appropriate smooth and compactly supported cutoff function such that $\rho_T \to 1$ as $T \to \infty$, and where $\vartheta_T := (\rho_T(\mathrm{D}))_*\vartheta$ is the law of the regularised field ϕ_T under ϑ. In other words ν_T is the law of ϕ_T under the probability measure $e^{-V_T(\phi_T)}\vartheta(\mathrm{d}\phi)/\mathscr{Z}_T$: this is the measure which is naturally controlled by the variational method below. The measures ϑ and ν_T are realised as probability measures on $\mathscr{S}'(\Lambda)$, the space of tempered distributions on Λ. They are supported on the Hölder–Besov space $\mathscr{C}^{(2-d)/2-\kappa}(\Lambda)$ for all small $\kappa > 0$. The existence of the limit ν is conditioned on the choice of a suitable sequence of *renormalisation constants* $(a_T, b_T)_{T>0}$. The constant b_T is not necessary, but is useful to decouple the behaviour of the numerator from that of the denominator in eq. (36).

The aim of the rest of these lecture notes is to give a proof of convergence using a variational formula for the partition function $\mathscr{Z}_T$ and for the generating function of the measure ν_T. As a byproduct we obtain also a variational description for the generating function of the limiting measure ν via Γ-convergence of the variational problem.

Define the normalised free energy $\mathcal{W}_T$ for the cutoff φ^4_d measure, as the functional

$$\mathcal{W}_T(f) := -\frac{1}{|\Lambda|}\log \int_{\mathscr{S}'(\Lambda)} \exp[-|\Lambda| f(\phi_T) - V_T(\phi_T)]\vartheta(\mathrm{d}\phi), \tag{37}$$

for all $f \in C(\mathscr{S}'(\Lambda); \mathbb{R})$.

In order to use the stochastic control problem we need to embed our problem into a setting where we have a Brownian motion. How to do this is quite arbitrary and mostly shaped by technical convenience. In two dimensions, where we have very mild singularities, we have adopted a simplistic approach, introducing a Brownian martingale with GFF marginals. To tackle the three dimensional case, where singularities are more severe, it is convenient to choose an embedding where the Brownian motion encodes a **scale decomposition** of the underlying GFF.

We begin by constructing a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ endowed with a process $(W_t)_{t\in[0,\infty]}$ belonging to $C([0,\infty], \mathscr{C}^{(2-d)/2-\kappa}(\Lambda))$ and such that $\mathrm{Law}_\vartheta(\phi_T) = \mathrm{Law}_{\mathbb{P}}(W_T)$ for all $T \geqslant 0$ and $\mathrm{Law}_{\mathbb{P}}(W_\infty) = \vartheta$, the Gaussian free field with covariance $(1-\Delta)^{-1}$.

Fix $\alpha < -d/2$ and let $\Omega := C(\mathbb{R}_+; H^\alpha)$, $(X_t)_{t\geqslant 0}$ the canonical process on Ω and $\mathscr{B}$ the Borel σ–algebra of Ω. On $(\Omega, \mathscr{B})$ consider the probability measure $\mathbb{P}$ which makes the canonical process X a cylindrical Brownian motion in $L^2(\Lambda)$. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}$ will denote expectations wrt. some other measure $\mathbb{Q}$. On the probability space $(\Omega, \mathscr{B}, \mathbb{P})$ there exists a collection $(B^n_t)_{n\in(L^{-1}\mathbb{Z})^d}$ of complex (2-dimensional) Brownian motions, such that $\overline{B^n_t} = B^{-n}_t$, B^n_t, B^m_t independent for $m \neq \pm n$ and $X_t = |\Lambda|^{-1/2}\sum_{n\in(L^{-1}\mathbb{Z})^d} e^{i\langle n,\cdot\rangle} B^n_t$.

Note that X has a.s. trajectories in $C(\mathbb{R}_+, \mathscr{C}^{-d/2-\varepsilon}(\Lambda))$ for any $\varepsilon > 0$ by standard arguments.

Fix some $\rho \in C^\infty_c(\mathbb{R}_+, \mathbb{R}_+)$, decreasing such that $\rho(s) = 1$ for any $s \leqslant 1/2$ and $\rho(s) = 0$ for any $s \geqslant 1$. For $x \in \mathbb{R}^d$, set, $\rho_t(x) := \rho(\langle x\rangle / t)$ and

$$\sigma_t(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho^2_t(x))\right)^{1/2} = (2\dot\rho_t(x)\rho_t(x))^{1/2} = (-2(\langle x\rangle/t)\rho(\langle x\rangle/t)\rho'(\langle x\rangle/t))^{1/2}/t^{1/2},$$

where $\dot\rho_t$ is the partial derivative of ρ_t with respect to t. Consider the process $(W_t)_{t\geqslant 0}$ defined by

$$W_t := \frac{1}{|\Lambda|^{1/2}} \sum_{n\in(L^{-1}\mathbb{Z})^d} \int_0^t \frac{\sigma_s(n)}{\langle n\rangle} e^{i\langle n,\cdot\rangle} \mathrm{d}B^n_s, \qquad t \geqslant 0. \tag{38}$$

It is a centred Gaussian process with covariance

$$\begin{aligned}
\mathbb{E}[\langle W_t, \varphi\rangle\langle W_s, \psi\rangle] &= \frac{1}{|\Lambda|} \sum_{n,m\in(L^{-1}\mathbb{Z})^d} \mathbb{E}\Big[\int_0^t \frac{\sigma_u(n)}{\langle n\rangle}\mathrm{d}B^n_u\, \hat\varphi(n) \overline{\int_0^s \frac{\sigma_u(m)}{\langle m\rangle}\mathrm{d}B^m_u\, \hat\psi(m)}\Big] \\
&= \frac{1}{|\Lambda|} \sum_{n\in(L^{-1}\mathbb{Z})^d} \frac{\rho^2_{\min(s,t)}(n)}{\langle n\rangle^2} \hat\varphi(n)\overline{\hat\psi(n)},
\end{aligned}$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Itô isometry. By dominated convergence

$$\lim_{t\to\infty} \mathbb{E}[\langle W_t, \varphi\rangle\langle W_t, \psi\rangle] = |\Lambda|^{-1} \sum_{n\in(L^{-1}\mathbb{Z})^d} \langle n\rangle^{-2} \hat\varphi(n)\overline{\hat\psi(n)}$$

for any $\varphi, \psi \in L^2(\Lambda)$.

Note that up to any finite time T the r.v. W_T has a bounded spectral support and the stopped process $W_t^T = W_{t\wedge T}$ for any fixed $T>0$, is in $C(\mathbb{R}_+, W^{k,2}(\Lambda))$ for any $k \in \mathbb{N}$. Furthermore $(W_t^T)_t$ only depends on a finite subset of the Brownian motions $(B^n)_n$. Denote

$$W_t = \int_0^t J_s \mathrm{d}X_s, \qquad t \geqslant 0, \tag{39}$$

with $J_s := \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D})$.

Observe that W_t has a distribution given by the push-forward $(\rho_t(\mathrm{D}))_* \vartheta$ of ϑ through $\rho_t(\mathrm{D})$. We write the measure ν_T in (36) in terms of expectations over $\mathbb{P}$ as

$$\int g(\phi) \nu_T(\mathrm{d}\phi) = \frac{\mathbb{E}[g(W_T)\, e^{-V_T(W_T)}]}{\mathscr{Z}_T}, \tag{40}$$

for any bounded measurable $g \colon \mathscr{S}'(\Lambda) \to \mathbb{R}$.

For fixed T the polynomial appearing in the expression for $V_T(W_T)$ is bounded below (since $\lambda > 0$) and $\mathscr{Z}_T$ is well defined and also bounded away from zero (this follows easily from Jensen's inequality). However as $T \to \infty$ we tend to lose both these properties due to the fact that we will be obliged to take $a_T \to +\infty$ to renormalise the non–linear terms. To obtain uniform upper and lower bounds we need a more detailed analysis and we proceed as follows.

Denote by $\mathbb{H}_a$ the space of progressively measurable processes which are $\mathbb{P}$–almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will need also (generalised) drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$, we denote the corresponding space with $\mathbb{H}_a^\alpha$. Consider the measure $\mathbb{Q}_T$ on $(\Omega, \mathscr{B})$ whose Radon–Nikodym derivative wrt. $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \frac{e^{-V_T(W_T)}}{\mathscr{Z}_T}.$$

For a drift $v \in \mathbb{H}_a^\alpha$ we denote by $I(v)$ the process

$$I_t(v) := \int_0^t J_s v_s \mathrm{d}s, \qquad t \geqslant 0,$$

(this is the shift of W induced by the shift $X \to X + \int_0^\cdot v_s \mathrm{d}s$ of the Brownian motion), and for a functional f we set $V_T^f := V_T + |\Lambda|\, f$.

The Boué–Dupuis formula takes the form:

Theorem 62. *For any $f \in C(\mathscr{C}^{(2-d)/2-\kappa}; \mathbb{R})$ with linear growth the Boué–Dupuis (BD) variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|} \log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \inf_{v \in \mathbb{H}_a} \mathbb{E}\left[\frac{1}{|\Lambda|} V_T^f(W_T + I_T(v)) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s\right],$$

where the expectation is taken wrt. the measure $\mathbb{P}$ on Ω.

Remark 63. The Boué–Dupuis formula gives $-\log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \inf_v \mathbb{E}\Big[V_T^f(W_T + I_T(v)) + \frac{1}{2}\int_0^\infty \int_\Lambda |v_s|^2 \mathrm{d}x\mathrm{d}s\Big]$, so that dividing by $|\Lambda|$ produces the factor $1/|\Lambda|$ also in front of the quadratic cost. To keep formulas light, here and in the rest of these notes the L^p, H^α and $\mathcal{H}$ norms appearing in the variational problems are *volume-averaged*, that is $\|v\|_{L^2}^2 := \fint_\Lambda |v|^2 = |\Lambda|^{-1}\int_\Lambda |v|^2$, consistently with the averaged integrals $\fint$ used in the potentials. With this convention the formula above, and all the variational formulas below, hold as written.

The next lemma provides a deterministic regularity result for $I(v)$: it says that the drift v generates shifts of the Gaussian free field in directions which belong to H^1 uniformly in the scale parameter up to ∞. The space H^1 is the Cameron–Martin space of the free field [38].

Lemma 64. *Let $\alpha \in \mathbb{R}$. For any $v \in L^2([0,\infty), H^\alpha)$ we have*

$$\sup_{0\leqslant t\leqslant T} \|I_t(v)\|^2_{H^{\alpha+1}} + \sup_{0\leqslant s<t\leqslant T} \frac{\|I_t(v)-I_s(v)\|^2_{H^{\alpha+1}}}{1\wedge(t-s)} \lesssim \int_0^T \|v_r\|^2_{H^\alpha}\mathrm{d}r.$$

Proof. Using the fact that $\sigma_s(\mathrm{D})$ is diagonal in Fourier space, and denoting with $(e_k)_{k\in(L^{-1}\mathbb{Z})^d}$ the basis of trigonometric polynomials, we have

$$\begin{aligned}
\left\|\int_r^t \sigma_s(\mathrm{D})v_s\mathrm{d}s\right\|^2_{H^\alpha} &= \frac{1}{|\Lambda|}\sum_{k\in(L^{-1}\mathbb{Z})^d}\langle k\rangle^{2\alpha}\left|\int_r^t\langle\sigma_s(\mathrm{D})e_k, v_s\rangle\mathrm{d}s\right|^2\\
&\leqslant \frac{1}{|\Lambda|}\sum_{k\in(L^{-1}\mathbb{Z})^d}\langle k\rangle^{2\alpha}\left(\int_r^t|\langle\sigma_s(\mathrm{D})e_k,e_k\rangle|^2\mathrm{d}s\right)\left(\int_r^t|\langle e_k,v_s\rangle|^2\mathrm{d}s\right)\\
&\leqslant \int_r^t\|v_s\|^2_{H^\alpha}\mathrm{d}s\sup_k\int_r^t\langle e_k,\sigma_s(\mathrm{D})^2e_k\rangle\mathrm{d}s\\
&\leqslant \int_r^t\|v_s\|^2_{H^\alpha}\mathrm{d}s\sup_k\langle e_k,\rho_t^2(\mathrm{D})e_k\rangle \leqslant \int_0^T\|v_s\|^2_{H^\alpha}\mathrm{d}s.
\end{aligned}$$

Since $I_t(v)-I_r(v) = \langle\mathrm{D}\rangle^{-1}\int_r^t\sigma_s(\mathrm{D})v_s\mathrm{d}s$, taking $r=0$ this gives the bound on $\sup_{t\leqslant T}\|I_t(v)\|^2_{H^{\alpha+1}}$, i.e. the first part of the statement, and also the bound on the increments when $t-r\geqslant 1$. On the other hand $\sigma_s(\mathrm{D})$ is a smooth Fourier multiplier and using Proposition 10 we have the estimate $\|\sigma_s(\mathrm{D})f\|_{H^\alpha}\lesssim\|f\|_{H^\alpha}/\langle s\rangle^{1/2}$ uniformly in $s\geqslant 0$, therefore, for all $0\leqslant r\leqslant t\leqslant T$, we have

$$\begin{aligned}
\left\|\int_r^t\sigma_s(\mathrm{D})v_s\mathrm{d}s\right\|^2_{H^\alpha} &\leqslant \left(\int_r^t\|\sigma_s(\mathrm{D})v_s\|_{H^\alpha}\mathrm{d}s\right)^2 \leqslant (t-r)\int_r^t\|\sigma_s(\mathrm{D})v_s\|^2_{H^\alpha}\mathrm{d}s\\
&\lesssim (t-r)\int_0^T\|v_s\|^2_{H^\alpha}\mathrm{d}s.
\end{aligned}$$

We conclude that

$$\|I_t(v)-I_r(v)\|^2_{H^{\alpha+1}} \lesssim \left\|\int_r^t\sigma_s(\mathrm{D})v_s\mathrm{d}s\right\|^2_{H^\alpha} \leqslant [1\wedge(t-r)]\int_0^T\|v_s\|^2_{H^\alpha}\mathrm{d}s.$$

□

6.1 Two dimensions

We consider first the case $d=2$ assuming that the test functional f satisfies the following estimate: for sufficiently small $\alpha>0$

$$|f(\varphi)| \leqslant C + \alpha\|\varphi\|^4_{B^{-\kappa}_{4,4}}. \tag{41}$$

From Theorem 62 we see that the relevant quantity to bound is of the form

$$F_T(u) := \mathbb{E}\left[f(W_T+I_T(u)) + \frac{1}{|\Lambda|}V_T(W_T+I_T(u)) + \frac{1}{2}\|u\|^2_{\mathcal{H}}\right], \tag{42}$$

for $u \in \mathbb{H}_a$. Theorem 62 is stated for f with linear growth, but this is not a restriction: the truncations $f_M := (f \wedge M) \vee (-M)$ are bounded and satisfy (41) with the same constants (since $|f_M| \leqslant |f|$), and by Fatou's lemma $\mathbb{E}[e^{-|\Lambda| f(W_T) - V_T(W_T)}] \leqslant \liminf_{M\to\infty} \mathbb{E}[e^{-|\Lambda| f_M(W_T) - V_T(W_T)}]$, so that the bounds below, which only depend on the constants in (41), transfer from f_M to f. From now on we leave implicit the integration variable over the spatial domain Λ and let $Z_t = I_t(u)$ for brevity. Choosing

$$a_T = 6\mathbb{E}[W_T(0)^2], \qquad b_T = 3\mathbb{E}[W_T(0)^2]^2, \tag{43}$$

we have

$$\frac{1}{|\Lambda|} V_T(W_T + Z_T) = \lambda \fint [\![W_T^4]\!] + 4\lambda \fint [\![W_T^3]\!] Z_T + 6\lambda \fint [\![W_T^2]\!] Z_T^2 + 4\lambda \fint W_T Z_T^3 + \lambda \fint Z_T^4,$$

where

$$\begin{aligned} [\![W_T^4]\!] &:= W_T^4 - 6\mathbb{E}[W_T^2]\, W_T^2 + 3\mathbb{E}[W_T^2]^2, \\ [\![W_T^3]\!] &:= W_T^3 - 3\mathbb{E}[W_T^2] W_T, \\ [\![W_T^2]\!] &:= W_T^2 - \mathbb{E}[W_T^2], \end{aligned}$$

denote the Wick powers of the Gaussian r.v. W_T [38].

These polynomials, when seen as stochastic processes in T, are $\mathbb{P}$–martingales wrt. the filtration of $(W_t)_t$:

$$[\![W_t^n]\!] = n \int_0^t [\![W_s^{n-1}]\!] \mathrm{d}W_s, \qquad n \geqslant 1.$$

In particular they have an expression as iterated stochastic integrals wrt. the Brownian motions $(B_t^n)_{t,n}$ introduced in eq. (38).

We will need to estimate the average size of the random fields $[\![W_T^n]\!]$ uniformly in T. Our computations in Section 2.3 show that for all $\kappa > 0$ we have, for any $N \geqslant 1$ and any $n \geqslant 1$,

$$\sup_T \mathbb{E} \big\| [\![W_T^n]\!] \big\|_{C^{-\kappa}}^N < +\infty. \tag{44}$$

Using Theorem 62 with $u = 0$ we readily have an upper bound for the free energy:

$$-\frac{1}{|\Lambda|} \log \mathscr{Z}_T \leqslant \lambda \mathbb{E}\Big[\fint [\![W_T^4]\!] \Big] = 0.$$

For a lower bound we need to estimate from below the average under $\mathbb{P}$ of the variational expression

$$\lambda \fint [\![W_T^4]\!] + 4\lambda \fint [\![W_T^3]\!] Z_T + 6\lambda \fint [\![W_T^2]\!] Z_T^2 + 4\lambda \fint W_T Z_T^3 + \lambda \fint Z_T^4 + \frac{1}{2} \|u\|_{\mathcal{H}}^2.$$

The strategy we adopt is to bound *path-wise*, and for a generic drift u, the contributions

$$\Phi_T(Z) := \underbrace{4\lambda \fint [\![W_T^3]\!] Z_T}_{\mathrm{I}} + \underbrace{6\lambda \fint [\![W_T^2]\!] Z_T^2}_{\mathrm{II}} + \underbrace{4\lambda \fint W_T Z_T^3}_{\mathrm{III}},$$

in terms of quantities involving only the Wick powers of W which we can control in expectation and the last two *positive terms* (our "good terms")

$$\frac{1}{2} \|u\|_{\mathcal{H}}^2 + \lambda \fint Z_T^4.$$

Any residual positive contribution depending on u can be dropped in the lower bound making the dependence on the drift disappear.

To control term I we see that by duality and Young's inequality, for any $\delta>0$,

$$\left|\fint \llbracket W_T^3\rrbracket Z_T\right| \leqslant \left\|\llbracket W_T^3\rrbracket\right\|_{C^{-\kappa}} \|Z_T\|_{B^\kappa_{1,1}} \leqslant C_\delta \left\|\llbracket W_T^3\rrbracket\right\|^2_{C^{-\kappa}} + \delta \|Z_T\|^2_{H^1}, \tag{45}$$

where we used that $\|Z_T\|_{B^\kappa_{1,1}} \lesssim \|Z_T\|_{H^1}$ and Young's inequality to have a small factor $0<\delta\ll 1$ in front of $\|Z_T\|^2_{H^1}$.

The other terms can be estimated with similar arguments, for example it will be useful to use interpolation and the following fractional Leibniz rule:

Proposition 65. *Let $1<p<\infty$ and $p_1,p_2,p_1',p_2'>1$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{1}{p}$. Then for every $s,\alpha\geqslant 0$ there exists a constant C such that*

$$\|\langle \mathrm{D}\rangle^s (fg)\|_{L^p} \leqslant C\|\langle \mathrm{D}\rangle^{s+\alpha} f\|_{L^{p_2}} \|\langle \mathrm{D}\rangle^{-\alpha} g\|_{L^{p_1}} + C\|\langle \mathrm{D}\rangle^{s+\alpha} g\|_{L^{p_1'}} \|\langle \mathrm{D}\rangle^{-\alpha} f\|_{L^{p_2'}}.$$

The end product of these analytic bounds are estimates

$$|\Phi_T(Z)| \leqslant Q_T + \delta\left[\frac{1}{2}\|u\|^2_{\mathcal{H}} + \lambda \fint Z_T^4\right], \tag{46}$$

where

$$Q_T = O(\lambda^2)[1+\|\llbracket W_T^3\rrbracket\|_{C^{-\varepsilon}} + \|\llbracket W_T^2\rrbracket\|_{C^{-\varepsilon}} + \|W_T\|_{C^{-\varepsilon}}]^K$$

for some integer $K\geqslant 1$. It remains to bound the term involving f, which we do with (41). Since $\|\Delta_j g\|_{L^4} \lesssim \|g\|_{L^4}$ uniformly in j, we have $\|g\|_{B^{-\kappa}_{4,4}} \leqslant C_0\|g\|_{L^4}$, with C_0 independent of Λ (all norms being normalised by the volume). Hence, using $(a+b)^4 \leqslant 8(a^4+b^4)$,

$$f(W_T+Z_T) \geqslant -C - 8\alpha\|W_T\|^4_{B^{-\kappa}_{4,4}} - 8\alpha C_0^4 \fint Z_T^4 \geqslant -C - 8\alpha\|W_T\|^4_{B^{-\kappa}_{4,4}} - \delta\lambda \fint Z_T^4,$$

provided $8\alpha C_0^4 \leqslant \delta\lambda$, which is our smallness condition on α. We include the term $C + 8\alpha\|W_T\|^4_{B^{-\kappa}_{4,4}}$ in Q_T; this does not affect the bounds below, since $\|W_T\|_{B^{-\kappa}_{4,4}} \lesssim \|W_T\|_{C^{-\kappa/2}}$, whose moments are bounded uniformly in T by (44). Therefore (for $\delta<1/2$)

$$F_T(u) \geqslant -\mathbb{E}[Q_T] + (1-2\delta)\mathbb{E}\left[\frac{1}{2}\|u\|^2_{\mathcal{H}} + \lambda\fint Z_T^4\right] \geqslant -\mathbb{E}[Q_T].$$

This last average does not depend anymore on the drift and using (44) we end up with

$$\sup_T \mathbb{E}[Q_T] < \infty.$$

We have established that

$$\sup_T \mathbb{E}\left[e^{-|\Lambda| f(W_T) - V_T(W_T)}\right] \leqslant \sup_T \exp(|\Lambda|\,\mathbb{E}[Q_T]) < +\infty \tag{47}$$

which in particular is enough to establish tightness for the family of measures $(\nu_T)_T$, as (apply (47) to $-f/|\Lambda|$, which still satisfies (41))

$$\sup_T \int e^{f(\varphi)} \nu_T(\mathrm{d}\varphi) = \sup_T \frac{\mathbb{E}\left[e^{f(W_T)-V_T(W_T)}\right]}{\mathbb{E}\left[e^{-V_T(W_T)}\right]} \leqslant \sup_T \frac{\mathbb{E}\left[e^{f(W_T)-V_T(W_T)}\right]}{e^{-\mathbb{E}V_T(W_T)}} < +\infty.$$

We can also observe that the argument leading to (47) remains valid upon replacing V_T by pV_T for any $p \geqslant 1$. This implies that $e^{-V_T(W_T)}$ is in all the L^p spaces wrt. the measure $\mathbb{P}$ uniformly in T and for any $p \geqslant 1$. Together with the a.s. convergence of the martingale $(V_T(W_T))_T$ (which is bounded in all L^p spaces), this allows to conclude also the existence of the limit measure

$$\nu^\infty = \lim_T \nu_T,$$

such that

$$\frac{\mathrm{d}\nu^\infty}{\mathrm{d}\vartheta}(W_\infty) = \frac{e^{-V_\infty(W_\infty)}}{\mathscr{Z}_\infty}, \qquad \mathscr{Z}_\infty := \mathbb{E}[e^{-V_\infty(W_\infty)}] = \lim_T \mathscr{Z}_T,$$

where $V_\infty(W_\infty) = \lim_T V_T(W_T)$ (a.s. and in all $L^p(\mathbb{P})$); this limit is the Wick ordered quartic potential of W_∞, hence a measurable function of W_∞, whose law is ϑ.

7 The quartic interaction in three dimensions

For the quartic interaction in three dimensions the strategy we used in two dimensions fails: the Wick powers are less regular: $[\![W_T^2]\!] \in \mathscr{C}^{-1-\kappa}$ uniformly in T for any small $\kappa > 0$ and $[\![W_T^3]\!]$ does not allow uniform bounds as a well-defined random distribution.

By this we mean that, for any $\alpha > 3/2$ and $N \geqslant 1$, the best bound we can find looks like

$$\mathbb{E}\,\|[\![W_T^3]\!]\|^N_{B^{-\alpha}_{\infty,\infty}} \lesssim (\log T)^{N/2},$$

and this logarithmic divergence cannot be removed (for $\alpha < 3/2$ the moments even diverge polynomially in T).

As a consequence there is no straightforward approach to control the terms

$$\int [\![W_T^3]\!] Z_T, \qquad \text{and} \qquad \int [\![W_T^2]\!] Z_T^2, \tag{48}$$

like we did in two dimensions.

The only a-priori estimate on the regularity of $Z_T = I_T(u)$ is in H^1, coming from Lemma 64 and the quadratic term in the variational functional $F_T(u)$.

An additional substantial difference is that we expect further divergences beyond the Wick ordering which have to be subtracted in order for the limiting measure to be non-trivial. For these reasons in the energy V_T we introduce further scale dependent renormalisation constants γ_T, δ_T to have

$$\frac{1}{|\Lambda|} V_T^f(Y_T) = f(Y_T) + \int (\lambda [\![Y_T^4]\!] - \lambda^2 \gamma_T [\![Y_T^2]\!] - \delta_T), \tag{49}$$

where the Wick products $[\![Y_T^4]\!], [\![Y_T^2]\!]$ are taken with respect to the variance of W_T (as opposed to Y_T).

Repeating the computation from Section 6.1 we arrive at

$$\begin{aligned} F_T(u) \;=\; & \mathbb{E}\Big[f(W_T + Z_T) + \lambda \int \mathbb{W}_T^3 Z_T + \frac{\lambda}{2} \int \mathbb{W}_T^2 Z_T^2 + 4\lambda \int W_T Z_T^3 \Big] \\ & - \mathbb{E}\Big[2\lambda^2 \gamma_T \int W_T Z_T + \lambda^2 \gamma_T \int Z_T^2 + \delta_T \Big] + \mathbb{E}\Big[\lambda \int Z_T^4 + \frac{1}{2} \|u\|^2_{\mathcal{H}} \Big]. \end{aligned} \tag{50}$$

where we introduced the convenient notations

$$\mathbb{W}_t^3 := 4[\![W_t^3]\!], \qquad \mathbb{W}_t^2 := 12[\![W_t^2]\!], \qquad t \geqslant 0,$$

and we recall that f is a fixed function belonging to $C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth.

We will perform a change of variables in the variational functional in order to obtain some explicit cancellations which will leave only quantities well behaved as $T \to \infty$. The main drawback is that the functional will have a less compact and canonical form.

We are going to absorb the mixed terms (48) via the quadratic cost function $\frac{1}{2}\|u\|_{\mathcal{H}}^2$. For the first we have, via Ito formula and then completing the square

$$\begin{aligned}\lambda \fint \mathbb{W}_T^3 Z_T + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s &= \lambda \int_0^T \fint \mathbb{W}_t^3 \dot{Z}_t \mathrm{d}t + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s + \text{martingale} \\ &= -\frac{\lambda^2}{2}\int_0^T \fint (\mathbb{W}_t^{\langle 3\rangle})^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_s\|_{L^2}^2 \mathrm{d}s + \text{martingale},\end{aligned}$$

where we used that J_t is self-adjoint, set $\mathbb{W}_t^{\langle 3\rangle} := J_t \mathbb{W}_t^3$ and let

$$w_t := u_t + \lambda \mathbb{1}_{t \leqslant T} \mathbb{W}_t^{\langle 3\rangle}, \qquad t \geqslant 0. \tag{51}$$

Taking into account (here and below) that the martingale term will average to zero, we have replaced the divergent term $\fint \mathbb{W}_T^3 Z_T$ with a divergent but purely stochastic term $\int_0^T \fint (\mathbb{W}_t^{\langle 3\rangle})^2 \mathrm{d}t$ which does not affect anymore the variational problem and can be explicitly removed by adding its average to δ_T.

The price to pay for this change of variables is that now the cost functional is relative to w and not u anymore. This is an important feature of the variational approach in three dimensions as we do not expect anymore that the limiting measure is absolutely continuous wrt. the GFF.

Defining

$$K_t := I_t(w), \qquad \mathbb{W}_T^{[3]} := \int_0^T J_s \mathbb{W}_s^{\langle 3\rangle} \mathrm{d}s$$

from (51) we deduce the relation

$$Z_T = -\lambda \mathbb{W}_T^{[3]} + K_T,$$

which can be used to expand the second mixed divergent term in (48) as

$$\frac{\lambda}{2}\fint \mathbb{W}_T^2 Z_T^2 = \frac{\lambda^3}{2}\fint \mathbb{W}_T^2 (\mathbb{W}_T^{[3]})^2 - \lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2}\fint \mathbb{W}_T^2 K_T^2. \tag{52}$$

The first term on the r.h.s. is a purely stochastic object and will give a contribution independent of the drift u and absorbed in δ_T.

We are still not done since this operation has left two new divergent terms on the r.h.s. of eq. (52): the H^1 regularity of K_T (guaranteed by the cost functional $\frac{1}{2}\|w\|_{\mathcal{H}}^2$) is not enough to control the products with $\mathbb{W}^2$ which has regularity $\mathscr{C}^{-1-\kappa}$, a bit below -1.

In order to proceed further we will isolate the divergent parts of these products via a *paraproduct* decomposition.

Let us pause for a moment in order to introduce **paraproducts**. The basic idea is that a product of Besov functions f, g naturally decomposes in various contributions involving products of LP blocks:

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_{i \ll j} \Delta_i f \Delta_j g + \sum_{i \gg j} \Delta_i f \Delta_j g + \sum_{i \approx j} \Delta_i f \Delta_j g$$

where we split contributions above and below a thick diagonal of the double sum. The key observation is that if $i \ll j$ then the product $\Delta_i f \Delta_j g$ has Fourier transform supported in an annulus at scale 2^j, while if $i \approx j$ then the same product is a-priori only spectrally supported in a ball of radius $\approx 2^j$. These different spectral supports imply a different behaviour of the bilinear maps:

$$f \prec g = \sum_{i \ll j} \Delta_i f \Delta_j g$$

and

$$f \circ g = \sum_{i \approx j} \Delta_i f \Delta_j g.$$

Definition 66. *Let $f, g \in \mathscr{S}(\Lambda)$. We define the paraproducts and resonant product*

$$f \succ g = g \prec f := \sum_{j < i-1} \Delta_i f \Delta_j g, \qquad \text{and} \qquad f \circ g := \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g.$$

Then

$$fg = f \prec g + f \circ g + f \succ g.$$

The basic analytic bounds for the paraproduct and resonant terms are

Proposition 67. *For any polynomial weight ρ, $\beta \leqslant 0, \alpha \in \mathbb{R}$ and $p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ we have the estimate*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \lesssim \| f \|_{B^{\alpha}_{p_1,\infty}(\rho)} \| g \|_{B^{\beta}_{p_2,q}(\rho)}.$$

For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$ the estimate

$$\| f \circ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \lesssim \| f \|_{B^{\alpha}_{p_1,\infty}(\rho)} \| g \|_{B^{\beta}_{p_2,q}(\rho)}.$$

For a proof see Theorem 3.17 and Remark 3.18 in [45].

Proposition 68. *For any polynomial weights ν, ρ and $\beta \leqslant 0, \alpha \in \mathbb{R}$ we have*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho^{p} \nu)} \lesssim \| f \|_{\mathscr{C}^{\alpha}(\rho)} \| g \|_{B^{\beta}_{p,q}(\nu)}.$$

The proof is an easy modification of the proof of Theorem 3.17 in [45].

Proposition 69. *Assume $m \leqslant 0$, $\alpha \in (0,1), \beta \in \mathbb{R}$. Let η be an S^m multiplier and $q, p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $f \in B^{\beta}_{p_1,\infty}$, $g \in B^{\alpha}_{p_2,\infty}$. Then for any $\delta > 0$,*

$$\| \eta(\mathrm{D})(f \succ g) - (\eta(\mathrm{D}) f \succ g) \|_{B^{\alpha+\beta-m-\delta}_{p,q}} \lesssim \| f \|_{B^{\beta}_{p_1,\infty}} \| g \|_{B^{\alpha}_{p_2,\infty}}.$$

The constant depends only on α, β, δ and the constants in (5).

For a proof see [3] Lemma 2.99.

To continue our algebraic analysis we expand

$$\begin{aligned}-\lambda^2\fint \mathbb{W}_T^2\mathbb{W}_T^{[3]}K_T+\frac{\lambda}{2}\fint \mathbb{W}_T^2K_T^2 \;=\; & \lambda\fint(\mathbb{W}_T^2\succ Z_T)K_T-\lambda^2\fint(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]})K_T\\ & -\lambda^2\fint(\mathbb{W}_T^2\prec\mathbb{W}_T^{[3]})K_T+\frac{\lambda}{2}\fint(\mathbb{W}_T^2\prec K_T)K_T\\ & +\frac{\lambda}{2}\Big(\fint(\mathbb{W}_T^2\circ K_T)K_T-\fint(\mathbb{W}_T^2\succ K_T)K_T\Big).\end{aligned}$$

The first two terms will require renormalisations which we put in place below. All the other terms will be well behaved and we collect them in $\Upsilon_T^{(1)}$. In particular we observe that the last one can be rewritten as

$$\frac{\lambda}{2}\Big(\fint(\mathbb{W}_T^2\circ K_T)K_T-\fint(\mathbb{W}_T^2\succ K_T)K_T\Big)=-\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2,K_T,K_T)$$

introducing the trilinear form $\mathfrak{K}_2$ whose properties are detailed in Proposition 74 below.

As we anticipated, the resonant term $\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]}$ requires renormalisation. In the expression of F_T in (50) we have the counterterm $-2\lambda^2\gamma_T\fint W_TZ_T$ available, which we put now in use writing

$$-\lambda^2\fint(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]})K_T-2\lambda^2\gamma_T\fint W_TZ_T=-\lambda^2\fint\underbrace{\big(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]}+2\gamma_TW_T\big)}_{\mathbb{W}_T^{2\diamond[3]}}K_T+2\lambda^3\gamma_T\fint W_T\mathbb{W}_T^{[3]}.$$

The first contribution is collected in $\Upsilon_T^{(6)}$ and the expectation of the second will contribute to δ_T.

As far as the term $\lambda\fint(\mathbb{W}_T^2\succ Z_T)K_T$ is concerned, we want to absorb it into $\int\|w_s\|^2\mathrm{d}s$. Before we can do this we must be sure that, after applying Itô's formula, it will be still possible to use $\fint Z_T^4$ to control some of the growth of this term. Indeed the quadratic dependence in K_T (via Z_T) cannot be fully taken care of by the quadratic cost $\int\|w_s\|^2\mathrm{d}s$.

We need to introduce a regularisation which makes compatible Fourier cutoffs with L^4 estimates. To introduce such a regularisation fix smooth functions $\tilde\theta,\eta\colon\mathbb{R}^d\to\mathbb{R}_+$ such that $\tilde\theta(\xi)=1$ if $|\xi|\leqslant 1/4$ and $\tilde\theta(\xi)=0$ if $|\xi|\geqslant 1/3$, $\eta(\xi)=1$ if $|\xi|\leqslant 1$ and $\eta(\xi)=0$ if $|\xi|\geqslant 2$. Set $\tilde\theta_t(\xi):=\tilde\theta(\xi/t)$, then define

$$\theta_t(\xi)=(1-\eta(\xi))\tilde\theta_t(\xi)+\zeta(t)\eta(\xi)\tilde\theta_t(\xi)$$

where $\zeta(t)\colon\mathbb{R}_+\to\mathbb{R}$ is a smooth function such that $\zeta(t)=0$ for $t\leqslant 10$ and $\zeta(t)=1$ for $t\geqslant 11$.

$$\begin{aligned}\theta_t(\xi)\sigma_s(\xi) &= 0 \text{ for } s\geqslant t,\\ \theta_t(\xi) &= 1 \text{ for } |\xi|\leqslant ct \text{ for some } c>0 \text{ provided that } t\geqslant 11.\end{aligned}\tag{53}$$

By the Mihlin-Hörmander theorem, the operator $\theta_t=\theta_t(\mathrm{D})$ is bounded on L^p for any $1<p<\infty$.

In the following, for any $f\in C([0,\infty],\mathscr{S}'(\Lambda))$ we define $f_t^\flat:=\theta_tf_t$ then

$$Z_t^\flat=\theta_tZ_t=\int_0^t\theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\int_0^T\theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\theta_tZ_T.$$

In this way we have $\|Z_t^\flat\|_{L^p} \lesssim \|Z_T\|_{L^p}$ for all $t \leqslant T$. In the sequel we will always assume $T \gg 1$.

Now decompose

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T = \lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T + \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T$$

and use the fact that the functions $Z_T - Z_T^\flat$ and $K_T - K_T^\flat$ are spectrally supported outside of a ball of radius cT to show that the second term is nice enough as $T \to \infty$ to not require further analysis and we collect it in $\Upsilon_T^{(2)}$.

For the first we apply Itô's formula to decompose it along the flow of scales as

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T = \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t + \text{martingale}.$$

The second term will be fine and we collect it in $\Upsilon_T^{(3)}$.

We are left with the singular term $\int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t$. We again absorb it by completing the square together with the cost $\frac{1}{2}\|w\|_{\mathcal{H}}^2$:

$$\begin{aligned}\lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|l_t\|_{L^2}^2 \mathrm{d}t \\ &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) \mathrm{d}t + \frac{1}{2}\|l\|_{\mathcal{H}}^2 \end{aligned} \tag{54}$$

where we introduced

$$l_t := u_t + \lambda \mathbb{1}_{t \leqslant T} \mathbb{W}_t^{\langle 3 \rangle} + \lambda \mathbb{1}_{t \leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat). \tag{55}$$

To renormalise the first term on the r.h.s. of (54) we observe that the remaining counterterm can be rewritten as

$$-\lambda^2 \gamma_T \fint Z_T^2 = -\lambda^2 \gamma_T \fint (Z_T^\flat)^2 - 2\lambda^2 \gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2 \gamma_T \fint (Z_T - Z_T^\flat)^2. \tag{56}$$

Differentiating in T the first term in the r.h.s. of eq. (56) we get

$$-\lambda^2 \gamma_T \fint (Z_T^\flat)^2 = -\lambda^2 \int_0^T \fint \dot{\gamma}_t (Z_t^\flat)^2 \mathrm{d}t - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t. \tag{57}$$

The last term in eq. (57) and the last two contributions in (56) are collected in $\Upsilon_T^{(5)}$. The first contribution in eq. (57) has the right form to be used as a counterterm for the resonant product in (54).

Using the commutator $\mathfrak{K}_{3,t}$ described in Proposition 75 below we have

$$\begin{aligned} &-\frac{\lambda^2}{2}\int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) + 2\dot{\gamma}_t (Z_t^\flat)^2] \mathrm{d}t \\ = &-\frac{\lambda^2}{2}\int_0^T \fint \underbrace{[(J_t \mathbb{W}_t^2) \circ (J_t \mathbb{W}_t^2) + 2\dot{\gamma}_t]}_{\mathbb{W}^{\langle 2 \rangle \diamond \langle 2 \rangle}} (Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat) \mathrm{d}t \end{aligned}$$

and we collect both terms in $\Upsilon_T^{(6)}$.

Finally, we are left with the cubic term which we rewrite as

$$4\lambda \fint W_T Z_T^3 = -4\lambda^4 \fint W_T (\mathbb{W}_T^{[3]})^3 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 4\lambda \fint W_T K_T^3.$$

The average of the first term is collected in δ_T while all the remaining terms are collected in $\Upsilon_T^{(4)}$. At last we have established the claimed decomposition since the residual cost functional, from eq. (54) is indeed $\|l\|_{\mathcal{H}}^2/2$.

Collecting all the terms, we proved the following

Lemma 70. *The functional $F_T(u)$ defined in eq. (50) takes the form*

$$F_T(u) \;=\; \mathbb{E}\Big[\Phi_T(\mathbb{W}, Z, K) + \lambda \fint (Z_T)^4 + \frac{1}{2}\|l\|_{\mathcal{H}}^2\Big],$$

where

$$w_t := u_t + \lambda \mathbb{1}_{t\leqslant T} \mathbb{W}_t^{\langle 3\rangle}, \quad Z_t = I_t(u), \quad K_t = I_t(w),$$

$$l_t := u_t + \lambda \mathbb{1}_{t\leqslant T} \mathbb{W}_t^{\langle 3\rangle} + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat),$$

while the stochastic objects are collected in the vector $\mathbb{W}$ defined as

$$\mathbb{W} := (\mathbb{W}^1, \mathbb{W}^2, \mathbb{W}^{\langle 3\rangle}, \mathbb{W}^{[3]\circ 1}, \mathbb{W}^{2\diamond[3]}, \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}),$$

with $\mathbb{W}^1 := W$,

$$\mathbb{W}_t^{\langle 3\rangle} := J_t \mathbb{W}_t^3, \quad \mathbb{W}_t^{[3]} := \int_0^t J_s \mathbb{W}_s^{\langle 3\rangle} \mathrm{d}s, \quad \mathbb{W}_t^{[3]\circ 1} := \mathbb{W}_t^1 \circ \mathbb{W}_t^{[3]},$$

$$\mathbb{W}_t^{2\diamond[3]} := \mathbb{W}_t^2 \circ \mathbb{W}_t^{[3]} + 2\gamma_t \mathbb{W}_t^1, \quad \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle} := (J_t \mathbb{W}_t^2) \circ (J_t \mathbb{W}_t^2) + 2\dot{\gamma}_t.$$

and moreover the potential has the explicit form

$$\Phi_T(\mathbb{W}, Z, K) := f(W_T + Z_T) + \sum_{i=1}^{6} \Upsilon_T^{(i)},$$

with

$$\begin{aligned}
\Upsilon_T^{(1)} &:= -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T) + \frac{\lambda}{2}\fint (\mathbb{W}_T^2 \prec K_T) K_T - \lambda^2 \fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T,\\
\Upsilon_T^{(2)} &:= \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T,\\
\Upsilon_T^{(3)} &:= \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t,\\
\Upsilon_T^{(4)} &:= 4\lambda \fint W_T K_T^3 - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T,\\
\Upsilon_T^{(5)} &:= -2\lambda^2 \fint \gamma_T Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2 \fint \gamma_T (Z_T - Z_T^\flat)^2 - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t,\\
\Upsilon_T^{(6)} &:= -\lambda^2 \fint \mathbb{W}_T^{2\diamond[3]} K_T - \frac{\lambda^2}{2}\int_0^T \fint \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle} (Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t,
\end{aligned}$$

for a suitable value of the renormalisation constant δ_T.

We do not need to include $\mathbb{W}^{[3]}$ in the vector $\mathbb{W}$ since it can be obtained as a function of $\mathbb{W}^{\langle 3\rangle}$ thanks to the bound

$$\|\mathbb{W}_t^{[3]}-\mathbb{W}_s^{[3]}\|_{\mathscr{C}^{1/2-2\kappa}}\leqslant\int_s^t\|J_r\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}}\mathrm{d}r\leqslant\Big[\int_0^T\|J_r\mathbb{W}_r^{\langle 3\rangle}\|^2_{\mathscr{C}^{1/2-2\kappa}}\mathrm{d}r\Big]^{1/2}|t-s|^{1/2}$$
$$\leqslant\Big[\int_0^T\|\mathbb{W}_r^{\langle 3\rangle}\|^2_{\mathscr{C}^{-1/2-\kappa}}\frac{\mathrm{d}r}{\langle r\rangle^{1+2\kappa}}\Big]^{1/2}|t-s|^{1/2}\lesssim\sup_{r\in[0,T]}\|\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{-1/2-\kappa}}|t-s|^{1/2},$$

valid for all $0\leqslant s\leqslant t\leqslant T$ which shows that the deterministic linear map $\mathbb{W}^{\langle 3\rangle}\mapsto\mathbb{W}^{[3]}$ is continuous from $C([0,\infty],\mathscr{C}^{-1/2-\kappa})$ to $C^{1/2}([0,\infty],\mathscr{C}^{1/2-2\kappa})$.

The path-wise regularity of all the other stochastic objects follows from the next lemma, provided the function γ is chosen appropriately.

Lemma 71. *There exists a function $\gamma_t\in C^1(\mathbb{R}_+,\mathbb{R})$ such that*

$$|\gamma_t|+\langle t\rangle|\dot\gamma_t|\lesssim\log\langle t\rangle,\qquad t\geqslant 0,\tag{58}$$

and such that the vector $\mathbb{W}$ is almost surely in $\mathfrak{S}$ where $\mathfrak{S}$ is the Banach space

$$\mathfrak{S}=C([0,\infty],\mathfrak{W})\cap\{\mathbb{W}^{\langle 3\rangle}\in L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa}),\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}\in L^1(\mathbb{R}_+,\mathscr{C}^{-\kappa})\}$$

with

$$\mathfrak{W}=\mathfrak{W}_\kappa:=\mathscr{C}^{-1/2-\kappa}\times\mathscr{C}^{-1-\kappa}\times\mathscr{C}^{-1/2-\kappa}\times\mathscr{C}^{-\kappa}\times\mathscr{C}^{-1/2-\kappa}\times\mathscr{C}^{-\kappa},$$

and equipped with the norm

$$\|\mathbb{W}\|_{\mathfrak{S}}:=\|\mathbb{W}\|_{C([0,\infty],\mathfrak{W})}+\|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa})}+\|\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}\|_{L^1(\mathbb{R}_+,\mathscr{C}^{-\kappa})}.$$

The norm $\|\mathbb{W}\|_{\mathfrak{S}}$ belongs to all $L^p(\mathbb{P})$ spaces. Moreover the averages of the Besov norms $B^\alpha_{q,r}$ of the components of $\mathbb{W}$ of regularity α are uniformly bounded in the volume $|\Lambda|$ if $q<\infty$ (with the L^q norms normalised by the volume).

Proof. The proof is based on the observation that one can choose γ in such a way that every component $\mathbb{W}^{(i)}$ of the vector $\mathbb{W}$ is such that $(\Delta_q\mathbb{W}_t^{(i)}(x))_{t\geqslant 0}$ for $q\geqslant -1$ and $x\in\Lambda$ is a martingale wrt. the Brownian filtration (possibly modulo a deterministic term we can control). This can be seen by writing these terms as iterated stochastic integrals. For example, introducing the notation $\mathrm{d}w_s(k)=\langle k\rangle^{-1}\sigma_s(k)\mathrm{d}B_s^k$ we can write

$$\mathbb{W}_T^2(x)\;=\;24\sum_{k_1,k_2}e^{i(k_1+k_2)\cdot x}\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)$$

so, recalling the definition of Littlewood-Paley kernels ϱ_i, we have

$$\Delta_i\mathbb{W}_T^2(x)=24\sum_{k_1,k_2}e^{i(k_1+k_2)\cdot x}\varrho_i(k_1+k_2)\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2).$$

By Burkholder's inequality and Fubini's theorem

$$\mathbb{E}\Big[\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|_{L^p}^p\Big]\;\lesssim\;\Big(\sum_{k_1,k_2}\varrho_i(k_1+k_2)\int_0^T\int_0^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\Big)^{p/2}$$
$$\lesssim\;2^{p(2+\kappa)i/2},$$

uniformly in T and so

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\leqslant T}\|\mathbb{W}_t^2\|^p_{B^{-1-\kappa}_{p,p}}\Big] &\leqslant \mathbb{E}\Big[\Big(\sum_i 2^{p(-1-\kappa)i}\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|^p_{L^p}\Big)\Big]\\ &\lesssim \sum_i 2^{p(-1-\kappa)i}\,\mathbb{E}\Big[\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|^p_{L^p}\Big]\\ &\lesssim \sum_i 2^{p(-1-\kappa)i}2^{p(1+\kappa/2)i}\lesssim\sum_i 2^{-pi\kappa/2}<+\infty\end{aligned}$$

By Besov embedding this implies that $\mathbb{E}[\sup_{T<\infty}\|\mathbb{W}_T^2\|^p_{B^{-1-\kappa}_{p,q}}]$ is finite for any $p,q<\infty$ uniformly in the volume and $\mathbb{E}[\|\mathbb{W}_T^2\|^p_{C\mathscr{C}^{-1-}}]$ is finite. Since $\mathbb{W}_T^2$ is a continuous, L^2-bounded martingale, it converges and therefore it belongs to $C([0,\infty],\mathscr{C}^{-1-})$. The same reasoning can be carried out for the more complicated terms $\mathbb{W}^{\langle 3\rangle},\mathbb{W}^{[3]\circ 1},\mathbb{W}^{2\diamond[3]},\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$. The details can be found in Section 7.5. □

For convenience of the reader we summarise the probabilistic estimates in Table 1.

$\mathbb{W}^1$	$\mathbb{W}^2$	$\mathbb{W}^{\langle 3\rangle}$	$\mathbb{W}^{[3]}$	$\mathbb{W}^{[3]\circ 1}$	$\mathbb{W}^{2\diamond[3]}$	$\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-}\cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{1/2-}$	$C\mathscr{C}^{0-}$	$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-}\cap L^1\mathscr{C}^{0-}$

Table 1. Regularities of the various stochastic objects, the domain of the time variable is understood to be $[0,\infty]$. Estimates in these norms hold a.s. and in $L^p(\mathbb{P})$ for all $p\geqslant 1$ (see Lemma 71).

Remark 72. The requirement that $\mathbb{W}^{\langle 3\rangle}\in L^2\mathscr{C}^{-1/2-}$ will be used in Section 7.3 to establish equicoercivity and to relax the variational problem to a suitable space of measures.

7.1 Commutators

We collect here the analysis of the various commutators introduced above. The properties of these commutators are crucial to the analysis of the resonant products in the paracontrolled analysis above. As such they have an important role in the argument. Their proof, however, is a routine application of basic ideas of harmonic analysis and LP decompositions. We decided to report the proofs as it can be useful for the reader to study them in order to adapt the ideas to other contexts.

Proposition 73. *Let $\alpha\in(0,1)$, $\beta,\gamma\in\mathbb{R}$ such that $\beta+\gamma<0$, $\alpha+\beta+\gamma>0$ and p_1,p_2,p_3, $p\in[1,\infty]$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{p}$. Then there exists a trilinear form $\mathfrak{K}_1(f,g,h)$ such that*

$$\|\mathfrak{K}_1(f,g,h)\|_{B^{\alpha+\beta+\gamma}_{p,\infty}}\lesssim\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,\infty}}\|h\|_{B^{\gamma}_{p_3,\infty}},$$

and when $f,g,h\in\mathscr{S}$ it has the form

$$\mathfrak{K}_1(f,g,h)=(f\succ g)\circ h-g(f\circ h).$$

Proof. The proof is a slight modification of the one given in [28]. Write $S_mg:=\sum_{-1\leqslant k<m}\Delta_kg$, so that $f\succ g=\sum_\ell S_{\ell-1}g\Delta_\ell f$. For smooth f,g,h we have $f\circ h=\sum_{|i-j|\leqslant 1}\Delta_jf\Delta_ih$ and similarly for $(f\succ g)\circ h$. Therefore, fixing an integer $N\geqslant 1$,

$$\mathfrak{K}_1(f,g,h)=\sum_{j\geqslant -1}\sum_{|i-j|\leqslant 1}A_j\Delta_ih-\sum_{j\geqslant -1}\sum_{|i-j|\leqslant 1}\sum_{k\geqslant j+N}\Delta_kg\Delta_jf\Delta_ih,$$

where $A_j := \Delta_j(f \succ g) - (S_{j+N}g)\Delta_j f$. Since the Fourier transform of $S_{\ell-1}g\Delta_\ell f$ is supported in an annulus $2^\ell\mathcal{A}$, there is $N_0 \geqslant 1$ such that $\Delta_j(S_{\ell-1}g\Delta_\ell f) = 0$ for $|\ell - j| > N_0$; moreover $\Delta_j\Delta_\ell = 0$ for $|\ell - j| > 1$. Hence

$$A_j = \sum_{|\ell-j|\leqslant N_0} ([\Delta_j, S_{\ell-1}g]\Delta_\ell f + (S_{\ell-1}g - S_{j+N}g)\Delta_j\Delta_\ell f).$$

By Lemma 2.97 from [3], $\|[\Delta_j, a]b\|_{L^r} \lesssim 2^{-j}\|\nabla a\|_{L^{p_1}}\|b\|_{L^{p_2}}$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, while, since $\alpha \in (0,1)$, Bernstein's inequality gives $\|\nabla S_{\ell-1}g\|_{L^{p_1}} \lesssim 2^{\ell(1-\alpha)}\|g\|_{B^\alpha_{p_1,\infty}}$ and $\|S_{\ell-1}g - S_{j+N}g\|_{L^{p_1}} \lesssim 2^{-j\alpha}\|g\|_{B^\alpha_{p_1,\infty}}$ for $|\ell - j| \leqslant N_0$. Together with $\|\Delta_\ell f\|_{L^{p_2}} \lesssim 2^{-\ell\beta}\|f\|_{B^\beta_{p_2,\infty}}$ this yields

$$\|A_j\|_{L^r} \lesssim 2^{-j(\alpha+\beta)}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}.$$

Now we estimate the two sums separately. For fixed j all the factors of $\sum_{|i-j|\leqslant 1} A_j\Delta_i h$ have frequencies of order at most 2^j, so this function has a Fourier transform supported in 2^jB for some ball B independent of j. Since $\alpha + \beta + \gamma > 0$, by the summation lemma for functions with Fourier support in dyadic balls (Lemma 2.84 in [3]) it is enough to get an estimate on

$$\sup_{j\geqslant -1} 2^{(\alpha+\beta+\gamma)j}\left\|\sum_{|i-j|\leqslant 1} A_j\Delta_i h\right\|_{L^p}$$

to bound the first sum in $B^{\alpha+\beta+\gamma}_{p,\infty}$, so by Hölder inequality,

$$\begin{aligned}\left\|\sum_{|i-j|\leqslant 1} A_j\Delta_i h\right\|_{L^p} &\lesssim \sum_{|i-j|\leqslant 1} \|A_j\|_{L^r}\|\Delta_i h\|_{L^{p_3}} \lesssim \\ &\sum_{|i-j|\leqslant 1} 2^{-j(\alpha+\beta)}2^{-i\gamma}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}\|h\|_{B^\gamma_{p_3,\infty}} \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}\|h\|_{B^\gamma_{p_3,\infty}}.\end{aligned}$$

For the second term observe that for fixed k the Fourier transform of

$$\sum_{-1\leqslant j\leqslant k-N} \sum_{|i-j|\leqslant 1} \Delta_k g\Delta_j f\Delta_i h$$

is supported in 2^kB. Now we can estimate again by Hölder inequality

$$\begin{aligned}&\left\|\sum_{-1\leqslant j\leqslant k-N} \sum_{|i-j|\leqslant 1} \Delta_k g\Delta_j f\Delta_i h\right\|_{L^p} \\ \lesssim\ & 2^{-\alpha k} \sum_{-1\leqslant j\leqslant k-N} 2^{-(\beta+\gamma)j}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}\|h\|_{B^\gamma_{p_3,\infty}} \\ \lesssim\ & 2^{-k(\alpha+\beta+\gamma)}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}\|h\|_{B^\gamma_{p_3,\infty}},\end{aligned}$$

where in the last step we used $\beta + \gamma < 0$. The same summation lemma (with k in place of j) then bounds the second sum in $B^{\alpha+\beta+\gamma}_{p,\infty}$, which concludes the proof. □

Proposition 74. *Assume $\beta \in (0,1)$, $\alpha, \gamma \in \mathbb{R}$ such that $\alpha + \gamma < 0$, and $\alpha + \beta + \gamma = 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then there exists a trilinear form $\mathfrak{K}_2(f,g,h)$ for which*

$$|\mathfrak{K}_2(f,g,h)| \lesssim \|f\|_{B^\alpha_{p_1,\infty}}\|g\|_{B^\beta_{p_2,q_1}}\|h\|_{B^\gamma_{p_3,q_2}},$$

and on smooth functions

$$\mathfrak{K}_2(f,g,h) = \fint [(f \succ g)h - g(f \circ h)].$$

Proof. This is a modification of the proof of Lemma A.6 in [25]. Repeating an algebraic computation given in the proof of that lemma, we get that for smooth f, g, h we have

$$\mathfrak{K}_2(f,g,h) = \left(\sum_{j \leqslant i-1, |i-k| \leqslant L} - \sum_{i \sim k, j < i+L} \right) \fint (\Delta_i f \Delta_j g \Delta_k h),$$

for some $L \geqslant 1$. Then we estimate

$$\begin{aligned} |\mathfrak{K}_2(f,g,h)| &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f \Delta_j g \Delta_k h\|_{L^1} \\ &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f\|_{L^{p_1}} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \sup_i (2^{\alpha i} \|\Delta_i f\|_{L^{p_1}}) \sum_{j \sim k} 2^{(\beta+\gamma)k} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \|f\|_{B^{\alpha}_{p_1,\infty}} \|g\|_{B^{\beta}_{p_2,q_1}} \|h\|_{B^{\gamma}_{p_3,q_2}}. \end{aligned}$$

□

Proposition 75. *There exists a family $(\mathfrak{K}_{3,t})_{t \geqslant 0}$ of bounded multilinear forms on $\mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1-\kappa} \times H^{1/2-\kappa} \times H^{1/2-\kappa}$ such that for smooth $\varphi, \psi, g^{(1)}, g^{(2)}$ it holds*

$$\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)}) = \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \circ J_t \psi) g^{(1)} g^{(2)}],$$

and

$$|\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)})| \lesssim \frac{1}{\langle t \rangle^{1+\delta}} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}.$$

Proof. Here $\delta > 0$ is a small parameter which we may (and do) take with $\delta \leqslant \kappa$, so that $\langle t \rangle^{-1-\kappa} \leqslant \langle t \rangle^{-1-\delta}$. Note that $\langle t \rangle^{1/2} J_t$ satisfies the assumptions of Proposition 69 with $m = -1$, therefore using also Proposition 7

$$\|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \lesssim \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}}.$$

Therefore

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) - (J_t \varphi \succ g^{(1)})] J_t(\psi \succ g^{(2)}) \right| \\ \lesssim\ & \|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \|J_t(\psi \succ g^{(2)})\|_{H^{-1/2+3\kappa}} \\ \lesssim\ & \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \langle t \rangle^{-1/2-\delta} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

and by symmetry also

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \succ g^{(1)})(J_t \psi \succ g^{(2)})] \right| \\ \lesssim\ & \langle t \rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

Furthermore from Proposition 74 and for sufficiently small $\kappa > 0$,

$$\begin{aligned} &\left| \not{\prec}(J_t\varphi \succ g^{(1)})(J_t\psi \succ g^{(2)}) - \not{\prec}((J_t\varphi \succ g^{(1)}) \circ J_t\psi) g^{(2)} \right| \\ \lesssim\ & \|J_t\varphi\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t\psi\|_{\mathscr{C}^{-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \\ \lesssim\ & \langle t \rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}. \end{aligned}$$

Applying Proposition 73 we get

$$\begin{aligned} &\|(J_t\varphi \succ g^{(1)}) \circ J_t\psi - (J_t\varphi \circ J_t\psi)(g^{(1)})\|_{H^{-1/2+\kappa}} \\ \lesssim\ & \|J_t\varphi\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t\psi\|_{\mathscr{C}^{-\kappa}} \\ \lesssim\ & \langle t \rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \end{aligned}$$

and putting things together gives the required estimate. □

7.2 Bounds

The aim of this section is to give upper and lower bounds on $\mathcal{W}_T(f)$ uniformly in T.

The symbol Q_T will denote a generic random variable measurable wrt. $\sigma((W_t)_{t\in[0,T]})$ and belonging to $L^p(\mathbb{P})$ uniformly in T and $|\Lambda|$ for any $1 \leqslant p < \infty$. This is a convenient way to express the dependence of bounds on the norms of the stochastic object $\mathbb{W}$.

Lemma 76. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\Upsilon_T^{(1)}| + T^\delta \left|\Upsilon_T^{(2)}\right| + \left|\Upsilon_T^{(3)}\right| + \left|\Upsilon_T^{(4)}\right| + \left|\Upsilon_T^{(6)}\right| \leqslant C_{\varepsilon,\delta} Q_T + \varepsilon \left[\sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^2 + \lambda \|Z_T\|_{L^4}^4 \right].$$

$$\left|\Upsilon_T^{(5)}\right| \leqslant C_\varepsilon \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t|\,\mathrm{d}t}{\langle t \rangle^{5/4}} \right]^2 + \varepsilon \|Z_T\|_{H^{1/2-\delta}}^2 + \varepsilon\lambda \|Z_T\|_{L^4}^4.$$

Now note that the definition of K_t gives

$$K_t = -\lambda \int_0^t \mathbb{1}_{s\leqslant T} J_s^2(\mathbb{W}_s^2 \succ Z_s^\flat)\mathrm{d}s + I_t(l)$$

therefore for any small $\kappa > 0$,

$$\begin{aligned} \left\| \int_0^t \mathbb{1}_{s\leqslant T} J_s^2(\mathbb{W}_s^2 \succ Z_s^\flat)\mathrm{d}s \right\|_{H^{1-\kappa}}^2 &\leqslant \int_0^{t\wedge T} \|J_s(\mathbb{W}_s^2 \succ Z_s^\flat)\|_{H^{-\kappa}}^2 \mathrm{d}s \\ &\lesssim \int_0^{t\wedge T} \frac{1}{\langle s \rangle^{1+\kappa}} \|\mathbb{W}_s^2 \succ Z_s^\flat\|_{H^{-1-\kappa/2}}^2 \mathrm{d}s \\ &\lesssim \int_0^{t\wedge T} \frac{1}{\langle s \rangle^{1+\kappa}} \|\mathbb{W}_s^2\|_{B_{4,\infty}^{-1-\kappa/4}}^2 \|Z_s^\flat\|_{L^4}^2 \mathrm{d}s \\ &\lesssim \int_0^{t\wedge T} \frac{1}{\langle s \rangle^{1+\kappa}} \|\mathbb{W}_s^2\|_{B_{4,\infty}^{-1-\kappa/4}}^2 \mathrm{d}s\ \|Z_T\|_{L^4}^2 \\ &\lesssim \lambda \left(\int_0^{t\wedge T} \frac{1}{\langle s \rangle^{1+\kappa}} \|\mathbb{W}_s^2\|_{B_{4,\infty}^{-1-\kappa/4}}^2 \mathrm{d}s \right)^2 + \frac{1}{\lambda} \|Z_T\|_{L^4}^4 \end{aligned}$$

where we used that $|\sigma_s(x)| \lesssim \langle s\rangle^{-1/2}\mathbb{1}_{\langle x\rangle\sim s}$, so that $\|J_sF\|_{H^{-\kappa}} \lesssim \langle s\rangle^{-1/2-\kappa/2}\|F\|_{H^{-1-\kappa/2}}$, and the inequality (valid for any $\alpha\in\mathbb{R}$ and $\varepsilon>0$; here $\alpha=-1-\kappa/2$ and $\varepsilon=\kappa/4$)

$$\begin{aligned}\|f\prec g\|_{H^\alpha}^2 = \|f\prec g\|_{B^\alpha_{2,2}}^2 &= \sum_i 2^{2i\alpha}\|\Delta_i(f\prec g)\|_{L^2}^2 \lesssim \sum_i 2^{2i\alpha}\sum_{j\sim i}\|\Delta_{\leqslant j-1}f\Delta_j g\|_{L^2}^2\\ &\lesssim \sum_i 2^{2i\alpha}\sum_{j\sim i}\|\Delta_j g\|_{L^4}^2\|\Delta_{\leqslant j-1}f\|_{L^4}^2 \lesssim \|f\|_{L^4}^2\sum_i 2^{2i\alpha}\sum_{j\sim i}\|\Delta_j g\|_{L^4}^2\\ &\lesssim \|f\|_{L^4}^2\|g\|_{B^\alpha_{4,2}}^2 \lesssim \|f\|_{L^4}^2\|g\|_{B^{\alpha+\varepsilon}_{4,\infty}}^2\end{aligned}$$

so that

$$\begin{aligned}\|K_t\|_{H^{1-\kappa}}^2 &\lesssim \lambda^2\Big\|\int_0^t \mathbb{1}_{s\leqslant T}J_s^2(\mathbb{W}_s^2\succ Z_s^\flat)\mathrm{d}s\Big\|_{H^{1-\kappa}}^2 + \|I_t(l)\|_{H^{1-\kappa}}^2\\ &\lesssim \lambda^3\Big(\int_0^t \frac{1}{\langle s\rangle^{1+\kappa}}\|\mathbb{W}_s^2\|_{B^{-1-\kappa/4}_{4,\infty}}^2\mathrm{d}s\Big)^2 + \lambda\|Z_T\|_{L^4}^4 + \int_0^t\|l_s\|_{L^2}^2\mathrm{d}s,\end{aligned} \tag{59}$$

which implies that quadratic functions of the norm $\|K_t\|_{H^{1-\kappa}}$ with small coefficients can always be controlled, uniformly in $t\in[0,\infty]$, by the coercive term

$$\lambda\not{f}Z_T^4 + \frac{1}{2}\int_0^\infty\|l_s\|_{L^2}^2\mathrm{d}s.$$

Remark 77. Note that

$$\sup_T\left[\frac{|\gamma_T|}{\langle T\rangle^{1/4}} + \int_0^T\frac{|\gamma_t|\,\mathrm{d}t}{\langle t\rangle^{5/4}}\right] < \infty,$$

provided γ_T does not grow too fast in T which is indeed guaranteed by the choice of renormalisation made in Lemma 103.

Observe that, from Lemma 70 and from the detailed analysis of the various terms (which we skip but can be found in [5]), we have that

$$|\Phi_T(\mathbb{W},Z,K)| \leqslant Q_T + \varepsilon\Big(\lambda\|Z_T\|_{L^4}^4 + \frac{1}{2}\int_0^\infty\|l_t^T(u)\|_{L^2}^2\mathrm{d}t\Big),$$

which immediately gives

$$-\mathbb{E}[Q_T] \leqslant -\mathbb{E}[Q_T] + (1-\varepsilon)\mathbb{E}\Big(\lambda\|Z_T\|_{L^4}^4 + \frac{1}{2}\int_0^\infty\|l_t^T(u)\|_{L^2}^2\mathrm{d}t\Big) \leqslant \mathcal{W}_T(f). \tag{60}$$

On the other hand for any suitable drift $\check{u}\in\mathbb{H}_a$ we get the bound

$$\mathcal{W}_T(f) \leqslant \mathbb{E}[Q_T] + (1+\varepsilon)\mathbb{E}\Big(\lambda\|I_T(\check{u})\|_{L^4}^4 + \frac{1}{2}\int_0^\infty\|l_t^T(\check{u})\|_{L^2}^2\mathrm{d}t\Big), \tag{61}$$

where

$$l_t^T(\check{u}) = \check{u}_t + \lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^3 + \mathbb{W}_t^2\succ(I_t(\check{u}))^\flat). \tag{62}$$

Therefore it remains to produce an appropriate drift $\check{u}$ for which the r.h.s. in eq. (61) is finite (and so uniformly in $|\Lambda|$ and of order $o(\lambda^3)$).

One possible strategy is to try and choose $\check{u}$ such that $l^T(\check{u})=0$, however this fails since estimates on this choice of drift via Gronwall's inequality would rely on the Besov-Hölder norm of $\mathbb{W}^2$ for which we do not have any uniform control in the volume. In order to overcome this problem we decompose $\mathbb{W}^2$ and use weighted estimates similarly as done in [26] in the SPDE context.

Consider the decomposition

$$\mathbb{W}_s^2 = \mathcal{U}_> \mathbb{W}_s^2 + \mathcal{U}_\leqslant \mathbb{W}_s^2,$$

where the random field $\mathcal{U}_> \mathbb{W}_s^2$ is constructed as follows. Let φ be a smooth function, positive and supported on $[-2,2]^3$ and such that $\sum_{m\in\Lambda\cap\mathbb{Z}^d}\varphi^2(\bullet - m) = 1$. Denote $\varphi_m := \varphi(\bullet - m)$. Let $\tilde{\chi}$ be a smooth function supported in $B(0,1)$, denote by $\mathcal{X}_{\leqslant N}$ the Fourier multiplier operator $\tilde{\chi}(\mathrm{D}/N)$ and similarly $\mathcal{X}_{>N} := (1-\tilde{\chi}(\mathrm{D}/N))$. Set $L_m(s) := (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}})^{\frac{1}{2\delta}}$, let

$$\mathcal{U}_> \mathbb{W}_s^2 := \sum_{m\in\Lambda\cap\mathbb{Z}^d} \varphi_m \mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)$$

and

$$\mathcal{U}_\leqslant \mathbb{W}_s^2 := \sum_{m\in\Lambda\cap\mathbb{Z}^d} \varphi_m \mathcal{X}_{\leqslant L_m(s)}(\varphi_m \mathbb{W}_s^2).$$

(with slight abuse of notation we drop the dependence on time of the operators $\mathcal{U}_\leqslant$, $\mathcal{U}_>$).

Observe that the laws of both $\mathcal{U}_> \mathbb{W}_s^2$ and $\mathcal{U}_\leqslant \mathbb{W}_s^2$ are translation invariant w.r.t. translations by $m \in \Lambda \cap \mathbb{Z}^d$. By [53], Theorem 2.4.7 and Bernstein inequality

$$\begin{aligned}
\|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} &\lesssim \sup_m \|\mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)\|_{\mathscr{C}^{-1-3\delta}} \\
&\lesssim \sup_m \frac{1}{1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}} \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}} \lesssim 1.
\end{aligned}$$

Furthermore for a polynomial weight ρ:

$$\begin{aligned}
\|\mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)} &\lesssim \sup_m \|\varphi_m \mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)} \\
&\lesssim \sup_m (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho^2)} \\
&\lesssim \sup_m \rho(m)\,(1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)} \qquad (63) \\
&\lesssim \sup_m (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)} \\
&\lesssim 1 + \|\mathbb{W}_s^2\|^2_{\mathscr{C}^{-1-\delta}(\rho)},
\end{aligned}$$

where we used the possibility to compare weighted and unweighted norms once localised via φ_m. We now let $\check{u}$ be the solution to the linear integral equation

$$\check{u}_t = -\lambda \mathbb{1}_{t\leqslant T}[\mathbb{W}_t^{\langle 3\rangle} + J_t \mathcal{U}_> \mathbb{W}_t^2 \succ \theta_t(I_t(\check{u}))], \qquad t \geqslant 0, \qquad (64)$$

which can be solved globally. For $2\delta < 1/2$, $p \geqslant 1$ and $t \in [0,T]$, we have

$$\begin{aligned}
\|I_t(\check{u})\|_{B^{1/2-2\delta}_{p,p}(\rho)} &\lesssim \lambda \int_0^t \Big[\|J_s^2 \mathbb{W}_s^3\|_{B^{1/2-2\delta}_{p,p}(\rho)} + \|J_s^2 \mathcal{U}_> \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{B^{1/2-2\delta}_{p,p}(\rho)} \Big] \mathrm{d}s \\
&\lesssim \lambda \int_0^t \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B^{-1/2-\delta}_{p,p}(\rho)} + \lambda \int_0^t \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} \|I_s(\check{u})\|_{B^{1/2-2\delta}_{p,p}(\rho)}.
\end{aligned}$$

Gronwall's lemma, together with the bound $\|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} \lesssim 1$ (which makes the exponential factor below bounded by a constant), implies that, for $t \in [0,T]$:

$$\begin{aligned}
\|I_t(\check{u})\|_{B^{1/2-2\delta}_{p,p}(\rho)} &\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B^{-1/2-\delta}_{p,p}(\rho)} \right) \exp\left(\lambda \int_0^T \frac{\|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} \mathrm{d}s}{\langle s\rangle^{1+\delta}} \right) \\
&\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B^{-1/2-\delta}_{p,p}(\rho)} \right) \qquad (65) \\
&\lesssim \lambda \|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+, B^{-1/2-\delta}_{p,p}(\rho))}.
\end{aligned}$$

Note that eq. (65) is also valid replacing the weighted norm $B^{1/2-2\delta}_{p,p}(\rho)$ with the standard (normalised) norm $B^{1/2-2\delta}_{p,p}$, from which, using Besov embedding we deduce:

$$\sup_T \mathbb{E}\|I_T(\check{u})\|^4_{L^4} \lesssim \lambda^4 \mathbb{E}\left(\int_0^\infty \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}} \|J_s \mathbb{W}^3_s\|_{B^{-1/2-\delta}_{4,4}}\right)^4 \lesssim \lambda^4.$$

Computing $l^T(\check{u})$ from eq. (62) and (64), we obtain

$$l^T_t(\check{u}) = \lambda \mathbb{1}_{t\leqslant T} J_t \mathcal{U}_\leqslant \mathbb{W}^2_t \succ \theta_t(I_t(\check{u})), \qquad t \geqslant 0.$$

It remains to prove that $\mathbb{E}[\|l^T(\check{u})\|^2_{\mathcal{H}}] \lesssim O(\lambda^3)$ uniformly in $T>0$. Note that, for $s\in[0,T]$,

$$\begin{aligned} \|J_s \mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|_{L^2(\rho^3)} &\lesssim \frac{1}{\langle s\rangle^{1/2+\delta/2}} \|\mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|_{B^{-1+\delta/2}_{2,2}(\rho^3)} \\ &\lesssim \frac{1}{\langle s\rangle^{1/2+\delta/2}} \|\mathcal{U}_\leqslant \mathbb{W}^2_s\|_{\mathscr{C}^{-1+\delta/2}(\rho^2)} \|I_s(\check{u})\|_{B^{1/2-3\delta}_{2,2}(\rho)}. \end{aligned} \tag{66}$$

We know that the distribution of $\check{u}$ is invariant under translation by $m\in\Lambda\cap\mathbb{Z}^d$. Recalling that $\sum_{m\in\Lambda\cap\mathbb{Z}^d}\varphi^2(\bullet - m) = 1$ and letting ρ be a polynomial weight with sufficient decay and such that $\rho^5 \geqslant \varphi^2$, we have

$$\begin{aligned} \mathbb{E}[\|l^T(\check{u})\|^2_{\mathcal{H}}] &= \lambda^2 \mathbb{E}[\|s\mapsto \mathbb{1}_{s\leqslant T} J_s \mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ &\leqslant \lambda^2 \sum_{m\in\Lambda\cap\mathbb{Z}^d} \mathbb{E}[\|s\mapsto \varphi(\bullet - m) J_s \mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ \text{(by trans. inv.)} \quad &\lesssim \lambda^2 |\Lambda| \mathbb{E}[\|s\mapsto \mathbb{1}_{s\leqslant T}\varphi J_s \mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ \text{(using } \rho^5\geqslant\varphi^2) \quad &\lesssim \lambda^2 \int_0^T \mathrm{d}s\, \mathbb{E}[\|J_s \mathcal{U}_\leqslant \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{L^2(\rho^5)}] \\ \text{(by eq. (66))} \quad &\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \mathbb{E}\Big[\|\mathcal{U}_\leqslant \mathbb{W}^2_s\|^2_{\mathscr{C}^{-1+\delta/2}(\rho^2)} \|I_s(\check{u})\|^2_{B^{1/2-3\delta}_{2,2}(\rho)}\Big] \\ &\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \mathbb{E}\Big[\lambda^2 \|\mathcal{U}_\leqslant \mathbb{W}^2_s\|^4_{\mathscr{C}^{-1+\delta/2}(\rho^2)} + \lambda^{-2}\|I_s(\check{u})\|^4_{B^{1/2-3\delta}_{2,2}(\rho)}\Big] \\ \text{(by eqs. (65),(63))} \quad &\lesssim \lambda^4 \int_0^\infty \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \Big[1 + \mathbb{E}\|\mathbb{W}^2_s\|^8_{\mathscr{C}^{-1-\delta/2}(\rho)} + \mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|^4_{L^2(\mathbb{R}_+, B^{-1/2-\delta}_{p,p}(\rho))}\Big] \\ &\lesssim \lambda^4 \Big[1 + \sup_{s\geqslant 0} \mathbb{E}\|\mathbb{W}^2_s\|^8_{\mathscr{C}^{-1-\delta/2}(\rho)} + \mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|^4_{L^2(\mathbb{R}_+, B^{-1/2-\delta}_{p,p}(\rho))}\Big] \\ &\lesssim O(\lambda^4). \end{aligned}$$

The last inequality is the consequence of bounds on the two expectations on the r.h.s. obtained as follows. For p sufficiently large (namely $3/p \leqslant \delta/4$, so that by Besov embedding $B^{-1-\delta/4}_{p,p}(\rho^p) \subset \mathscr{C}^{-1-\delta/2}(\rho)$) we have

$$\big[\mathbb{E}\|\mathbb{W}^2_s\|^8_{\mathscr{C}^{-1-\delta/2}(\rho)}\big]^{p/8} \leqslant \mathbb{E}\|\mathbb{W}^2_s\|^p_{\mathscr{C}^{-1-\delta/2}(\rho)} \leqslant \mathbb{E}\|\mathbb{W}^2_s\|^p_{B^{-1-\delta/4}_{p,p}(\rho^p)}$$

$$= \sum_{i\geqslant -1} 2^{i(-1-\delta/4)p} \int_\Lambda \mathrm{d}x\, |\rho(x)|^p \mathbb{E}|\Delta_i \mathbb{W}^2_s(x)|^p \lesssim \sum_{i\geqslant -1} 2^{i(-1-\delta/4)p} \mathbb{E}|\Delta_i \mathbb{W}^2_s(0)|^p \lesssim 1,$$

uniformly in $s\geqslant 0$. Similarly, we have

$$\Big[\mathbb{E}\|\mathbb{W}^3_s\|^4_{B^{-\delta/2}_{p,p}(\rho)}\Big]^{p/4} \leqslant \mathbb{E}\|\mathbb{W}^3_s\|^p_{B^{-\delta/2}_{p,p}(\rho)} \lesssim \mathbb{E}|\mathbb{W}^3_s(0)|^p.$$

By Lemma 104

$$\mathbb{E}|\mathbb{W}^3_s(0)|^p \lesssim (\mathbb{E}|\mathbb{W}^3_s(0)|^2)^{p/2} \lesssim \langle s\rangle^{3p/2},$$

and using the standard multiplier bounds for J_s we conclude

$$\begin{aligned}
\mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|^4_{L^2(\mathbb{R}_+,B^{-1/2-\delta}_{p,p}(\rho))} &\lesssim \mathbb{E}\bigg(\int_0^\infty \|J_s\mathbb{W}^3_s\|^2_{B^{-1/2-\delta}_{p,p}(\rho)}\mathrm{d}s\bigg)^2\\
&\lesssim \mathbb{E}\bigg(\int_0^\infty \Big\|\frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D}\rangle}\mathbb{W}^3_s\Big\|^2_{B^{-1/2-\delta}_{p,p}(\rho)}\mathrm{d}s\bigg)^2\\
&\lesssim \mathbb{E}\bigg(\int_0^\infty \langle s\rangle^{-1-\delta}\Big(\langle s\rangle^{-3/2}\|\mathbb{W}^3_s\|_{B^{-\delta/2}_{p,p}(\rho)}\Big)^2\mathrm{d}s\bigg)^2\\
&\lesssim \int_0^\infty \langle s\rangle^{-1-\delta}\mathbb{E}\Big(\langle s\rangle^{-3/2}\|\mathbb{W}^3_s\|_{B^{-\delta/2}_{p,p}(\rho)}\Big)^4\mathrm{d}s\\
&\lesssim 1.
\end{aligned}$$

Remark 78. The decomposition of the noise is similar to the one given in [26] but differs in the fact that we choose the frequency cutoff dependent on the size of the noise instead of the point, to preserve translation invariance. The price to pay is that the decomposition is nonlinear in the noise, however this does not present any inconvenience in our context.

7.3 Removal of the UV cutoff

From Lemma 70 we learned that

$$\mathcal{W}_T(f) = \inf_{u\in\mathbb{H}_a} F_T(u),$$

with

$$F_T(u) = \mathbb{E}\bigg[\Phi_T(\mathbb{W}, Z(u), K(u)) + \lambda\|Z_T(u)\|^4_{L^4} + \frac{1}{2}\|l^T(u)\|^2_{\mathcal{H}}\bigg],$$

where $l^T(u), Z(u), K(u)$ are functions of u. The goal of this section is to establish the $T\to\infty$ limit of this variational problem:

Theorem 79. *We have*

$$\lim_{T\to\infty}\mathcal{W}_T(f) = \mathcal{W}(f) := \inf_{u\in\mathbb{H}_a} F_\infty(u),$$

where

$$F_\infty(u) = \mathbb{E}\bigg[\Phi_\infty(\mathbb{W}, Z(u), K(u)) + \lambda\|Z_\infty(u)\|^4_{L^4} + \frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg],$$

with Φ_∞ and l^∞ introduced in Lemma 81 below.

Proof. The statement will be a direct consequence of Theorem 93 below. □

The functions $\Phi_\infty, Z_\infty, l^\infty$ are suitable limits of Φ_T, Z_T, l^T. Therefore our first task is to analyse their continuity and to identify suitable topologies. As $T\to\infty$ the variational functional tends to keep l in $\mathcal{H}$ but not u, so we need at first to understand in which space u will live in the limit.

We denote

$$\mathcal{H}^{\alpha,p} := L^2([0,\infty); W^{\alpha,p}), \qquad \alpha\in\mathbb{R}, 1<p<\infty,$$

and by $\mathcal{H}^{\alpha,p}_w$ the reflexive Banach space $\mathcal{H}^{\alpha,p}$ endowed with the weak topology. With these definitions we have $\mathcal{H}^\alpha = \mathcal{H}^{\alpha,2}$ and $\mathcal{H} = \mathcal{H}^{0,2}$. Moreover for small enough $\kappa>0$ (fixed once and for all) we let

$$\mathcal{L} := \mathcal{H}^{-1/2-\kappa,3}.$$

This is the space in which we will put u, and next we show that it is sufficient to get control over Z.

Lemma 80. *For κ small enough, $u \mapsto Z(u)$ is a compact map $\mathcal{L} \to C([0,\infty], L^4)$.*

Proof. By definition of Z we have for any $0 < \varepsilon < 1/8 - \kappa/2$,

$$\begin{aligned}
\|Z_{t_2}(u) - Z_{t_1}(u)\|_{W^{\varepsilon,4}} &= \left\| \int_{t_1}^{t_2} J_s u_s \mathrm{d}s \right\|_{W^{\varepsilon,4}} \leqslant \int_{t_1}^{t_2} \left\| \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} u_s \right\|_{W^{\varepsilon,4}} \mathrm{d}s \\
&\lesssim \int_{t_1}^{t_2} \|\langle \mathrm{D} \rangle^{-1+\varepsilon} u_s\|_{W^{\varepsilon,4}} \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\varepsilon}} \\
&\lesssim \int_{t_1}^{t_2} \|\langle \mathrm{D} \rangle^{-1+\varepsilon} u_s\|_{W^{1/4+\varepsilon,3}} \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\varepsilon}} \\
&\lesssim \left(\int_0^{\infty} \|u_s\|^2_{W^{-1/2-\kappa,3}} \mathrm{d}s \right)^{1/2} \left(\int_{t_1}^{t_2} \frac{\mathrm{d}s}{\langle s \rangle^{1+2\varepsilon}} \right)^{1/2} \\
&\lesssim \left(\int_{t_1}^{t_2} \frac{\mathrm{d}s}{\langle s \rangle^{1+2\varepsilon}} \right)^{1/2} \|u\|_{\mathcal{L}}.
\end{aligned}$$

where we have used the Sobolev embedding $W^{1/4+\varepsilon,3} \hookrightarrow W^{\varepsilon,4}$. Since

$$\lim_{t_1 \to t_2} \int_{t_1}^{t_2} \frac{\mathrm{d}s}{\langle s \rangle^{1+2\varepsilon}} = 0, \qquad \int_0^{\infty} \frac{\mathrm{d}s}{\langle s \rangle^{1+2\varepsilon}} < \infty,$$

for any $t_2 \in [0,\infty]$, we can conclude by the Rellich–Kondrachov embedding theorem and the Ascoli–Arzelà theorem, that bounded sets in $\mathcal{L}$ are mapped to compact sets in $C([0,\infty], L^4)$, proving the claim. □

In the sequel, by an abuse of notation, we will denote both a generic element of $\mathfrak{S}$ and the canonical random variable on $\mathfrak{S}$ by

$$\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2, \mathbb{X}^{\langle 3 \rangle}, \mathbb{X}^{[3] \circ 1}, \mathbb{X}^{\langle 2 \rangle \diamond \langle 2 \rangle}, \mathbb{X}^{2 \diamond [3]}).$$

The following lemma establishes point-wise convergence for the functional Φ_T defined in Lemma 70.

Lemma 81. *Define $l^{\infty}(u) = l^{\infty}(\mathbb{X}, u) \in \mathbb{H}_a$ by*

$$l^{\infty}_t(u) := u_t + \lambda \mathbb{X}^{\langle 3 \rangle}_t + \lambda J_t(\mathbb{X}^2_t \succ Z^{\flat}_t(u)), \qquad t \geqslant 0. \tag{67}$$

For any sequence $(\mathbb{X}^T, u^T)_T$ such that $u^T \to u$ in $\mathcal{L}_w$, $l^T = l^T(\mathbb{X}^T, u^T) \to l = l^{\infty}(\mathbb{X}, u)$ in $\mathcal{H}_w$ and

$$\begin{array}{rcl}
\mathbb{X}^T & = & (\mathbb{X}^{T,1}, \mathbb{X}^{T,2}, \mathbb{X}^{T,\langle 3 \rangle}, \mathbb{X}^{T,[3] \circ 1}, \mathbb{X}^{T,\langle 2 \rangle \diamond \langle 2 \rangle}, \mathbb{X}^{T,2 \diamond [3]}) \\
\downarrow & & \\
\mathbb{X} & = & (\mathbb{X}^1, \mathbb{X}^2, \mathbb{X}^{\langle 3 \rangle}, \mathbb{X}^{[3] \circ 1}, \mathbb{X}^{\langle 2 \rangle \diamond \langle 2 \rangle}, \mathbb{X}^{2 \diamond [3]})
\end{array}$$

in $\mathfrak{S}$ we have

$$\lim_{T \to \infty} \Phi_T(\mathbb{X}^T, Z(u^T), K(u^T)) = \Phi_{\infty}(\mathbb{X}, Z(u), K(u)).$$

Here $Z_t(u) = I_t(u)$, we let $K_t(u) := Z_t(u) + \lambda \mathbb{X}^{[3]}_t$ and Φ_{∞} is defined by

$$\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) := f(\mathbb{X}^1_{\infty} + Z_{\infty}(u)) + \sum_{i=1}^{6} \Upsilon^{(i)}_{\infty}(\mathbb{X}, Z(u), K(u)),$$

with $\Upsilon^{(i)}_\infty(\mathbb{X}, Z, K) = \Upsilon^{(i)}_\infty$ *given by*

$$\begin{aligned}
\Upsilon^{(1)}_\infty &:= -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{X}^2_\infty, K_\infty, K_\infty) + \frac{\lambda}{2}\!\!\!\!\!\!\int (\mathbb{X}^2_\infty \prec K_\infty)K_\infty - \lambda^2 \!\!\!\!\!\!\int (\mathbb{X}^2_\infty \prec \mathbb{X}^{[3]}_\infty)K_\infty,\\
\Upsilon^{(2)}_\infty &= 0,\\
\Upsilon^{(3)}_\infty &:= \lambda \int_0^\infty \!\!\!\!\!\!\int (\mathbb{X}^2_t \succ \dot{Z}^\flat_t)K_t \mathrm{d}t,\\
\Upsilon^{(4)}_\infty &:= 4\lambda \!\!\!\!\!\!\int \mathbb{X}^1_\infty K^3_\infty - 12\lambda^2 \!\!\!\!\!\!\int (\mathbb{X}^1_\infty \mathbb{X}^{[3]}_\infty)K^2_\infty + 12\lambda^3 \!\!\!\!\!\!\int \mathbb{X}^1_\infty (\mathbb{X}^{[3]}_\infty)^2 K_\infty,\\
\Upsilon^{(5)}_\infty &:= -2\lambda^2 \int_0^\infty \!\!\!\!\!\!\int \gamma_t Z^\flat_t \dot{Z}^\flat_t \mathrm{d}t,\\
\Upsilon^{(6)}_\infty &:= -\lambda^2 \!\!\!\!\!\!\int \mathbb{X}^{2\diamond[3]}_\infty K_\infty - \frac{\lambda^2}{2}\int_0^\infty \!\!\!\!\!\!\int \mathbb{X}^{\langle 2\rangle \diamond \langle 2\rangle}_t (Z^\flat_t)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^\infty \mathfrak{K}_{3,t}(\mathbb{X}^2_t, \mathbb{X}^2_t, Z^\flat_t, Z^\flat_t)\mathrm{d}t,
\end{aligned}$$

where $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_{3,t}$ *are the multilinear forms defined in Proposition 73, Proposition 74 and Proposition 75 respectively and where, with abuse of notation, we let*

$$\begin{aligned}
\mathbb{X}^1_\infty \mathbb{X}^{[3]}_\infty &:= \mathbb{X}^1_\infty \succ \mathbb{X}^{[3]}_\infty + \mathbb{X}^1_\infty \prec \mathbb{X}^{[3]}_\infty + \mathbb{X}^{[3]\circ 1}_\infty,\\
\mathbb{X}^1_\infty (\mathbb{X}^{[3]}_\infty)^2 &:= \mathbb{X}^1_\infty (\mathbb{X}^{[3]}_\infty \circ \mathbb{X}^{[3]}_\infty) + 2\mathbb{X}^{[3]\circ 1}_\infty \mathbb{X}^{[3]}_\infty + 2\mathfrak{K}_1(\mathbb{X}^{[3]}_\infty, \mathbb{X}^{[3]}_\infty, \mathbb{X}^1_\infty)\\
&\quad + 2\mathbb{X}^1_\infty \succ (\mathbb{X}^{[3]}_\infty \succ \mathbb{X}^{[3]}_\infty) + 2\mathbb{X}^1_\infty \prec (\mathbb{X}^{[3]}_\infty \succ \mathbb{X}^{[3]}_\infty).
\end{aligned} \tag{68}$$

Proof. Lemma 80 implies that for any $u^T \to u$ in $\mathcal{L}_w$ we have $Z(u^T) \to Z(u)$ in $C([0,\infty], L^4)$ and by the convergence of $l^T \to l$ in $\mathcal{H}_w$ we have also $K(u^T) \to K(u)$ in $C([0,\infty], H^{1-\kappa})$. The products $\mathbb{X}^{T,1}_T \mathbb{X}^{T,[3]}_T$ and $\mathbb{X}^{T,1}_T (\mathbb{X}^{T,[3]}_T)^2$ can be decomposed using paraproducts and, after identifying the resonant products with the corresponding stochastic objects in $\mathbb{X}^T$, we obtain the finite T analogs of the expressions in eq. (68). After this preprocessing, it is easy to see by continuity that we have $\mathbb{X}^{T,1}_T \mathbb{X}^{T,[3]}_T \to \mathbb{X}^1_\infty \mathbb{X}^{[3]}_\infty$ and $\mathbb{X}^{T,1}_T (\mathbb{X}^{T,[3]}_T)^2 \to \mathbb{X}^1_\infty (\mathbb{X}^{[3]}_\infty)^2$ in $\mathscr{C}^{-1/2-\kappa}$. For $\Upsilon^{(1)}$ and $\Upsilon^{(4)}$ and the first term of $\Upsilon^{(6)}$ the statement follows from the fact that they are bounded multilinear forms on $\mathfrak{S} \times C([0,\infty], H^{1/2-\kappa}) \times C([0,\infty], H^{1-\kappa})$. For the remaining terms $\Upsilon^{(2)}, \Upsilon^{(3)}, \Upsilon^{(5)}, \Upsilon^{(6)}$ the convergence follows from the detailed analysis in [5]. □

This settles the framework for the point-wise convergence of the cost function. In order to use Γ-convergence, we need to modify the variational setting to guarantee enough compactness as $T \to \infty$.

As long as T is finite, the original potential V_T is bounded below so in particular we have

$$-C_T + \mathbb{E}\left[\frac{1}{2}\|u\|^2_{\mathcal{H}}\right] \leqslant F_T(u), \tag{69}$$

which quantifies the coercivity of F_T. Unfortunately, this estimate does not survive the limit. However the analytic estimates of the renormalised control problem allow to infer that there exists a small $\delta \in (0,1)$, and a finite constant $C > 0$ independent of T, such that

$$-C + (1-\delta)\mathbb{E}\left[\lambda\|Z_T(u)\|^4_{L^4} + \frac{1}{2}\|l^T(u)\|^2_{\mathcal{H}}\right] \leqslant F_T(u), \tag{70}$$

and

$$F_T(u) \leqslant C + (1+\delta)\mathbb{E}\bigg[\lambda\|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\bigg]. \tag{71}$$

Moreover the cost functional $\lambda\|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2$ controls the $\mathcal{L}$ norm of u uniformly in T, modulo constants depending only on $\|\mathbb{W}\|_{\mathfrak{S}}$ and which are bounded in average uniformly in T. More precisely we have (in a more general setting, useful below)

Lemma 82. *Let μ be a probability measure on $\mathfrak{S}\times\mathcal{L}$ with first marginal $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ and denote with $(\mathbb{X}, u)$ the canonical variable on $\mathfrak{S}\times\mathcal{L}$. Then there exists a constant C, depending only on λ, such that*

$$\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] \lesssim C + 2\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].$$

Proof. We use $\|l^T(u)\|_{\mathcal{L}} \lesssim \|l^T(u)\|_{\mathcal{H}}$ in the bound

$$\begin{aligned}
\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] &\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \lambda\mathbb{E}_\mu[\|s\mapsto J_s(\mathbb{X}_s^2 \succ \theta_s Z_T(u))\|_{\mathcal{L}}^2] + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] \\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \lambda\mathbb{E}_\mu\bigg[\int_0^\infty \frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^2}{\langle s\rangle^{1+\kappa}}\|Z_T(u)\|_{L^4}^2 \mathrm{d}s\bigg] \\
&\quad + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] \\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \frac{\lambda}{2}\mathbb{E}_\mu\bigg[\int_0^\infty \frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^4}{\langle s\rangle^{1+\kappa}}\mathrm{d}s\bigg] + 2\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] \\
&\quad + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].
\end{aligned}$$

□

From this we conclude that we can relax the optimisation problem and ask that $u \in \mathbb{L}_a$ where $\mathbb{L}_a$ is the space of predictable processes in $\mathcal{L}$:

$$\mathcal{W}_T(f) = \inf_{u\in\mathbb{L}_a} F_T(u).$$

For future reference note that eq. (70) implies also that for any sequence $(u^T)_T$ such that $F_T(u^T)$ remains bounded we must have that also

$$\sup_T \mathbb{E}[\|l^T(u^T)\|_{\mathcal{H}}^2] < \infty. \tag{72}$$

To prove Γ-convergence we need to set up the problem in a space with a topology which, on the one hand is strong enough to enable to prove the Γ-liminf inequality, and on the other hand allows to obtain enough compactness from F_T. Almost sure convergence on $\mathfrak{S}\times\mathcal{L}$ would allow for the former but is too strong for the latter. For this reason we need a setting based on convergence in law as made precise in the following definition.

Definition 83. *Denote by $(\mathbb{X}, u)$ the canonical variables on $\mathfrak{S}\times\mathcal{L}$ and consider the space of probability measures*

$$\mathcal{Y} := \{\mu \in \mathcal{P}(\mathfrak{S}\times\mathcal{L}) \mid \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] < \infty\}$$

equipped with the following topology: $\mu_n \to \mu$ iff

a) μ_n converges to μ weakly on $\mathfrak{S}\times\mathcal{L}_w$,

b) $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$.

Let

$$\mathcal{X} := \{\mu \in \mathcal{Y} \mid \mu = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, u) \text{ for some } u \in \mathbb{L}_a\}$$

and denote by $\bar{\mathcal{X}} \subseteq \mathcal{Y}$ the closure of $\mathcal{X}$ in $\mathcal{Y}$.

Remark 84. Condition (b) allows to exclude pathological points in $\bar{\mathcal{X}}$ and makes possible Lemma 89 below.

With these new notations we have

$$\mathcal{W}_T(f) = \inf_{\mu \in \mathcal{X}} \breve{F}_T(\mu), \tag{73}$$

where

$$\breve{F}_T(\mu) := \mathbb{E}_\mu\Big[\Phi_T(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\Big]$$

and where $\mathbb{E}_\mu$ denotes the expectation on $\mathfrak{S} \times \mathcal{L}$ wrt. the probability measure μ. We also define the corresponding limiting functional as

$$\breve{F}_\infty(\mu) := \mathbb{E}_\mu\Big[\Phi_\infty(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\Big]. \tag{74}$$

As far as the topology of $\bar{\mathcal{X}}$ is concerned, we have the following tightness and equicoercivity results.

Lemma 85. *Let $G \subseteq \bar{\mathcal{X}}$ such that $\sup_{\mu \in G} \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] < \infty$. Then G is tight on $\mathfrak{S} \times \mathcal{L}_w$ and in particular compact in $\bar{\mathcal{X}}$.*

Proof. Observe that for all $\mu \in G$, $\mathrm{Law}_\mu(\mathbb{X}) = \mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ and that $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ on $\mathfrak{S}$ is tight since $\mathfrak{S}$ is a separable metric space, so for any $\varepsilon > 0$, we can find a compact set $\mathcal{K}_\varepsilon^1 \subset \mathfrak{S}$ such that $\mu((\mathfrak{S} \setminus \mathcal{K}_\varepsilon^1) \times \mathcal{L}) < \varepsilon/2$. Now let $\mathcal{K}_\varepsilon^2 := \mathcal{K}_\varepsilon^1 \times B(0, C) \subset \mathfrak{S} \times \mathcal{L}$, for some large C to be chosen later. Then $\mathcal{K}_\varepsilon^2$ is a compact subset of $\mathfrak{S} \times \mathcal{L}_w$ and

$$\mathbb{P}_\mu[(\mathbb{X}, u) \notin \mathcal{K}_\varepsilon^2] \leqslant \frac{\varepsilon}{2} + \frac{1}{C}\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].$$

Choosing $C > \sup_{\mu \in G} 2\,\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]/\varepsilon$ gives tightness of the family G. □

Lemma 86. *The family $(\breve{F}_T)_T$ is equicoercive on $\bar{\mathcal{X}}$.*

Proof. Define for some $K > 0$ large enough

$$\mathcal{K} := \{\mu \in \bar{\mathcal{X}} : \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] \leqslant K\}.$$

Note that $\mathcal{K}$ is compact from Lemma 85. From eq. (70) we have

$$\lambda \mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] + \frac{1}{2}\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] \leqslant C + 2\breve{F}_T(\mu).$$

Indeed, note that the analytic estimates of Section 7.2 are path-wise and hold also wrt. $(\mathbb{X}, u)$ under the measure μ (the point is that here u is not necessarily adapted to $\mathbb{X}$), while for the probabilistic estimates on $Q_T(\mathbb{W})$ we have $\mathbb{E}[Q_T(\mathbb{W})] = \mathbb{E}_\mu[Q_T(\mathbb{X})]$ since $\mathrm{Law}_\mu(\mathbb{X}) = \mathrm{Law}_{\mathbb{P}}(\mathbb{W})$. From this we deduce that for some $C, c > 0$

$$\begin{aligned} & \breve{F}_T(\mu) \\ \geqslant\ & \frac{\lambda}{2}\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] + \frac{1}{4}\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] - C \\ \geqslant\ & c\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] - C \end{aligned}$$

where in the last line we have used Lemma 82. Therefore $\inf_{\mu\in\mathcal{K}^c}\breve{F}_T(\mu)\geqslant cK-C$. On the other hand from eq. (71) it follows that $\sup_T\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu)<\infty$. So for K large enough

$$\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu)=\inf_{\mu\in\mathcal{K}}\breve{F}_T(\mu).$$

□

To be able to use this equicoercivity we will need to show that we can extend the infimum in (73) to $\bar{\mathcal{X}}$. For this we will first need some properties of the space $\bar{\mathcal{X}}$.

In particular we will need to show that measures with sufficiently high moments are dense in $\bar{\mathcal{X}}$ in a way which behaves well with respect to $\breve{F}_T$.

We skip the proofs of the next three lemmas.

Lemma 87. *Let $\mu\in\bar{\mathcal{X}}$. There exist $(\mu_n)_n$ in $\mathcal{X}$ such that $\mu_n\to\mu$ on $\mathfrak{S}\times\mathcal{L}$ (now with the norm topology) and $\sup_n\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]<\infty$.*

Lemma 88. *Let $\mu_n\to\mu$ on $\mathfrak{S}\times\mathcal{L}$, such that $\sup_n\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]<\infty$. Then*

1. *for every Lipschitz function f on $\mathcal{L}$, $\mathbb{E}_{\mu_n}[f(u)]\to\mathbb{E}_{\mu}[f(u)]$;*
2. *for every Lipschitz function f on $C([0,\infty],L^4)$ we have $\mathbb{E}_{\mu_n}[f(Z(u))]\to\mathbb{E}_{\mu}[f(Z(u))]$.*

The next lemma proves that we can approximate measures in $\bar{\mathcal{X}}$ by measures with bounded support in the second marginal which are still in $\bar{\mathcal{X}}$.

Lemma 89. *Let $\mu\in\bar{\mathcal{X}}$ such that $\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4]+\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]<\infty$. For any $L>0$ there exists $\mu_L\in\bar{\mathcal{X}}$ such that $\|u\|_{\mathcal{L}}\leqslant L$, μ_L-almost surely, $\mu_L\to\mu$ weakly on $\mathfrak{S}\times\mathcal{L}$ as $L\to\infty$,*

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4]\to\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4],\quad \text{and}\quad \mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2]\to\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].$$

Furthermore for any μ_L there exists $(\mu_{L,n})_n\subset\mathcal{X}$ such that $\|u\|_{\mathcal{L}}\leqslant L$, $\mu_{L,n}$-almost surely and $\mu_{L,n}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{L}_w$.

With this setup we are ready to extend the variational problem. To illustrate the proofs we need some approximation procedures.

Definition 90. *Let $u\in\mathcal{L}$, $N\in\mathbb{N}$, and $(\eta_\varepsilon)_{\varepsilon>0}$ be a smooth Dirac sequence on Λ and $(\varphi_\varepsilon)_{\varepsilon>0}$ be another smooth Dirac sequence compactly supported on $\mathbb{R}_+\times\Lambda$. Denote by $*_\Lambda$ the convolution only wrt the space variable, and by $*$ the space-time convolution. Define the following approximations of the identity:*

$$\begin{aligned}(\mathrm{reg}_{x,\varepsilon}(u)) &:= u*_\Lambda\eta_\varepsilon,\\ (\mathrm{reg}_{t:x,\varepsilon}(u))(t) &:= e^{-\varepsilon t}u*\varphi_\varepsilon(t)=e^{-\varepsilon t}\int_0^t u(t-s)*_\Lambda\varphi_\varepsilon(s)\,\mathrm{d}s.\end{aligned}$$

Let

$$\tilde{T}^N(u):=\inf\left\{t\geqslant 0\middle|\int_0^t\|u(s)\|_{W^{-1/2-\kappa,3}}^2\mathrm{d}s\geqslant N\right\},$$

and

$$(\mathrm{cut}_N(u))(t) := u(t)\mathbb{1}_{\{t\leqslant \tilde{T}^N(u)\}}.$$

Observe the following properties of these maps:

- $\mathrm{reg}_{x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w \to \mathcal{H}_w$ *and* $\mathcal{L}\to\mathcal{H}$*;*
- $\mathrm{reg}_{t:x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w\to\mathcal{H}$*;*
- cut_N *is continuous as a map* $\mathcal{L}\to B(0,N^{1/2})\subset\mathcal{L}$*;*
- *if* u *is a predictable process then* $\mathrm{reg}_{x,\varepsilon}(u)$, $\mathrm{reg}_{t:x,\varepsilon}(u)$, $\mathrm{cut}_N(u)$ *will also be predictable.*

Furthermore we have the bounds

$$\|\mathrm{reg}_{x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{reg}_{t:x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{cut}_N(u)\|_{\mathcal{L}}\leqslant\|u\|_{\mathcal{L}}.$$

uniformly in ε, N*, and for every* $u\in\mathcal{L}$*,*

$$\lim_{\varepsilon\to0}\|\mathrm{reg}_{x,\varepsilon}(u)-u\|_{\mathcal{L}}=\lim_{\varepsilon\to0}\|\mathrm{reg}_{t:x,\varepsilon}(u)-u\|_{\mathcal{L}}=\lim_{N\to\infty}\|\mathrm{cut}_N(u)-u\|_{\mathcal{L}}=0.$$

With abuse of notation, for $\mu\in\mathcal{P}(\mathfrak{S}\times\mathcal{L})$ *and* $f:\mathcal{L}\to\mathcal{L}$*, we let*

$$f_*\mu=(\mathrm{Id},f)_*\mu=\mathrm{Law}_\mu(\mathbb{X},f(u)).$$

Remark 91. Let us briefly comment on the rationale for these approximations. $\mathrm{reg}_{t:x,\varepsilon}$ will be used when one wants to obtain a sequence of weakly convergent measures on $\mathfrak{S}\times\mathcal{H}$ or $\mathfrak{S}\times\mathcal{L}$ from a sequence of measures weakly convergent on $\mathfrak{S}\times\mathcal{L}_w$. $\mathrm{reg}_{x,\varepsilon}$ will be used when one wants to obtain a measure on $\mathfrak{S}\times\mathcal{H}$ from one on $\mathfrak{S}\times\mathcal{L}$, while preserving the estimates on the moments of $Z(u)$ since $Z(u*_\Lambda\eta_\varepsilon)=Z(u)*_\Lambda\eta_\varepsilon$.

Let us now proceed.

Lemma 92. *If* $T<\infty$ *we have*

$$\inf_{\mu\in\mathcal{X}}\breve{F}_T(\mu)=\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu).$$

Proof. To prove the claim it is enough to show that for any $\mu\in\bar{\mathcal{X}}$, for any $\alpha>0$, there exists a sequence $\mu_n\in\mathcal{X}$ such that $\mathrm{limsup}_{n\to\infty}\breve{F}_T(\mu_n)\leqslant\breve{F}_T(\mu)+\alpha$. W.l.o.g we can assume that $\breve{F}_T(\mu)<\infty$. Observe that, as long as $T<\infty$ we can also express

$$\breve{F}_T(\mu)=\mathbb{E}_\mu\bigg[\frac{1}{|\Lambda|}V_T^f(\mathbb{X}_T^1+Z_T(u))+\frac12\|u\|_{\mathcal{H}}^2\bigg],$$

and deduce that $\mathbb{E}_\mu[\|u\|_{\mathcal{H}}^2]<\infty$ since V_T is bounded below at fixed T. By Lemma 89 there exists a sequence $(\mu_L)_L\subset\bar{\mathcal{X}}$, such that $\mu_L(\|u\|_{\mathcal{L}}\leqslant L)=1$, $\mu_L\to\mu$ on $\mathfrak{S}\times\mathcal{L}$ and by weak convergence and domination,

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4]\to\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4],\qquad \mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2]\to\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].$$

First we have to improve the regularity of μ_L to get convergence on $\mathfrak{S}\times\mathcal{H}_w$ but without affecting our control on the moments of Z_T, so let $\mu_L^\varepsilon:=(\mathrm{reg}_{x,\varepsilon})_*\mu_L$ and $\mu^\varepsilon:=(\mathrm{reg}_{x,\varepsilon})_*\mu$. Then

$$\mathbb{E}_{\mu_L^\varepsilon}[\|Z_T(u)\|_{L^4}^4]\to\mathbb{E}_{\mu^\varepsilon}[\|Z_T(u)\|_{L^4}^4],\qquad \mathbb{E}_{\mu_L^\varepsilon}[\|u\|_{\mathcal{H}}^2]\to\mathbb{E}_{\mu^\varepsilon}[\|u\|_{\mathcal{H}}^2],$$

and $\mu_L^\varepsilon \to \mu^\varepsilon$ on $\mathfrak{S}\times\mathcal{H}$. By continuity of $\breve{F}_T$ and the bound (71), $\breve{F}_T(\mu_L^\varepsilon)\to\breve{F}_T(\mu^\varepsilon)$ as $L\to\infty$ and $\breve{F}_T(\mu^\varepsilon)\to\breve{F}_T(\mu)$ as $\varepsilon\to 0$. In particular we can find L and ε such that $|\breve{F}_T(\mu_L^\varepsilon)-\breve{F}_T(\mu)|<\alpha/2$. By Lemma 89 there exists a sequence $(\mu_{n,L})_{n,L}\subset\mathcal{X}$ such that each measure $\mu_{n,L}$ is supported on $\mathfrak{S}\times B(0,L)$ and $\mu_{n,L}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{L}_w$. Setting $\mu_{n,L}^{\varepsilon,\delta}:=(\mathrm{reg}_{t:x,\delta})_*(\mathrm{reg}_{x,\varepsilon})_*\mu_{n,L}$ and $\mu_L^{\varepsilon,\delta}:=(\mathrm{reg}_{t:x,\delta})_*(\mathrm{reg}_{x,\varepsilon})_*\mu_L$ we have $\mu_{n,L}^{\varepsilon,\delta}\to\mu_L^{\varepsilon,\delta}$ on $\mathfrak{S}\times\mathcal{H}$ with norm topology. It is not hard to see that $V_T(\mathbb{X}_T^1+Z_T(u))\lesssim_T\|\mathbb{X}\|_{\mathfrak{S}}^4+\|u\|_{\mathcal{H}}^4$ and since on the support of $\mu_{n,L}^{\varepsilon,\delta}$, $\|u\|_{\mathcal{H}}\leqslant L$ and the first marginal of $\mu_{n,L}^{\varepsilon,\delta}$ is fixed we have again by domination and weak convergence

$$\lim_{n\to\infty}\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|}V_T^f(\mathbb{X}_T^1+Z_T(u))+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]=\mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|}V_T^f(\mathbb{X}_T^1+Z_T(u))+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]$$

and by dominated convergence (since $\mu_L^{\varepsilon,\delta}$ is supported on $\mathfrak{S}\times B(0,L)$) we can find a δ such that $|\breve{F}_T(\mu_L^{\varepsilon,\delta})-\breve{F}_T(\mu_L^\varepsilon)|<\alpha/2$ which proves the statement. □

Finally we can state the key result of this section.

Theorem 93. *The family $(\breve{F}_T)_T$ Γ–converges to $\breve{F}_\infty$ on $\bar{\mathcal{X}}$. Moreover*

$$\lim_T\mathcal{W}_T(f)=\lim_T\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu)=\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_\infty(\mu)=\mathcal{W}(f).$$

Proof.

Step 1. *(Relaxation)* We will prove below that:

a) the family $(\breve{F}_T)_T$ is indeed equicoercive on $\bar{\mathcal{X}}$ (Lemma 86);

b) the variational problems for $\breve{F}_T$ (with $T<\infty$ or $T=\infty$) on $\mathcal{X}$ and on $\bar{\mathcal{X}}$ are equivalent (Lemma 92 and Lemma 96).

Step 2. *(liminf inequality)* Consider a sequence $\mu^T\to\mu$ in $\bar{\mathcal{X}}$. We need to prove that

$$\liminf_{T\to\infty}\breve{F}_T(\mu^T)\geqslant\breve{F}_\infty(\mu).$$

It is enough to prove this statement for a subsequence, the full statement follows from the fact that every sequence has a subsequence satisfying the inequality. Take a subsequence (not relabelled) such that

$$\sup_T\breve{F}_T(\mu^T)<\infty. \tag{75}$$

If there is no such subsequence there is nothing to prove. Otherwise tightness for the subsequence follows like in the proof of equicoercivity in Lemma 86 above. Then invoking the Skorokhod representation theorem of [37] we can extract a subsequence (again, not relabelled) and find random variables $(\tilde{\mathbb{X}}^T,\tilde{u}^T)_T$ and $(\tilde{\mathbb{X}},\tilde{u})$ on some probability space $(\tilde{\Omega},\tilde{\mathbb{P}})$ such that $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}}^T,\tilde{u}^T)=\mu^T$, $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}},\tilde{u})=\mu$ and almost surely $\tilde{\mathbb{X}}^T\to\tilde{\mathbb{X}}$ in $\mathfrak{S}$, $\tilde{u}^T\to\tilde{u}$ in $\mathcal{L}_w$. Note that $\tilde{l}^T:=l^T(\tilde{\mathbb{X}}^T,\tilde{u}^T)\to l:=l^\infty(\tilde{\mathbb{X}},\tilde{u})$ in $\mathcal{L}_w$ and using (75) we deduce the almost sure convergence $l^T\to l$ in $\mathcal{H}_w$, maybe modulo taking another subsequence, again not relabelled. Note that, by the analytic estimates of Section 7.2 (which hold point-wise on the probability space) we have

$$\Phi_T(\tilde{\mathbb{X}}^T,Z(\tilde{u}^T),K(\tilde{u}^T))+\lambda\|Z_T(\tilde{u}^T)\|_{L^4}^4+\frac{1}{2}\|l^T(\tilde{u}^T)\|_{\mathcal{H}}^2+Q(\tilde{\mathbb{X}}^T)\geqslant 0,$$

for some $L^1(\tilde{\mathbb{P}})$ random variable $Q(\tilde{\mathbb{X}}^T)$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[Q(\tilde{\mathbb{X}}^T)] = \mathbb{E}[Q(\mathbb{W})]$ (for example we can take $Q(\tilde{\mathbb{X}}^T) = C(1 + \|\tilde{\mathbb{X}}^T\|_{\mathfrak{S}}^p)$ for some large enough p). Fatou's lemma and Lemma 81 then give

$$\begin{aligned}
\liminf_{T\to\infty} \breve{F}_T(\mu^T) &= \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\Big[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|_{L^4}^4 + \frac{1}{2}\|l^T\|_{\mathcal{H}}^2\Big] \\
&= \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\Big[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|_{L^4}^4 + \frac{1}{2}\|l^T\|_{\mathcal{H}}^2 + Q(\tilde{\mathbb{X}}^T)\Big] - \mathbb{E}[Q(\mathbb{W})] \\
&\geqslant \mathbb{E}_{\tilde{\mathbb{P}}} \liminf_{T\to\infty}\Big[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|_{L^4}^4 + \frac{1}{2}\|l^T\|_{\mathcal{H}}^2 + Q(\tilde{\mathbb{X}}^T)\Big] - \mathbb{E}[Q(\mathbb{W})] \\
&\geqslant \mathbb{E}_{\tilde{\mathbb{P}}}\Big[\Phi_\infty(\tilde{\mathbb{X}}, Z(\tilde{u}), K(\tilde{u})) + \lambda\|Z_\infty(\tilde{u})\|_{L^4}^4 + \frac{1}{2}\|l^\infty(\tilde{u})\|_{\mathcal{H}}^2\Big] = \breve{F}_\infty(\mu),
\end{aligned}$$

which is the Γ-liminf inequality.

Step 3. *(limsup inequality)* Now all that remains is constructing a recovery sequence, for this we can again assume w.l.o.g that $\breve{F}_\infty(\mu) < \infty$. From Lemma 95 there is μ_L such that $|\breve{F}_\infty(\mu) - \breve{F}_\infty(\mu_L)| < \frac{1}{L}$ and (81) is satisfied. Then choosing $\mu_L^T = \mathrm{Law}_{\mu_L}(\mathbb{X}, \mathbb{1}_{\{t\leqslant T\}}u_t)$ we obtain that $l^T(\mathbb{1}_{\{\cdot\leqslant T\}}u) = \mathbb{1}_{\{\cdot\leqslant T\}}l^\infty(u)$, so $\|l^T(\mathbb{1}_{\{\cdot\leqslant T\}}u)\|_{\mathcal{H}} \leqslant \|l^\infty(u)\|_{\mathcal{H}}$, and $\|Z_T(\mathbb{1}_{\{\cdot\leqslant T\}}u)\|_{L^4}^4 = \|Z_T(u)\|_{L^4}^4 \lesssim \|u\|_{\mathcal{L}}^4$, which is integrable by (81). By dominated convergence and Lemma 81 we obtain $\lim_{T\to\infty}\breve{F}_T(\mu_L^T) = \breve{F}_\infty(\mu_L)$. Extracting a suitable diagonal sequence gives the required recovery sequence. □

We will deal now with the case that $T = \infty$.

The proof of Lemma 92 does not apply when $T = \infty$. An additional difficulty derives from the fact that in approximating the drift u we might destroy the regularity of $l^\infty(u)$, since now $l^\infty(u)$ needs to be more regular than u, contrary to the finite T case. To resolve this problem we need to be able to smooth out the remainder without destroying the bound on $Z_T(u)$. To do so smoothing $l^\infty(u)$ directly, and constructing a corresponding new u will not work, since $l^\infty(u)$ by itself does not give enough control on u and $Z(u)$. However we are still able to prove the following lemma by regularising an "augmented" version of $l^\infty(u)$.

Lemma 94. *There exists a family of continuous functions* $\mathrm{rem}_\varepsilon\colon \mathfrak{S}\times\mathcal{L}\to\mathcal{L}$*, which are also continuous* $\mathfrak{S}\times\mathcal{L}_w\to\mathcal{L}_w$*, such that for any* $T\in[0,\infty]$*,*

$$\begin{aligned}
\|\mathrm{rem}_\varepsilon(\mathbb{X},u)\|_{\mathcal{L}} &\lesssim \|\mathbb{X}\|_{\mathfrak{S}} + \|u\|_{\mathcal{L}}, \\
\|Z_T(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{L^4} &\lesssim \|\mathbb{X}\|_{\mathfrak{S}} + \|Z_T(u)\|_{L^4}, \\
\|l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{\mathcal{H}}^2 &\lesssim_\varepsilon (1+\|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_\infty(u)\|_{L^4}^4 + \|u\|_{\mathcal{L}}^2,
\end{aligned}$$

and $\|l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{\mathcal{H}}$ *depends continuously on* $(\mathbb{X},u)\in\mathfrak{S}\times\mathcal{L}$*. Furthermore*

$$\mathrm{rem}_\varepsilon(\mathbb{X},u)\to u \ \text{ in } \mathcal{L},$$

and if $l^\infty(u)\in\mathcal{H}$

$$l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\to l^\infty(u) \ \text{ in } \mathcal{H} \text{ as } \varepsilon\to 0.$$

Proof. Let $\mathbb{X}^2 = \mathcal{U}_{\leqslant}\mathbb{X}^2 + \mathcal{U}_{>}\mathbb{X}^2$ be the decomposition introduced in Section 7.2, and observe that for any $c>0$ we can easily modify it to ensure that $\|\mathcal{U}_{>}\mathbb{X}^2\|_{\mathscr{C}^{-1-\kappa}} < c$, almost surely for any $\mu \in \bar{\mathcal{X}}$ and for any $1 \leqslant p < \infty$, $\mathbb{E}_\mu[\|\mathcal{U}_{\leqslant}\mathbb{X}^2\|^p_{\mathscr{C}^{-1+\kappa}}] \leqslant C$ where C depends on $|\Lambda|, \kappa, c, p$. Now set $\tilde{l}_t(u) = -\lambda J_t(\mathcal{U}_{\leqslant}\mathbb{X}^2_t \succ Z^\flat_t(u)) + l^\infty_t(u)$. Then u satisfies

$$u_s = -\lambda \mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^\flat_s) + \tilde{l}_s(u).$$

From this equation we can see that, like in Section 7.2,

$$\|u\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \lambda \int_0^\infty \frac{1}{\langle s\rangle^{1+\varepsilon}} \|\mathcal{U}_{>}\mathbb{X}^2_s\|_{\mathscr{C}^{-1-\kappa}} \mathrm{d}s \|u\|_{\mathcal{L}} + \|\tilde{l}(u)\|_{\mathcal{L}},$$

and choosing c small enough we get

$$\|u\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|\tilde{l}(u)\|_{\mathcal{L}}. \tag{76}$$

Similarly we observe that

$$Z_T(u) = -\lambda \mathbb{X}^{[3]}_T - \lambda \int_0^T J^2_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^\flat_s(u))\mathrm{d}s + Z_T(\tilde{l}(u)),$$

so again with c small enough and since $Z^\flat_s = \theta_s Z_T$ for $s \leqslant T$:

$$\|Z_T(u)\|_{L^4} \lesssim \lambda \big\|\mathbb{X}^{[3]}_T\big\|_{L^4} + \|Z_T(\tilde{l}(u))\|_{L^4}. \tag{77}$$

Conversely, it is not hard to see that we have the inequalities

$$\|Z_T(\tilde{l}(u))\|_{L^4} \;\lesssim\; \lambda \big\|\mathbb{X}^{[3]}_T\big\|_{L^4} + \|Z_T(u)\|_{L^4}, \tag{78}$$

and

$$\|\tilde{l}(u)\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}. \tag{79}$$

Clearly the map $(\mathbb{X}, u) \mapsto (\mathbb{X}, \tilde{l}(u))$ is continuous as a map $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$ and using Lemma 80 also as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$, and the inverse is clearly continuous $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$. We now show that it is also continuous as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$. Assume that $\tilde{l}(u^n) \to \tilde{l}(u)$ weakly, since then $\|\tilde{l}(u^n)\|_{\mathcal{L}}$ bounded, this implies by (76) that also $\|u^n\|_{\mathcal{L}}$ is bounded, and so we can select a weakly convergent subsequence, converging to $u^\star$. Then $u^\star$ solves the equation

$$u^\star_s = -\lambda \mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^\flat_s(u^\star)) + \tilde{l}_s(u),$$

(which can be seen for example by testing with some $h \in \mathcal{L}^*$) which implies that $u^\star = u$ (e.g. by Gronwall). Now define $\mathrm{rem}_\varepsilon(u)$ to be the solution to the equation

$$\mathrm{rem}_\varepsilon(u)_s = -\lambda \mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^\flat_s(\mathrm{rem}_\varepsilon(u))) + \mathrm{reg}_{x,\varepsilon}(\tilde{l}_s(u)).$$

Then by the properties discussed above $(\mathbb{X}, u) \mapsto (\mathbb{X}, \mathrm{rem}_\varepsilon(u))$ is continuous in both the weak and the norm topology and we also have from (76) and (79) that

$$\|\mathrm{rem}_\varepsilon(u)\|_{\mathcal{L}} \;\lesssim\; \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.$$

From (77) we have

$$\|Z_T(\mathrm{rem}_\varepsilon(u))\|_{L^4} \lesssim \lambda \big\|\mathbb{X}^{[3]}_T\big\|_{L^4} + \|Z_T(u)\|_{L^4},$$

and by definition of $\mathrm{rem}_\varepsilon(u)$

$$\begin{aligned}\|\tilde{l}(\mathrm{rem}_\varepsilon(u))\|_{\mathcal{H}} &= \|\mathrm{reg}_{x,\varepsilon}(\tilde{l}(u))\|_{\mathcal{H}}\\ &\lesssim_\varepsilon \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.\end{aligned} \tag{80}$$

Now observe that

$$\begin{aligned}\|l^\infty(\mathrm{rem}_\varepsilon(u))\|^2_{\mathcal{H}} &\lesssim \|s\mapsto \lambda J_s(\mathcal{U}_\leqslant \mathbb{X}^2_s \succ Z^\flat_s(\mathrm{rem}_\varepsilon(u)))\|^2_{\mathcal{H}} + \|\tilde{l}(\mathrm{rem}_\varepsilon(u))\|^2_{\mathcal{H}}\\ &\lesssim_\varepsilon \lambda\int_0^\infty \frac{1}{\langle s\rangle^{1+\kappa}}\|\mathcal{U}_\leqslant\mathbb{X}^2_s\|^2_{\mathscr{C}^{-1+\kappa}}\|Z^\flat_s(\mathrm{rem}_\varepsilon(u))\|^2_{L^4}\mathrm{d}s + \lambda\|\mathbb{X}^{\langle 3\rangle}\|^2_{\mathcal{L}} + \|u\|^2_{\mathcal{L}}\\ &\lesssim \lambda(1+\|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_\infty(\mathrm{rem}_\varepsilon(u))\|^4_{L^4} + \|u\|^2_{\mathcal{L}}\\ &\lesssim \lambda(1+\|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_\infty(u)\|^4_{L^4} + \|u\|^2_{\mathcal{L}}.\end{aligned}$$

Since also $\|\lambda J_t(\mathcal{U}_\leqslant\mathbb{X}^2_t \succ Z^\flat_t(\mathrm{rem}_\varepsilon(u)))\|_{\mathcal{H}}$ depends continuously on $(\mathbb{X},u)$ (both in the weak and strong topology on $\mathcal{L}$) we obtain the statement. □

Lemma 95. *For any $\mu\in\bar{\mathcal{X}}$ such that $\breve{F}_\infty(\mu)<\infty$ there exists a sequence of measures $\mu_L\in\bar{\mathcal{X}}$ such that*

i. For any $p<\infty$,

$$\mathbb{E}_{\mu_L}[\|u\|^p_{\mathcal{L}}] + \mathbb{E}_{\mu_L}[\|l^\infty(u)\|^p_{\mathcal{H}}] < \infty, \tag{81}$$

ii. $\mu_L\to\mu$ weakly on $\mathfrak{S}\times\mathcal{L}$ and $\mathrm{Law}_{\mu_L}(l^\infty(u))\to\mathrm{Law}_\mu(l^\infty(u))$ weakly on $\mathcal{H}$,

iii.

$$\lim_{L\to\infty}\breve{F}_\infty(\mu_L) = \breve{F}_\infty(\mu),$$

iv. For any μ_L there exists a sequence $\mu_{n,L}\in\mathcal{X}$ such that

$$\sup_n\left(\mathbb{E}_{\mu_{n,L}}[\|u\|^p_{\mathcal{L}}] + \mathbb{E}_{\mu_{n,L}}[\|l^\infty(u)\|^p_{\mathcal{H}}]\right) < \infty, \tag{82}$$

$\mu_{n,L}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{L}_w$ and $\mathrm{Law}_{\mu_{n,L}}(l^\infty(u))\to\mathrm{Law}_{\mu_L}(l^\infty(u))$ weakly on $\mathcal{H}_w$.

Proof. By Lemma 89 there exists a sequence $\mu_{\tilde{L}}\to\mu$ weakly on $\mathfrak{S}\times\mathcal{L}$ such that

$$\mathbb{E}_{\mu_{\tilde{L}}}[\|Z_T(u)\|^4_{L^4}]\to\mathbb{E}_\mu[\|Z_T(u)\|^4_{L^4}],\qquad \mathbb{E}_{\mu_{\tilde{L}}}[\|u\|^2_{\mathcal{L}}]\to\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}],$$

and $\mu_{\tilde{L}}$ is supported on $\mathfrak{S}\times B(0,\tilde{L})\subset\mathfrak{S}\times\mathcal{L}$. Now set $\mu^\varepsilon_{\tilde{L}} := (\mathrm{rem}_\varepsilon)_*\mu_{\tilde{L}}$. Then $\mu^\varepsilon_{\tilde{L}}\to\mu^\varepsilon := (\mathrm{rem}_\varepsilon)_*\mu$ on $\mathfrak{S}\times\mathcal{L}$ and by the bounds from Lemma 94 also $\mathbb{E}_{\mu^\varepsilon_{\tilde{L}}}[\|Z_T(u)\|^4_{L^4}]\to\mathbb{E}_{\mu^\varepsilon}[\|Z_T(u)\|^4_{L^4}]$ and $\mathbb{E}_{\mu^\varepsilon_{\tilde{L}}}[\|l^\infty(u)\|^2_{\mathcal{H}}]\to\mathbb{E}_{\mu^\varepsilon}[\|l^\infty(u)\|^2_{\mathcal{H}}]$. The bounds from Lemma 94 imply also $\mathbb{E}_{\mu^\varepsilon}[\|Z_T(u)\|^4_{L^4}]\to\mathbb{E}_\mu[\|Z_T(u)\|^4_{L^4}]$, $\mathbb{E}_{\mu^\varepsilon}[\|l^\infty(u)\|^2_{\mathcal{H}}]\to\mathbb{E}_\mu[\|l^\infty(u)\|^2_{\mathcal{H}}]$, and furthermore

$$\mathbb{E}_{\mu^\varepsilon_{\tilde{L}}}[\|u\|^p_{\mathcal{L}}]\lesssim\mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|^p_{\mathfrak{S}}+\|u\|^p_{\mathcal{L}})\lesssim\mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|^p_{\mathfrak{S}})+\tilde{L}^p,$$

and similarly

$$\mathbb{E}_{\mu^\varepsilon_{\tilde{L}}}[\|l^\infty(u)\|^p_{\mathcal{H}}]\lesssim\mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|^p_{\mathfrak{S}}+\|u\|^p_{\mathcal{L}})\lesssim\mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|^p_{\mathfrak{S}})+\tilde{L}^p,$$

and by continuity of $\breve{F}_\infty$ and domination using (71) we are also able to deduce that we can find ε small enough and $\tilde{L}$ large enough depending on ε such that $|\breve{F}_\infty(\mu^\varepsilon)-\breve{F}_\infty(\mu)|<1/(2L)$ and $|\breve{F}_\infty(\mu^\varepsilon_{\tilde{L}})-\breve{F}_\infty(\mu^\varepsilon)|<1/(2L)$. Choosing $\mu_L=\mu^\varepsilon_{\tilde{L}}$ we obtain the first three points of the Lemma. For the fourth point recall that from Lemma 89 we have sequences $\mu_{n,\tilde{L}}\to\mu_{\tilde{L}}$ weakly on $\mathfrak{S}\times\mathcal{L}_w$, and $\mu_{n,\tilde{L}}\in\mathcal{X}$, which have support in $\mathfrak{S}\times B(0,\tilde{L})$ and since rem_ε is continuous on $\mathfrak{S}\times\mathcal{L}_w$ setting $\mu^\varepsilon_{n,\tilde{L}} := (\mathrm{rem}_\varepsilon)_*\mu_{n,\tilde{L}}$ we obtain the desired sequence. □

Lemma 96. *If $T=\infty$ we have*

$$\inf_{\mu\in\mathcal{X}}\breve{F}_\infty(\mu)=\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_\infty(\mu).$$

Proof. One can now proceed very similarly to the proof of Lemma 92. Let $\mu\in\bar{\mathcal{X}}$ such that $\breve{F}_\infty(\mu)<\infty$. By Lemma 95, for any L, $\mu\in\bar{\mathcal{X}}$, there exists a μ_L such that $|\breve{F}_\infty(\mu)-\breve{F}_\infty(\mu_L)|<1/L$, and a sequence $(\mu_{n,L})_n$ such that $\mu_{n,L}\in\mathcal{X}$, $\mu_{n,L}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{L}_w$, and such that (82) is satisfied. Define $\mu^{\varepsilon,\delta}_{n,L}:=\mathrm{Law}(\mathbb{X},\mathrm{rem}_\varepsilon(\mathrm{reg}_{t:x,\delta}(u)))$, and observe that now $\mu^{\varepsilon,\delta}_{n,L}\to\mu^{\varepsilon,\delta}_L$ on $\mathfrak{S}\times\mathcal{L}$, $\mathrm{Law}_{\mu^{\varepsilon,\delta}_{n,L}}(\mathbb{X},l^\infty(u))\to\mathrm{Law}_{\mu^{\varepsilon,\delta}_L}(\mathbb{X},l^\infty(u))$ on $\mathfrak{S}\times\mathcal{H}$, and that we have $\sup_n\left(\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}[\|u\|^p_{\mathcal{L}}]+\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}[\|l^\infty(u)\|^p_{\mathcal{H}}]\right)<\infty$. Then for some $\chi\in C(\mathbb{R},\mathbb{R})$, $\chi=1$ on $B(0,1)$ supported on $B(0,2)$, for any $N\in\mathbb{N}$, the function

$$\begin{aligned}&\chi\bigg(\frac{\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}}{N}\bigg)\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\\ =:\ &\tilde{\chi}_N(\mathbb{X},u)\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\end{aligned}$$

is bounded and continuous on $\mathfrak{S}\times\mathcal{L}$, and so by weak convergence

$$\begin{aligned}&\lim_{n\to\infty}|\breve{F}_\infty(\mu^{\varepsilon,\delta}_{n,L})-\breve{F}_\infty(\mu^{\varepsilon,\delta}_L)|\\ \leqslant\ &\lim_{n\to\infty}\bigg|\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}\bigg[\tilde{\chi}_N(\mathbb{X},u)\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\bigg]-\\ &\qquad-\mathbb{E}_{\mu^{\varepsilon,\delta}_L}\bigg[\tilde{\chi}_N(\mathbb{X},u)\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\bigg]\bigg|\\ &+\sup_n\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}\bigg[\bigg|(1-\tilde{\chi}_N(\mathbb{X},u))\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\bigg|\bigg]\\ &+\mathbb{E}_{\mu^{\varepsilon,\delta}_L}\bigg[\bigg|(1-\tilde{\chi}_N(\mathbb{X},u))\bigg(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|^4_{L^4}+\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg)\bigg|\bigg]\\ \leqslant\ &2\sup_n\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}\bigg[\mathbb{1}_{\{\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}>N\}}\bigg|\Phi_\infty(\mathbb{X},\ Z(u),\ K(u))\ +\ \lambda\|Z_\infty(u)\|^4_{L^4}\ +\\ &\frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\bigg|\bigg]\\ \lesssim\ &\sup_n\Big(\mu^{\varepsilon,\delta}_{n,L}(\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}>N)^{1/2}\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}[\|\mathbb{X}\|^p_{\mathfrak{S}}+\|u\|^8_{\mathcal{L}}+\|l^\infty(u)\|^4_{\mathcal{H}}]^{1/2}\Big)\\ \lesssim\ &\sup_n\bigg(\bigg(\frac{1}{N}\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}[\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}]\bigg)^{1/2}\mathbb{E}_{\mu^{\varepsilon,\delta}_{n,L}}[\|\mathbb{X}\|^p_{\mathfrak{S}}+\|u\|^8_{\mathcal{L}}+\|l^\infty(u)\|^4_{\mathcal{H}}]^{1/2}\bigg)\\ \to\ &0\quad\text{ as } N\to\infty\end{aligned}$$

As we can find ε,δ such that $|\breve{F}_\infty(\mu^{\varepsilon,\delta}_L)-\breve{F}_\infty(\mu_L)|<1/L$ we conclude. □

7.4 Large deviations

In this section we want to discuss a Laplace principle for the Φ^4_3 measure in the "small noise limit". We introduce the family $\nu^\hbar_T$ of measures given by

$$\int g(\phi)\nu^\hbar_T(\mathrm{d}\phi)=\frac{\mathbb{E}\Big[\,g(\hbar^{1/2}W_T)\,e^{-\frac{1}{\hbar}V^\hbar_T(\hbar^{1/2}W_T)}\Big]}{\mathscr{Z}^\hbar_T},\tag{83}$$

where similarly to above

$$V_T^\hbar(\varphi) := \lambda\int_\Lambda (|\varphi(\xi)|^4 - a_T^\hbar|\varphi(\xi)|^2 + b_T^\hbar)\mathrm{d}\xi \qquad \mathscr{Z}_T^\hbar := \mathbb{E}\Big[e^{-\frac{1}{\hbar}V_T^\hbar(\hbar^{1/2}W_T)}\Big] = \int e^{-\frac{1}{\hbar}V_T^\hbar(\hbar^{1/2}\phi_T)}\vartheta(\mathrm{d}\phi)$$

for any bounded measurable $g\colon \mathscr{S}'(\Lambda)\to\mathbb{R}$. In this section we will take all integrals to be non-normalised for simplicity since we wish to derive a large deviations principle for fixed volume.

This corresponds (modulo renormalisation) to the measure heuristically defined by

$$e^{-\frac{1}{\hbar}\int\lambda\varphi(\xi)^4+\frac{1}{2}\varphi(\xi)^2+\frac{1}{2}|\nabla\varphi(\xi)|^2\mathrm{d}\xi}\mathrm{d}\varphi$$

Our goal is now to show that $\nu^\hbar$ given as the weak limit of $\nu_T^\hbar$ satisfies a Laplace principle according to the following definition.

Definition 97. *A sequence of measures $\rho_\hbar$ on $\mathscr{S}'(\Lambda)$ satisfies the Laplace principle with rate function I if for any continuous bounded function $f\colon\mathscr{S}'(\Lambda)\to\mathbb{R}$*

$$\lim_{\hbar\to 0} -\hbar\log\int e^{-\frac{1}{\hbar}f(\psi)}\mathrm{d}\rho_\hbar(\psi) = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi)+I(\psi)\}.$$

Our goal for the rest of this section will be to prove the following theorem:

Theorem 98. *The sequence of measures $\nu_T^\hbar$ converges to a unique limit $\nu^\hbar$ as $T\to\infty$. Furthermore $\nu^\hbar$ satisfies a Laplace principle with rate function*

$$I(\psi) = \lambda\int\psi^4 + \frac{1}{2}\int\psi^2 + \frac{1}{2}\int|\nabla\psi|^2 \tag{84}$$

as $\hbar\to 0$.

We will have to analyse the quantity

$$\mathcal{W}_T^\hbar(f) = -\hbar\log\mathbb{E}\Big[e^{-\frac{1}{\hbar}(f(\hbar^{1/2}W_T)+V_T^\hbar(\hbar^{1/2}W_T))}\Big]$$

First we have to choose $V_T^\hbar$ appropriately. Set $W_T^\hbar = \hbar^{1/2}W_T$. Again we choose $a_T^\hbar, b_T^\hbar$ so that

$$V_T^\hbar(W_T^\hbar) = \lambda\int[\![(W_T^\hbar)^4]\!] - \lambda^2\gamma_T^\hbar\int[\![(W_T^\hbar)^2]\!] - \delta_T^\hbar,$$

where now the Wick product is taken with respect to the law of $W_T^\hbar$ and not W_T, and $\gamma^\hbar$, $\delta^\hbar$ will be fixed below. Now most of the analysis from Sections 7.2, 7.3 carries over into this situation for fixed $\hbar$. By the BD formula we have

$$\mathcal{W}_T^\hbar(f) = \inf_{u\in\mathbb{H}_a}\tilde{F}_T^\hbar(u)$$

with

$$\tilde{F}_T^\hbar(u) = \mathbb{E}\Big[f(\hbar^{1/2}W_T+\hbar^{1/2}I_T(u)) + V_T^\hbar(W_T^\hbar+\hbar^{1/2}I_T(u)) + \frac{\hbar}{2}\int_0^T\|u_s\|_{L^2}^2\Big].$$

And by the change $\hbar^{1/2}u\to u$ of variables we get

$$\mathcal{W}_T^\hbar(f) = \inf_{u\in\mathbb{H}_a}F_T^\hbar(u),$$

with

$$F_T^\hbar(u) = \mathbb{E}\Big[f(\hbar^{1/2}W_T + I_T(u)) + V_T^\hbar(W_T^\hbar + I_T(u)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \Big].$$

Lemma 99. *Define $l = l^{T,\hbar}(u) \in \mathbb{H}_a$, $Z = Z(u) \in C([0,\infty], H^{1/2-\kappa})$, $K = K^\hbar(u) \in C([0,\infty], H^{1-\kappa})$ such that*

$$\begin{aligned} &Z_t(u) := I_t(u), \\ &l_t^{T,\hbar}(u) := u_t + \lambda\hbar^{3/2}\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda\hbar\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)), \qquad t \geqslant 0. \\ &K_t^\hbar(u) := I_t(w^\hbar(u)), \text{ with } w_t^\hbar(u) := -\lambda\hbar\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)) + l_t^{T,\hbar}(u), \end{aligned} \tag{85}$$

Then the functional $F_T^\hbar(u)$ takes the form

$$F_T^\hbar(u) \;=\; \mathbb{E}\Big[f(\hbar^{1/2}W_T + Z_T) + \Phi_T^\hbar(\mathbb{W}, Z(u), K(u)) + \lambda \fint (Z_T(u))^4 + \frac{1}{2}\|l^{T,\hbar}(u)\|_{\mathcal{H}}^2 \Big],$$

where

$$\Phi_T^\hbar(\mathbb{W}, Z, K^\hbar) := \sum_{i=1}^6 \Upsilon_T^{(i),\hbar},$$

$$\begin{aligned}
\Upsilon_T^{(1),\hbar} &:= -\frac{\lambda}{2}\hbar\mathfrak{K}_2(\mathbb{W}_T^2, K_T^\hbar, K_T^\hbar) + \frac{\lambda}{2}\hbar\fint (\mathbb{W}_T^2 \prec K_T^\hbar)K_T^\hbar - \lambda^2\hbar^{5/2}\fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]})K_T^\hbar, \\
\Upsilon_T^{(2),\hbar} &:= \lambda\hbar\int (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat))K_T^\hbar, \\
\Upsilon_T^{(3),\hbar} &:= \lambda\hbar\int_0^T\int (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t^\hbar \mathrm{d}t, \\
\Upsilon_T^{(4),\hbar} &:= 4\lambda\hbar^{1/2}\int W_T(K_T^\hbar)^3 - 12\lambda^2\hbar^2\int W_T\mathbb{W}_T^{[3]}(K_T^\hbar)^2 + 12\lambda^3\hbar^{7/2}\int W_T(\mathbb{W}_T^{[3]})^2K_T^\hbar, \\
\Upsilon_T^{(5),\hbar} &:= -2\lambda^2\hbar^2\int \gamma_T Z_T^\flat(Z_T - Z_T^\flat) - \lambda^2\hbar^2\int \gamma_T(Z_T - Z_T^\flat)^2 - 2\lambda^2\hbar^2\int_0^T\int \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t, \\
\Upsilon_T^{(6),\hbar} &:= -\lambda^2\hbar^{5/2}\int \mathbb{W}_T^{2\diamond[3]}K_T^\hbar - \frac{\lambda^2}{2}\hbar^2\int_0^T\int \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t - \frac{\lambda^2}{2}\hbar^2\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, \\
&\quad Z_t^\flat)\mathrm{d}t.
\end{aligned}$$

Moreover we have chosen $\gamma_T^\hbar = \hbar^2\gamma_T$ and the renormalisation constant $\delta_T^\hbar$ to be

$$\begin{aligned}
\delta_T^\hbar \;:=\; &-\frac{\lambda^2}{2}\hbar^3\mathbb{E}\int_0^T\int (\mathbb{W}_t^{\langle 3\rangle})^2\mathrm{d}t + \frac{\lambda^3}{2}\hbar^4\mathbb{E}\int \mathbb{W}_T^2(\mathbb{W}_T^{[3]})^2 \\
&+2\lambda^3\hbar^2\gamma_T^\hbar\mathbb{E}\int W_T\mathbb{W}_T^{[3]} - 4\lambda^4\hbar^5\mathbb{E}\int W_T(\mathbb{W}_T^{[3]})^3.
\end{aligned} \tag{86}$$

The proof of Lemma 99 is analogous to the proof of Lemma 70.

Now for fixed $\hbar$ reasoning analogous to Theorem 79 we can conclude that

$$\begin{aligned}
&\lim_{T\to\infty} \mathcal{W}_T^\hbar(f) \\
=\;& \mathcal{W}^\hbar(f) \\
=\;& \inf_{u\in\mathbb{H}_a} \mathbb{E}\Big[f(\hbar^{1/2}W_\infty + I_\infty(u)) + \Phi_\infty^\hbar(\mathbb{W}, Z(u), K^\hbar(u)) + \lambda\|Z_\infty\|_{L^4(\Lambda)}^4 + \frac{1}{2}\|l^\hbar(u)\|_{L^2(\mathbb{R}_+\times\Lambda)}^2 \Big]
\end{aligned}$$

where

$$l_t^\hbar(u) = u_t + \lambda\hbar^{3/2}\mathbb{W}_t^{\langle 3\rangle} + \lambda\hbar J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)).$$

and $\Phi_\infty^\hbar(\mathbb{W}, Z, K^\hbar) := \sum_{i=1}^6 \Upsilon_\infty^{(i),\hbar}$

$$\begin{aligned}
\Upsilon_\infty^{(1),\hbar} &:= -\frac{\lambda}{2}\hbar\mathfrak{K}_2(\mathbb{W}_\infty^2, K_\infty^\hbar, K_\infty^\hbar) + \frac{\lambda}{2}\hbar\int(\mathbb{W}_\infty^2 \prec K_\infty^\hbar)K_\infty^\hbar - \lambda^2\hbar^{5/2}\int(\mathbb{W}_\infty^2 \prec \mathbb{W}_\infty^{[3]})K_\infty^\hbar,\\
\Upsilon_\infty^{(2),\hbar} &:= 0\\
\Upsilon_\infty^{(3),\hbar} &:= \lambda\hbar\int_0^\infty\int(\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t^\hbar \mathrm{d}t,\\
\Upsilon_\infty^{(4),\hbar} &:= 4\lambda\hbar^{1/2}\int W_\infty(K_\infty^\hbar)^3 - 12\lambda^2\hbar^2\int W_\infty \mathbb{W}_\infty^{[3]}(K_\infty^\hbar)^2 + 12\lambda^3\hbar^{7/2}\int W_\infty(\mathbb{W}_\infty^{[3]})^2 K_\infty^\hbar,\\
\Upsilon_\infty^{(5),\hbar} &:= -2\lambda^2\hbar^2\int_0^\infty\int\gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t,\\
\Upsilon_\infty^{(6),\hbar} &:= -\lambda^2\hbar^{5/2}\int\mathbb{W}_\infty^{2\diamond[3]}K_\infty^\hbar - \frac{\lambda^2}{2}\hbar^2\int_0^\infty\int\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t\\
&\quad -\frac{\lambda^2}{2}\hbar^2\int_0^\infty\mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t.
\end{aligned}$$

Now it remains to prove

$$\lim_{\hbar\to 0}\mathcal{W}^\hbar(f) = \inf_{\psi\in\mathscr{D}'(\Lambda)}\{f(\psi) + I(\psi)\}$$

with I defined by (84). For this in analogy with Section 7.3 we introduce

$$\breve{F}^\hbar(\mu) := \mathbb{E}_\mu\Big[f(\hbar^{1/2}W_\infty + I_\infty(u)) + \Phi_\infty^\hbar(\mathbb{X}, Z(u), K^\hbar(u)) + \lambda\|Z_\infty\|_{L^4}^4 + \frac{1}{2}\|l^\hbar(u)\|_{\mathcal{H}}^2\Big]$$

where the functional is again defined on the space $\mathcal{X}$ and

$$\mathcal{W}^\hbar(f) = \inf_{\mu\in\mathcal{X}}\breve{F}^\hbar(\mu)$$

and in the same way as for Lemma 96 we can show that

$$\mathcal{W}^\hbar(f) = \inf_{\mu\in\bar{\mathcal{X}}}\breve{F}^\hbar(\mu)$$

Next we claim that taking

$$\breve{F}^0(\mu) = \mathbb{E}_\mu\Big[f(Z_\infty(u)) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|u\|_{\mathcal{H}}^2\Big]$$

we have the following statements

Lemma 100. *The family $\breve{F}^\hbar(\mu)$ is equicoercive on $\bar{\mathcal{X}}$*

Proof. It is not hard to see that

$$\breve{F}^\hbar(\mu) \geqslant -C + \frac{1}{2}\mathbb{E}\Big[\lambda\|Z_\infty\|_{L^4}^4 + \frac{1}{2}\|l^\hbar(u)\|_{\mathcal{H}}^2\Big]$$

and also that $\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] \lesssim C + 2\lambda\mathbb{E}_\mu[\|Z_\infty(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^\hbar(u)\|_{\mathcal{H}}^2]$. Define the set

$$\mathcal{K} = \{\mu\colon \mathbb{E}_\mu[\|Z_\infty(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^\hbar(u)\|_{\mathcal{H}}^2] \leqslant K\}$$

note that $\mathcal{K}$ is compact from Lemma 85. Now we can prove in the same way as in Section 7.2 that

$$\sup_{\hbar} \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu) < \infty.$$

So choosing K large enough we have that

$$\inf_{\mu \notin \mathcal{K}} \breve{F}^{\hbar}(\mu) \geqslant cK - C > \sup_{\hbar} \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu)$$

which implies

$$\inf_{\mu \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu) = \inf_{\mu \in \mathcal{K}} \breve{F}^{\hbar}(\mu)$$

□

Lemma 101. *$\breve{F}^{\hbar}(\mu)$ Γ-converges to $\breve{F}^{0}(\mu)$ on $\bar{\mathcal{X}}$*

Proof.

Step 1. First we prove the liminf inequality. Consider a sequence $\mu_{\hbar} \to \mu$ in $\bar{\mathcal{X}}$, our aim is to prove that

$$\liminf_{\hbar \to 0} \breve{F}^{\hbar}(\mu^{\hbar}) \geqslant \breve{F}^{0}(\mu).$$

As before it is enough to prove the statement for a subsequence of $\mu^{\hbar}$. W.l.o.g. we can assume that

$$\sup_{\hbar} \left(\mathbb{E}_{\mu^{\hbar}}[\| Z_{\infty}(u) \|_{L^4}^4] + \mathbb{E}_{\mu^{\hbar}}[\| l^{\hbar}(u) \|_{\mathcal{H}}^2] \right) < \infty. \tag{87}$$

Now from the estimates in Section 7.2 and the definition of $\Phi_{\infty}^{\hbar}$ we observe that for any $\mu_{\hbar} \in \bar{\mathcal{X}}$

$$\mathbb{E}_{\mu^{\hbar}}[|\Phi_{\infty}^{\hbar}|] \leqslant \hbar^{1/2} (\mathbb{E}_{\mu^{\hbar}}[\| Z_{\infty}(u) \|_{L^4}^4] + \mathbb{E}_{\mu^{\hbar}}[\| l^{\hbar}(u) \|_{\mathcal{H}}^2]),$$

so for any sequence satisfying (87) $\mathbb{E}_{\mu^{\hbar}}[|\Phi_{\infty}^{\hbar}|] \to 0$. By the Portmanteau Lemma we have

$$\liminf_{\hbar \to 0} \mathbb{E}_{\mu^{\hbar}}[\| Z_{\infty}(u) \|_{L^4}^4] \geqslant \mathbb{E}_{\mu}[\| Z_{\infty}(u) \|_{L^4}^4].$$

We claim that also

$$\liminf_{\hbar \to 0} \mathbb{E}_{\mu^{\hbar}}[\| l^{\hbar}(u) \|_{\mathcal{H}}^2] \geqslant \mathbb{E}_{\mu}[\| u \|_{\mathcal{H}}^2].$$

For this we find a subsequence of $\mu_{\hbar}$ (not relabelled) and a sequence of random variables $(\mathbb{X}^{\hbar}, u^{\hbar})$ on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ such that $\mathrm{Law}(\mathbb{X}^{\hbar}, u^{\hbar}) = \mu_{\hbar}$ and $\mathbb{X}^{\hbar} \to \mathbb{X}$ in $\mathfrak{S}$ and $u^{\hbar} \to u^{0}$ in $\mathcal{L}_w$, where $\mathrm{Law}(\mathbb{X}, u^{0}) = \mu$. Then from eq. (87) we can pick a further subsequence such that $l^{\hbar}(u^{\hbar}) \to l^{\star}$ in $\mathcal{H}_w$. The definition of $l^{\hbar}$ implies

$$l_t^{\hbar}(u) \;=\; u_t + \lambda \hbar^{3/2} \mathbb{X}_t^{\langle 3 \rangle, \hbar} + \lambda \hbar J_t(\mathbb{X}_t^{2,\hbar} \succ Z_t^{\flat}(u))$$

and testing and taking limits this implies

$$l^{\star} = u^{0},$$

which implies our claim. Now from weak convergence it follows that provided that f is bounded and continuous on $H^{-\delta}(\Lambda)$,

$$\lim_{\hbar \to 0} \mathbb{E}_{\mu_{\hbar}}[f(\hbar^{1/2} X_{\infty} + Z_{\infty}(u))] = \mathbb{E}_{\mu}[f(Z_{\infty}(u))],$$

which completes Step 1.

Step 2. Now we construct the recovery sequence, more precisely we prove that for every μ there exists $\mu^{\hbar} \to \mu$ such that

$$\limsup_{\hbar \to 0} \breve{F}^{\hbar}(\mu^{\hbar}) \leqslant \breve{F}^{0}(\mu).$$

By Lemma 89 for any μ such that $\breve{F}^0(\mu) < \infty$ (otherwise there is nothing to prove) we have a sequence μ^n such that $\|u\|_{\mathcal{L}} \leqslant n$ μ^n-almost surely and $\lim_{n\to\infty} \breve{F}^0(\mu^n) = \breve{F}^0(\mu)$. By a diagonal argument it is enough to find sequences $\mu^{n,\hbar}$ such that

$$\limsup_{\hbar\to 0} \breve{F}^\hbar(\mu^{n,\hbar}) \leqslant \breve{F}^0(\mu^n).$$

For this consider $u^\hbar$ to be the solution to

$$u_t \;=\; u_t^\hbar + \lambda\hbar^{3/2}\mathbb{X}_t^{\langle 3\rangle} + \lambda\hbar J_t(\mathbb{X}_t^2 \succ Z_t^\flat(u^\hbar)).$$

This can be proven to exist by a standard fix-point and Gronwall argument. Applying J and integrating we obtain

$$Z_t(u^\hbar) = Z_t(u) - \lambda\hbar^{3/2}\mathbb{X}_t^{[3]} - \lambda\hbar\int_0^t J_s^2(\mathbb{X}_s^2 \succ Z_s^\flat(u^\hbar))\mathrm{d}s \tag{88}$$

so, setting $g_t := \int_0^t \langle s\rangle^{-(2-\delta)}\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\delta}(\Lambda)}\mathrm{d}s$, by Gronwall's lemma (recall also $\|Z_s(u)\|_{L^4} \lesssim \|u\|_{\mathcal{L}}$)

$$\begin{aligned}\sup_{s\leqslant t}\|Z_s(u^\hbar)\|_{L^4(\Lambda)} &\leqslant C\bigg(\sup_{s\leqslant t}\|Z_s(u)\|_{L^4} + \lambda\hbar^{3/2}\sup_{s\leqslant t}\big\|\mathbb{X}_s^{[3]}\big\|_{L^4(\Lambda)}\bigg)e^{C\lambda\hbar g_t}\\ &\leqslant C\bigg(\|u\|_{\mathcal{L}}^2 + \lambda^2\hbar^3\sup_{s\leqslant t}\big\|\mathbb{X}_s^{[3]}\big\|_{L^4(\Lambda)}^2\bigg) + e^{2C\lambda\hbar g_t}.\end{aligned} \tag{89}$$

This together with (88) implies that

$$\begin{aligned}&\|Z_t(u^\hbar) - Z_t(u)\|_{L^4}^4\\ \leqslant\;& C\lambda^4\hbar^6\big\|\mathbb{X}_t^{[3]}\big\|_{L^4}^4 + C\lambda^4\hbar^4 g_t^4\sup_{s\leqslant t}\|Z_s(u^\hbar)\|_{L^4}^4\\ \leqslant\;& C\lambda^4\hbar^4\bigg(\hbar^2\big\|\mathbb{X}_t^{[3]}\big\|_{L^4}^4 + g_t^8 + \|u\|_{\mathcal{L}}^{16} + \lambda^{16}\hbar^{24}\sup_{s\leqslant t}\big\|\mathbb{X}_s^{[3]}\big\|_{L^4}^{16} + e^{16C\lambda\hbar g_t}\bigg).\end{aligned}$$

Since the stochastic terms have moments of all orders and g_t (an integral of Besov norms of a second chaos field) satisfies $\mathbb{E}[g_\infty^p]^{1/p} \lesssim p$, hence has exponential moments of small order, the expectation of the right-hand side is $O(\hbar^4)$ for $\hbar$ small, uniformly in t, and

$$\mathbb{E}_{\mu^n}\|Z_t(u^\hbar) - Z_t(u)\|_{L^4}^4 \to 0$$

since $\|u\|_{\mathcal{L}} \leqslant n$ μ^n almost surely. Now set $\mu^{n,\hbar} = \mathrm{Law}_{\mu^n}(\mathbb{X}, u^\hbar)$. Clearly $l^\hbar(u^\hbar) = u$, so $\mathbb{E}_{\mu^{n,\hbar}}[\|l^\hbar(u)\|_{\mathcal{H}}^2] = \mathbb{E}_{\mu^n}[\|u\|_{\mathcal{H}}^2]$ and we have already established that $\mathbb{E}_{\mu^{n,\hbar}}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4] \to \mathbb{E}_{\mu^n}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4]$.

Now together with the fact that

$$\mathbb{E}_{\mu^{n,\hbar}}[|\Phi_\infty^\hbar(\mathbb{W}^\hbar, Z(u), K^\hbar(u))|] \leqslant \hbar^{1/2}(\mathbb{E}_{\mu^{n,\hbar}}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4] + \mathbb{E}_{\mu^{n,\hbar}}[\|l^\hbar(u)\|_{\mathcal{H}}^2]) \to 0$$

we can conclude. □

Lemma 102. $$\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}^0(\mu) = \inf_{u\in\mathbb{H}_a} F(u) = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi) + I(\psi)\}$$

where

$$F(u) = \mathbb{E}\bigg[f(Z_\infty(u)) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|u\|_{\mathcal{H}}^2\bigg]$$

Proof. The first equality can easily be proven in the same fashion as Lemma 96. We only prove the second equality.

Step 1. First we prove

$$\inf_{u\in\mathbb{H}_a} F(u) \leqslant \inf_{\psi\in\mathscr{S}'(\Lambda)} \{f(\psi)+I(\psi)\}.$$

Restricting the infimum to processes of the form

$$u_s = J_s\langle\nabla\rangle^2\psi$$

with $\psi\in H^2(\Lambda)$, we see that

$$Z_\infty(u) = \int_0^\infty J_s u_s \mathrm{d}s = \int_0^\infty J_s^2\langle\nabla\rangle^2\psi \mathrm{d}s = \psi$$

we also compute

$$\|u\|_{\mathcal{H}}^2 = \int_0^\infty \int_\Lambda u_s^2 \mathrm{d}s = \int_0^\infty \langle J_s^2\langle\nabla\rangle^2\psi, \langle\nabla\rangle^2\psi\rangle_{L^2(\Lambda)}\mathrm{d}s = \langle\psi,\langle\nabla\rangle^2\psi\rangle_{L^2(\Lambda)} = \|\psi\|_{H^1}^2$$

so

$$\inf_{u\in\mathbb{H}_a} F(u) \leqslant \inf_{u_s = J_s\langle\nabla\rangle^2\psi} F(u) = \inf_{\psi\in H^2}\{f(\psi)+I(\psi)\} = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi)+I(\psi)\}$$

where the last equality follows from approximation of $\psi\in H^1(\Lambda)$ with $H^2(\Lambda)$ functions, and $f(\psi)+I(\psi)$ is understood to be $=\infty$ if $\psi\notin H^1(\Lambda)$.

Step 2. We now prove the converse inequality

$$\inf_{u\in\mathbb{H}_a} F(u) \geqslant \inf_{\psi\in\mathscr{S}'(\Lambda)} \{f(\psi)+I(\psi)\}.$$

Recall that from the proof of Lemma 64 $\|u\|_{\mathcal{H}} \geqslant \|Z_\infty(u)\|_{H^1}$ so

$$\begin{aligned}\inf_{u\in\mathbb{H}_a} F(u) &\geqslant \inf_{u\in\mathbb{H}_a} \mathbb{E}\Big[f(Z_\infty(u)) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|Z_\infty(u)\|_{H^1}^2\Big] \\ &\geqslant \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi)+I(\psi)\}.\end{aligned}$$

□

7.5 Stochastic estimates

In this section we close our argument proving the following lemmas which give uniform estimates as $T\to\infty$ of some of the stochastic terms appearing in our analytic estimates.

Lemma 103. *For any $\varepsilon>0$ and any $p>1, r<\infty, q\in[1,\infty]$, there exists a constant $C(\varepsilon,p,q)$ which does not depend on Λ such that*

$$\sup_T \mathbb{E}\Big[\big\| W_T \circ W_T^{[3]}\big\|_{B_{r,q}^{-\varepsilon}}^p\Big] \leqslant C(\varepsilon,p,q). \tag{90}$$

Moreover there exists a function $\gamma_t\in C^1(\mathbb{R}_+,\mathbb{R})$ such that for any $\varepsilon>0$ and any $p>1$,

$$\sup_T \mathbb{E}\Big[\big\| (\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]} + 2\gamma_T W_T)\big\|_{B_{r,q}^{-1/2-\varepsilon}}^p\Big] \leqslant C(\varepsilon,p,q), \tag{91}$$

$$\mathbb{E}\Big[\Big(\int_0^\infty \|J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2 + 2\dot\gamma_t\|_{B_{r,q}^{-\varepsilon}}\mathrm{d}t\Big)^p\Big] \leqslant C(\varepsilon,p,q). \tag{92}$$

$$\sup_t \mathbb{E}\Big[\|J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2 + 2\dot\gamma_t\|_{B_{r,q}^{-\varepsilon}}\Big] \leqslant C(\varepsilon,p,q)$$

and

$$|\gamma_t| + \langle t\rangle |\dot\gamma_t| \lesssim 1 + \log\langle t\rangle, \qquad t \geqslant 0. \tag{93}$$

Furthermore γ is independent of Λ. By Besov embedding, the Besov-Hölder norms of these objects are also uniformly bounded in T (but not uniformly in Λ).

Proof. We will concentrate on proving the bounds on the renormalised terms in eqs. (91) and (92) and leave to the reader to fill the details for the easier term in eq. (90). Recall the representation of $(W_t)_t$ in terms of the family of Brownian motions $(B^n_t)_{t,n}$ in eq. (38). Wick's products of the Gaussian field W_T can be represented as iterated stochastic integrals wrt. $(B^n_t)_{t,n}$. In particular, if we let $\mathrm{d}w_s(k) = \langle k\rangle^{-1}\sigma_s(k)\mathrm{d}B^k_s$, we have

$$\mathbb{W}^2_T(x) = 12[\![W^2_T]\!](x) = 24\sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x}\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2),$$

$$\mathbb{W}^{[3]}_T(x) = 24\sum_{k_1,k_2,k_3} e^{ik_{(123)}\cdot x}\int_0^T\int_0^{s_3}\int_0^{s_2}\left(\int_{s_3}^T\frac{\sigma^2_u(k_{(123)})}{\langle k_{(123)}\rangle^2}\mathrm{d}u\right)\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3),$$

where we denote $k_{(1\dots n)} := k_1 + \dots + k_n$ for any $n \geqslant 2$. Now products of iterated integrals can be decomposed in sums of iterated integrals and we get

$$\begin{aligned}
\Delta_q(\mathbb{W}^{2\diamond[3]}_T)(x) &= \Delta_q\big(\mathbb{W}^2_T\circ\mathbb{W}^{[3]}_T + 2\gamma_T W_T\big)(x)\\
&= \sum_{k_1,\dots,k_5}\int_{A^5_T} G^{2\diamond[3]}_{0,q}((s,k)_{1\dots5})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_5}(k_5)\\
&\quad+\sum_{k_1,\dots,k_3}\int_{A^3_T} G^{2\diamond[3]}_{1,q}((s,k)_{1\dots3})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_3}(k_3)\\
&\quad+\sum_{k_1}\int_{A^1_T} G^{2\diamond[3]}_{2,q}((s,k)_1)\mathrm{d}w_{s_1}(k_1),
\end{aligned}\tag{94}$$

where $A^n_T := \{0\leqslant s_1<\dots<s_n\leqslant T\}\subseteq[0,T]^n$ and where the deterministic kernels are given by

$$\begin{aligned}
G^{2\diamond[3]}_{0,q}((s,k)_{1\dots5}) &:= (24^2)\,\varrho_q(k_{(1\dots5)})e^{i(k_{(1\dots5)})\cdot x}\sum_{\sigma\in\mathrm{Sh}(2,3)}\sum_{i\sim j}\times\\
&\quad\times\varrho_i(k_{(\sigma_1\sigma_2)})\,\varrho_j(k_{(\sigma_3\sigma_4\sigma_5)})\left(\int_{s_{\sigma_5}}^T\frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2}\mathrm{d}u\right),\\
G^{2\diamond[3]}_{1,q}((s,k)_{1\dots3}) &:= (24^2)\,\varrho_q(k_{(1\dots3)})e^{i(k_{(1\dots3)})\cdot x}\sum_{\sigma\in\mathrm{Sh}(1,2)}\sum_{i\sim j}\sum_p\int_0^T\mathrm{d}r\frac{\sigma_r(p)^2}{\langle p\rangle^2}\times\\
&\quad\times\varrho_i(k_{\sigma_1}+p)\,\varrho_j(k_{(\sigma_2\sigma_3)}-p)\left(\int_{s_{\sigma_3}\vee r}^T\frac{\sigma_u(k_{(\sigma_2\sigma_3)}-p)^2}{\langle k_{(\sigma_2\sigma_3)}-p\rangle^2}\mathrm{d}u\right),\\
G^{2\diamond[3]}_{2,q}((s,k)_1) &:= (24^2)\,\varrho_q(k_1)e^{ik_1\cdot x}\sum_{i\sim j}\sum_{p_1,p_2}\int_0^T\mathrm{d}r_1\int_0^T\mathrm{d}r_2\frac{\sigma_{r_1}(p_1)^2}{\langle p_1\rangle^2}\frac{\sigma_{r_2}(p_2)^2}{\langle p_2\rangle^2}\times\\
&\quad\times\varrho_i(p_1+p_2)\,\varrho_j(k_1-p_1-p_2)\left(\int_{r_1\vee r_2\vee s_1}^T\frac{\sigma_u(k_1-p_1-p_2)^2}{\langle k_1-p_1-p_2\rangle^2}\mathrm{d}u\right),\\
G^{2\diamond[3]}_{2,q}((s,k)_1) &:= G^{2\circ[3]}_{2,q}((s,k)_1) + 2\gamma_T\varrho_q(k_1)\,e^{ik_1\cdot x},
\end{aligned}$$

where $\mathrm{Sh}(k,l)$ is the set of permutations σ of $\{1,...,k+l\}$ keeping the orders $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(k+l)$ and where, for any symbol z, we denote with expression of the form $z_{1\cdots n}$ the vector $(z_1,\dots,z_n)$. Estimation of $\Delta_q(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]})(x)$ reduces then to estimate each of the three iterated integrals using BDG inequalities to get, for any $p\geqslant 2$,

$$I_{0,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1,\dots,k_5}\int_{A_T^5}G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_5}(k_5)\right|^{2p}\right]\right\}^{1/p}$$

$$\lesssim\sum_{k_1,\dots,k_5}\int_{A_T^5}\left|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

The kernel $G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5})$ being a symmetric function of its argument, we can simplify this expression into an integral over $[0,T]^5$:

$$I_{0,q}\lesssim\sum_{k_1,\dots,k_5}\int_{[0,T]^5}\left|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

Under the measure $\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_5$, we have

$$\left|\int_{s_{\sigma_5}}^T\frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2}\mathrm{d}u\right|\lesssim\frac{\mathbb{1}_{\langle k_{\sigma_5}\rangle\leqslant2\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle}}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2},$$

since $\sigma_u(k)\neq0$ only for $\langle k\rangle<u<2\langle k\rangle$, $\int_0^\infty\sigma_u(k)^2\mathrm{d}u=1$, and $\langle k_{\sigma_5}\rangle<s_{\sigma_5}$ on the support of the measure. Therefore with some standard estimates (using the symmetry in σ) we can reduce this to

$$I_{0,q}\lesssim\sum_{k_1,\dots,k_5}\int_{[0,T]^5}\frac{\varrho_q(k_{(1\cdots5)})^2}{\langle k_{(345)}\rangle^4}\mathbb{1}_{\langle k_5\rangle\leqslant2\langle k_{(345)}\rangle}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5$$

$$\lesssim\sum_{k_1,\dots,k_5}\frac{\varrho_q(k_{(1\cdots5)})^2}{\langle k_{(345)}\rangle^4}\mathbb{1}_{\langle k_5\rangle\leqslant2\langle k_{(345)}\rangle}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim\sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\frac{\varrho_q(p_1+p_2)^2}{\langle p_2\rangle^4}\sum_{k_1,\dots,k_5}\mathbb{1}_{k_{(12)}=p_1,k_{(345)}=p_2}\mathbb{1}_{\langle k_5\rangle\leqslant2\langle p_2\rangle}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim\sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\varrho_q(p_1+p_2)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle p_2\rangle^4}\lesssim\sum_{p_1,r}\mathbb{1}_{p_1\sim r-p_1}\varrho_q(r)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle r-p_1\rangle^4}\lesssim\sum_r\varrho_q(r)^2\frac{1}{\langle r\rangle^2}\lesssim2^q.$$

Here we used $\sum_{k_1+k_2=m}\langle k_1\rangle^{-2}\langle k_2\rangle^{-2}\lesssim\langle m\rangle^{-1}$ in $d=3$, so that the sum over k_3,k_4,k_5 is bounded by $\sum_{\langle k_5\rangle\leqslant2\langle p_2\rangle}\langle k_5\rangle^{-2}\langle p_2-k_5\rangle^{-1}\lesssim1$, and that $p_1\sim r-p_1$ forces $\langle p_1\rangle\sim\langle r-p_1\rangle\gtrsim\langle r\rangle$. Now by similar reasoning we also have

$$\left|G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots3})\right|\lesssim\sum_{\sigma\in\mathrm{Sh}(1,2)}|\varrho_q(k_{(1\cdots3)})|\sum_{i\sim j}\sum_p\int_0^T\mathrm{d}r\frac{\sigma_r(p)^2|\varrho_i(k_{\sigma_1}+p)\,\varrho_j(k_{(\sigma_2\sigma_3)}-p)|}{\langle p\rangle^2\langle k_{\sigma_1}+p\rangle^2}$$

$$\lesssim\sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\cdots3)})|}{\langle k_{\sigma_1}\rangle+2^q}$$

since the constraint $i\sim j$ and the support of ϱ_q force $\langle k_{\sigma_1}+p\rangle \gtrsim 2^q$ and, in $d=3$, $\sum_{p:\langle k+p\rangle\gtrsim 2^q}\langle p\rangle^{-2}\langle k+p\rangle^{-2}\lesssim(\langle k\rangle+2^q)^{-1}$, so

$$I_{1,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1,\dots,k_3}\int_{A_T^3}G_{1,q}^{2\diamond[3]}((s,k)_{1\dots 3})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_3}(k_3)\right|^{2p}\right]\right\}^{1/p}$$

$$\lesssim\sum_{k_1,\dots,k_3}\int_{[0,T]^3}\left|\sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\dots 3)})|}{\langle k_{\sigma_1}\rangle+2^q}\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_3$$

$$\lesssim\sum_{k_1,\dots,k_3}\frac{|\varrho_q(k_{(1\dots 3)})|^2}{(\langle k_1\rangle+2^q)^2\langle k_1\rangle^2\langle k_2\rangle^2\langle k_3\rangle^2}\lesssim\sum_{k_1,r}\frac{\varrho_q(r)^2}{(\langle k_1\rangle+2^q)^2\langle k_1\rangle^2\langle r-k_1\rangle}\lesssim 2^q,$$

where, by symmetry, we took $\sigma_1=1$ and used $\sum_r\varrho_q(r)^2\langle r-k_1\rangle^{-1}\lesssim 2^{2q}$ uniformly in k_1 and $\sum_{k_1}(\langle k_1\rangle+2^q)^{-2}\langle k_1\rangle^{-2}\lesssim 2^{-q}$.

Finally, we note that the same strategy cannot be applied to the first chaos, since the kernel $G_{2,q}^{2\circ[3]}$ cannot be uniformly bounded. We let

$$\begin{aligned}A_T(s_1,k_1) \;:=\; & (24^2)\sum_{i\sim j}\sum_{q_1,q_2}\int_0^T\mathrm{d}r_1\int_0^T\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times\\ &\times\varrho_i(q_1+q_2)\varrho_j(k_1-q_1-q_2)\bigg(\int_{r_1\vee r_2\vee s_1}^T\frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u\bigg),\end{aligned}$$

so

$$G_{2,q}^{2\diamond[3]}((s,k)_1)=\varrho_q(k_1)e^{ik_1\cdot x}[A_T(s_1,k_1)+2\gamma_T].$$

Observe that

$$\begin{aligned}A_T(0,0) \;=\; & (24^2)\sum_{q_1,q_2}\int_0^T\mathrm{d}r_1\int_0^T\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times\\ &\times\int_{r_1\vee r_2}^T\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u\sum_{i\sim j}\varrho_i(q_1+q_2)\varrho_j(-q_1-q_2).\end{aligned}$$

We choose γ_T as (note that $\gamma_T\leqslant 0$: the first chaos of $\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]}$ has a positive divergent coefficient, so that with the sign convention of (49) the counterterm $-\lambda^2\gamma_T[\![Y_T^2]\!]$ is a positive mass term)

$$\gamma_T=-\frac{1}{2}A_T(0,0)=-(12^2\cdot 2)\sum_{q_1,q_2}\int_0^T\mathrm{d}u\int_0^u\mathrm{d}r_1\int_0^u\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\tag{95}$$

where we used the fact that for all $q\in\mathbb{R}^d$ we have $\sum_{i\sim j}\varrho_i(q)\varrho_j(q)=1$, since $\int f\circ g=\int fg$. Note that, as claimed,

$$|\gamma_T|\lesssim\sum_{q_1,q_2}\frac{\mathbb{1}_{|q_1|,|q_2|,|q_1+q_2|\lesssim T}}{\langle q_1\rangle^2\langle q_2\rangle^2\langle q_1+q_2\rangle^2}\lesssim 1+\log\langle T\rangle.$$

Now

$$A_T(s_1,k_1)+2\gamma_T=(24^2)\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\sum_{i\sim j}\varrho_i(q_1+q_2)\times$$

$$\times\bigg(\varrho_j(k_1-q_1-q_2)\int_{s_1\vee r_1\vee r_2}^T\frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u-\varrho_j(q_1+q_2)\int_{r_1\vee r_2}^T\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u\bigg)$$

so there exists a constant C such that, when $|q_1+q_2|\geqslant C|k_1|$, the quantity in round brackets can be estimated by $|k_1|\langle q_1+q_2\rangle^{-3}$ while when $|q_1+q_2|\leqslant C|k_1|$ it is estimated by $\langle q_1+q_2\rangle^{-2}$. We have

$$|A_T(s_1,k_1)+2\gamma_T|\lesssim\sum_{q_1,q_2}\frac{1}{\langle q_1\rangle^2}\frac{1}{\langle q_2\rangle^2}\frac{1}{\langle q_1+q_2\rangle^2}\bigg(\mathbb{1}_{|q_1+q_2|\leqslant C|k_1|}+\mathbb{1}_{|q_1+q_2|\geqslant C|k_1|}\frac{|k_1|}{\langle q_1+q_2\rangle}\bigg)$$

$$\lesssim 1+\log\langle k_1\rangle.$$

And then with this choice of γ_T the kernel $G_{2,q}^{2\diamond[3]}$ stays uniformly bounded as $T\to\infty$ and satisfies

$$\big|G_{2,q}^{2\diamond[3]}((s,k)_1)\big|\lesssim\varrho_q(k_1)\log\langle k_1\rangle.$$

From this we easily deduce that

$$I_{2,q}=\bigg\{\mathbb{E}\bigg[\bigg|\sum_{k_1}\int_{A_T}G_{2,q}^{2\diamond[3]}((s,k)_1)\mathrm{d}w_{s_1}(k_1)\bigg|^{2p}\bigg]\bigg\}^{1/p}\lesssim q^2\,2^q,\qquad q\geqslant -1.$$

All together these estimates imply that

$$\mathbb{E}\big\|\Delta_q\mathbb{W}_T^{2\diamond[3]}\big\|_{L^{2p}}^{2p}\lesssim(q2^{q/2})^{2p},\qquad q\geqslant -1.$$

Standard argument allows to deduce eq. (91). The analysis of the other renormalised product proceeds similarly. Let

$$V(t):=\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}=J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2+2\dot\gamma_t,\qquad t\geqslant 0.$$

First note that by definition of Besov spaces we have

$$\mathbb{E}\bigg[\bigg(\int_0^\infty\|V(t)\|_{B_{r,r}^{-\varepsilon-d/r}}\mathrm{d}t\bigg)^p\bigg]\lesssim\mathbb{E}\bigg[\bigg(\int_0^\infty\bigg(\sum_q 2^{-qr(\varepsilon+d/r)}\|\Delta_qV(t)\|_{L^r}^r\bigg)^{1/r}\mathrm{d}t\bigg)^p\bigg].$$

By Minkowski's integral inequality this is bounded by

$$\lesssim\bigg(\int_0^\infty\mathrm{d}t\bigg\{\mathbb{E}\bigg[\bigg(\sum_q 2^{-qr(\varepsilon+d/r)}\|\Delta_qV(t)\|_{L^r}^r\bigg)^{p/r}\bigg]\bigg\}^{1/p}\bigg)^p.$$

When $r \geqslant p$ Jensen's inequality and Fubini's theorem give

$$\lesssim \left(\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon+d/r)} \int_\Lambda \frac{\mathrm{d}x}{|\Lambda|} \mathbb{E}[|\Delta_q V(t)(x)|^r] \right\}^{1/r} \right)^p .$$

Finally hypercontractivity and stationarity allow to reduce this to bound

$$\lesssim \left(\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon+d/r)} \left(\mathbb{E}[|\Delta_q V(t)(0)|^2]\right)^{r/2} \right\}^{1/r} \right)^p .$$

Letting $I_q(t) = \mathbb{E}[|\Delta_q V(t)(0)|^2]$ we have

$$\mathbb{E}\left[\left(\int_0^\infty \|\mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle}\|_{B^{-\varepsilon-d/r}_{r,r}} \mathrm{d}t \right)^p \right] \lesssim \left(\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon+d/r)} \left(I_q(t)\right)^{r/2} \right\}^{1/r} \right)^p .$$

Now we decompose the random field $\Delta_q(\mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle})(x)$ into homogeneous stochastic integrals as above and obtain

$$\begin{aligned} \Delta_q(\mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle})(x) \;=\; & \sum_{k_1,\dots,k_4} \int_{A_t^4} G_{0,q}^{\langle 2\rangle \diamond \langle 2\rangle}((s,k)_{1\dots 4}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_4}(k_4) \\ & + \sum_{k_1,k_2} \int_{A_t^2} G_{1,q}^{\langle 2\rangle \diamond \langle 2\rangle}((s,k)_{12}) \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \\ & + G_{2,q}^{\langle 2\rangle \diamond \langle 2\rangle} \end{aligned} \tag{96}$$

with

$$\begin{aligned} G_{0,q}^{\langle 2\rangle \diamond \langle 2\rangle}((s,k)_{1\dots 4}) \;=\; & (24^2)\, \varrho_q(k_{(1\dots 4)}) e^{i(k_{(1\dots 4)})\cdot x} \times \\ & \times \sum_{\sigma \in \mathrm{Sh}(2,2)} \sum_{i\sim j} \varrho_i(k_{(\sigma_1\sigma_2)})\, \varrho_j(k_{(\sigma_3\sigma_4)}) \frac{\sigma_t(k_{(\sigma_1\sigma_2)})}{\langle k_{(\sigma_1\sigma_2)}\rangle} \frac{\sigma_t(k_{(\sigma_3\sigma_4)})}{\langle k_{(\sigma_3\sigma_4)}\rangle} \\ G_{1,q}^{\langle 2\rangle \diamond \langle 2\rangle}((s,k)_{12}) \;=\; & (24^2)\, \varrho_q(k_{(12)}) e^{i(k_{(12)})\cdot x} \sum_{\sigma \in \mathrm{Sh}(1,1)} \sum_{i\sim j} \sum_p \times \\ & \times \int_0^t \mathrm{d}r \frac{\sigma_r^2(p)}{\langle p\rangle^2} \varrho_i(k_{\sigma_1}+p)\, \varrho_j(k_{\sigma_2}-p) \left(\frac{\sigma_t(k_{\sigma_1}+p)}{\langle k_{\sigma_1}+p\rangle} \frac{\sigma_t(k_{\sigma_2}-p)}{\langle k_{\sigma_2}-p\rangle} \right) \\ G_{2,q}^{\langle 2\rangle \diamond \langle 2\rangle} \;=\; & (24^2) \mathbb{1}_{q=-1} \sum_{i\sim j} \sum_{q_1,q_2} \int_0^t \mathrm{d}r_1 \int_0^t \mathrm{d}r_2 \times \\ & \times \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2} \frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2} \varrho_i(q_1+q_2)\, \varrho_j(-q_1-q_2) \frac{\sigma_t(q_1+q_2)^2}{\langle q_1+q_2\rangle^2} \\ & + 2\dot\gamma_t \mathbb{1}_{q=-1}. \end{aligned}$$

Using our choice of γ_T in eq. (95) we have that

$$\dot\gamma_t = -(12^2\cdot 2) \sum_{q_1,q_2} \int_0^t \mathrm{d}r_1 \int_0^t \mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2} \frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2} \frac{\sigma_t^2(q_1+q_2)}{\langle q_1+q_2\rangle^2},$$

which implies also that

$$G_{2,q}^{\langle 2\rangle\diamond\langle 2\rangle}=0,\quad \text{and}\qquad |\dot\gamma_t|\lesssim\frac{1+\log\langle t\rangle}{\langle t\rangle}.$$

as claimed. We pass now to estimate the other two chaoses. The technique is the same we used above. Consider first

$$I_{0,q}(t):=\mathbb{E}\bigg[\bigg|\sum_{k_1,\dots,k_4}\int_{A_t^4}G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\dots 4})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_4}(k_4)\bigg|^2\bigg]$$

$$\lesssim\sum_{k_1,\dots,k_4}\int_{A_t^4}\big|G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\dots 4})\big|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4$$

$$\lesssim\sum_{k_1,\dots,k_4}\int_{[0,t]^4}\big|G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\dots 4})\big|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4$$

$$\lesssim\sum_{k_1,\dots,k_4}\varrho_q(k_{(1\cdots 4)})^2\int_{[0,t]^4}\frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)}\rangle^2}\frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)}\rangle^2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4$$

$$\lesssim\sum_{k_1,\dots,k_4}\varrho_q(k_{(1\cdots 4)})^2\frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)}\rangle^2}\frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)}\rangle^2}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_4\rangle^2}$$

$$\lesssim\frac{\mathbb{1}_{2^q\lesssim t}}{\langle t\rangle^6}\sum_{|k_{(12)}|,|k_{(34)}|\sim t}\frac{\varrho_q(k_{(1\cdots 4)})^2}{\langle k_1\rangle^2\langle k_2\rangle^2\langle k_3\rangle^2\langle k_4\rangle^2}$$

$$\lesssim\frac{\mathbb{1}_{2^q\lesssim t}}{\langle t\rangle^6}\sum_{l,m:|l|,|m|\sim t}\frac{\varrho_q(l+m)^2}{\langle l\rangle\langle m\rangle}\lesssim\frac{\mathbb{1}_{2^q\lesssim t}}{\langle t\rangle^5}2^{3q}$$

where we used that $|\sigma_t(x)|\lesssim t^{-1/2}\mathbb{1}_{x\sim t}$ (which also forces $2^q\lesssim t$), the bound $\sum_{k_1+k_2=l}\langle k_1\rangle^{-2}\langle k_2\rangle^{-2}\lesssim\langle l\rangle^{-1}$ in $d=3$, and that there are $O(2^{3q})$ values of $l+m$ in the support of ϱ_q and $O(t^3)$ values of l with $|l|\sim t$. (The constraint $|k_{(12)}|,|k_{(34)}|\sim t$ cannot be dropped, since the four-fold convolution of $\langle k\rangle^{-2}$ diverges in $d=3$.) Now taking $0<\varepsilon+d/r<3/2$ we have

$$\int_0^\infty\mathrm{d}t\bigg\{\sum_q 2^{-qr(\varepsilon+d/r)}\,(I_{0,q}(t))^{r/2}\bigg\}^{1/r}\lesssim\int_0^\infty\mathrm{d}t\bigg\{\sum_{q:2^q\lesssim t}\frac{2^{qr(3/2-\varepsilon-d/r)}}{\langle t\rangle^{5r/2}}\bigg\}^{1/r}$$

$$\lesssim\int_0^\infty\frac{\mathrm{d}t}{\langle t\rangle^{1+\varepsilon+d/r}}\lesssim 1.$$

Taking into account that $|k_1|,|k_2|\lesssim t$ we can estimate

$$\big|G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\big|\lesssim|\varrho_q(k_{(12)})|\sum_p\frac{\mathbb{1}_{|p|\lesssim t}}{\langle p\rangle^2}\bigg(\frac{\sigma_t(k_1+p)}{\langle k_1+p\rangle}\frac{\sigma_t(k_2-p)}{\langle k_2-p\rangle}\bigg)\lesssim|\varrho_q(k_{(12)})|\langle t\rangle^{-2},$$

from which we deduce that

$$I_{1,q}(t) := \mathbb{E}\Bigg[\Bigg|\sum_{k_1,k_2}\int_{A_t^2} G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\Bigg|^2\Bigg]$$

$$\lesssim \sum_{k_1,k_2}\int_{A_t^2}\big|G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\big|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2$$

$$\lesssim \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\int_{[0,t]^2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2$$

$$\lesssim \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\frac{\mathbb{1}_{k_1\lesssim t}}{\langle k_1\rangle^2}\frac{\mathbb{1}_{k_2\lesssim t}}{\langle k_2\rangle^2}\lesssim \langle t\rangle^{-4}2^{2q}\mathbb{1}_{2^q\lesssim t},$$

and then, as for $I_{0,q}$, we have

$$\int_0^\infty \mathrm{d}t\Bigg(\sum_q 2^{-qr(\varepsilon+d/r)}\left(I_{1,q}(t)\right)^{r/2}\Bigg)^{1/r}\lesssim\int_0^\infty\frac{\mathrm{d}t}{\langle t\rangle^2}\Bigg(\sum_q 2^{qr(1-\varepsilon-d/r)}\mathbb{1}_{2^q\lesssim t}\Bigg)^{1/r}\lesssim 1,$$

as claimed. From these estimates standard arguments give eq. (92). □

Lemma 104. *We have*

$$\mathbb{E}[\|\mathbb{W}_T^3\|_{L^p}^p]^{1/p}\lesssim T^{3/2}.$$

This implies that $\mathbb{W}^{\langle 3\rangle}\in C\big([0,\infty],B_{p,p}^{-1/2-\kappa}\big)\cap L^2\big(\mathbb{R}_+,B_{p,p}^{-1/2-\kappa}\big)$ *for any* $p<\infty$ *uniformly in the volume and* $\mathbb{W}^{\langle 3\rangle}\in C([0,\infty],\mathscr{C}^{-1/2-\kappa})\cap L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa})$.

Proof. Observe that

$$\mathbb{W}_T^3(x)=4[\![W_T^3]\!](x)=24\sum_{k_1,k_2,k_3}e^{i(k_1+k_2+k_3)\cdot x}\int_0^T\int_0^{s_3}\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3).$$

By space homogeneity, we get for any p,

$$\begin{aligned}
\mathbb{E}[\|\mathbb{W}_T^3\|_{L^p}^p] &= \mathbb{E}[|\mathbb{W}_T^3(0)|^p]\\
&= 24^p\mathbb{E}\Bigg[\Bigg|\sum_{k_1,k_2,k_3}\int_0^T\int_0^{s_3}\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\Bigg|^p\Bigg]\\
&\lesssim \Bigg(\mathbb{E}\Bigg[\Bigg|\sum_{k_1,k_2,k_3}\int_0^T\int_0^{s_3}\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\Bigg|^2\Bigg]\Bigg)^{p/2}\\
&= \Bigg(\sum_{k_1,k_2,k_3}\int_0^T\int_0^{s_3}\int_0^{s_2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_3\Bigg)^{p/2}\\
&\lesssim (T^{3/2})^p.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_3} \int_0^{s_2} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
\leqslant &\sum_{k_1,k_2,k_3} \int_0^T \int_0^T \int_0^T \frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
= &\left(\sum_k \int_0^T \frac{\sigma_s(k)^2}{\langle k\rangle^2} \mathrm{d}s \right)^3 \lesssim T^3.
\end{aligned}$$

Now the remaining properties follow by the fact that σ_t is supported in an annulus of radius t, so

$$\left\|\mathbb{W}_t^{\langle 3\rangle}\right\|_{B_{p,p}^{-1/2-\kappa}} = \left\| \frac{\sigma_t(\mathrm{D})}{\langle \mathrm{D}\rangle} \mathbb{W}_t^3 \right\|_{B_{p,p}^{-1/2-\kappa}} \lesssim \|\sigma_t(\mathrm{D})\mathbb{W}_t^3\|_{B_{p,p}^{-3/2-\kappa}} \lesssim \langle t\rangle^{-1/2-\kappa} \langle t\rangle^{-3/2} \|\mathbb{W}_t^3\|_{L^p}$$

and the Hölder estimates follow by Besov embedding (but with constants which depend on the volume). □

Bibliography

[1] Sergio Albeverio, Francesco C. De Vecchi, and Massimiliano Gubinelli. Elliptic stochastic quantization. *Annals of Probability*, 48(4):1693–1741, jul 2020.

[2] Sergio Albeverio, Francesco C. De Vecchi, and Massimiliano Gubinelli. The elliptic stochastic quantization of some two dimensional Euclidean QFTs. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 57(4), nov 2021.

[3] Hajer Bahouri, Jean-Yves Chemin, and Raphael Danchin. *Fourier analysis and nonlinear partial differential equations*. Springer, 2011.

[4] Ismael Bailleul, Ilya Chevyrev, and Massimiliano Gubinelli. Wilson-Itô diffusions. jul 2023.

[5] N. Barashkov and M. Gubinelli. A variational method for ϕ^4_3. *Duke Mathematical Journal*, 169(17):3339–3415, nov 2020.

[6] Nikolay Barashkov. *A Variational Approach to Gibbs Measures on Function Spaces*. PhD thesis, Bonn, 2021.

[7] Nikolay Barashkov and Massimiliano Gubinelli. The ϕ^4_3 measure via Girsanov's theorem. *Electronic Journal of Probability*, 26(none), jan 2021.

[8] Roland Bauerschmidt and Michael Hofstetter. Maximum and coupling of the sine-Gordon field. *Annals of Probability*, page 49, 2022.

[9] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:12, 2017. Id/No 27.

[10] Nathanaël Berestycki and Ellen Powell. Gaussian free field and Liouville quantum gravity. feb 2025.

[11] C. Borell. Diffusion equations and geometric inequalities. *Potential Analysis*, 12(1):49–71, 2000.

[12] M. Boué and P. Dupuis. A variational representation for certain functionals of Brownian motion. *The Annals of Probability*, 26(4):1641–1659, oct 1998. 10.1214/aop/1022855876.

[13] A. Braides. *Gamma-convergence for beginners*. Oxford Lecture Series in Mathematics and Its Applications, 22. Clarendon Press, 1 edition, 2002.

[14] Bjoern Bringmann. Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity I: Measures. *ArXiv:2009.04609*, 2020.

[15] Bjoern Bringmann. Invariant Gibbs measures for the three-dimensional wave equation with a hartree nonlinearity II: Dynamics. *ArXiv:2009.04616*, 2020.

[16] Bjoern Bringmann, Yu Deng, Andrea R. Nahmod, and Haitian Yue. Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation. jun 2022.

[17] Ajay Chandra, Trishen S. Gunaratnam, and Hendrik Weber. Phase transitions for ϕ^4_3. *ArXiv:2006.15933*, 2020. ArXiv: 2006.15933.

[18] A. Dal Maso. *An introduction to Γ-Convergence*. Birkhäuser Boston, 1993. 10.1007/978-1-4612-0327-8.

[19] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, Second edition, 2010.

[20] Jin Feng and Thomas G. Kurtz. *Large Deviations for Stochastic processes*. American Mathematical Society, 2015.

[21] Wendell H. Fleming and Halil Mete Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer, 2nd edition, nov 2005.

[22] H. Föllmer. An entropy approach to the time reversal of diffusion processes. In Prof M. Metivier and Prof E. Pardoux, editors, *Stochastic Differential Systems Filtering and Control*, number 69 in Lecture Notes in Control and Information Sciences, pages 156–163. Springer Berlin Heidelberg, 1985.

[23] Hans Föllmer. Random fields and diffusion processes. In Paul-Louis Hennequin, editor, *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, number 1362 in Lecture Notes in Mathematics, pages 101–203. Springer Berlin Heidelberg, jan 1988.

[24] James Glimm and Arthur Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer-Verlag, New York, 2 edition, 1987.

[25] M. Gubinelli, B. Ugurcan, and I. Zachhuber. Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2019. 10.1007/s40072-019-00143-9.

[26] Massimiliano Gubinelli and Martina Hofmanová. Global Solutions to Elliptic and Parabolic ϕ^4 Models in Euclidean Space. *Communications in Mathematical Physics*, 368(3):1201–1266, 2019. 10.1007/s00220-019-03398-4.

[27] Massimiliano Gubinelli and Martina Hofmanová. A PDE Construction of the Euclidean ϕ^4_3 Quantum Field Theory. *Communications in Mathematical Physics*, apr 2021.

[28] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics. Pi*, 3:0, 2015. 10.1017/fmp.2015.2.

[29] Massimiliano Gubinelli, Herbert Koch, and Tadahiro Oh. Renormalization of the two-dimensional stochastic nonlinear wave equations. *Transactions of the American Mathematical Society*, page 1, 2018.

[30] Massimiliano Gubinelli, Herbert Koch, and Tadahiro Oh. Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity. *Journal of the European Mathematical Society*, 26(3):817–874, jan 2023.

[31] Massimiliano Gubinelli, Herbert Koch, Tadahiro Oh, and Leonardo Tolomeo. Global Dynamics for the Two-dimensional Stochastic Nonlinear Wave Equations. *International Mathematics Research Notices*, page rnab084, aug 2021. https://academic.oup.com/imrn/advance-article/doi/10.1093/imrn/rnab084/6343523.

[32] Massimiliano Gubinelli and Sarah-Jean Meyer. The FBSDE approach to sine-Gordon up to 6π. *ArXiv:2401.13648*, jan 2024.

[33] M Hairer. Introduction to Malliavin Calculus. 2021. Available at https://www.hairer.org/notes/Malliavin.pdf.

[34] Yuu Hariya and Sou Watanabe. The Boué–Dupuis formula and the exponential hypercontractivity in the Gaussian space. *ArXiv:2110.14852 [math]*, nov 2021. ArXiv: 2110.14852.

[35] Raphael Høegh-Krohn. A general class of quantum fields without cut-offs in two space-time dimensions. *Communications in Mathematical Physics*, 21(3):244–255, sep 1971.

[36] Masato Hoshino, Hiroshi Kawabi, and Seiichiro Kusuoka. Stochastic quantization associated with the $\exp(\phi)_2$-quantum field model driven by space-time white noise on the torus in the full l^1-regime. 2020.

[37] A. Jakubowski. The Almost Sure Skorokhod Representation for Subsequences in Nonmetric Spaces. *Theory Probab. Appl.*, (42(1)):167–174, 1998.

[38] S. Janson. *Gaussian Hilbert Spaces*. Cambridge University Press, jun 1997.

[39] Jean-Pierre Kahane. Sur le chaos multiplicatif. (On multiplicative chaos). *Ann. Sci. Math. Qué.*, 9:105–150, 1985.

[40] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Semiclassical limit of liouville field theory. *Journal of Functional Analysis*, 273:0, 05 2017.

[41] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. The semiclassical limit of liouville conformal field theory. 2019.

[42] J. Lehec. Representation formula for the entropy and functional inequalities. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 49(3):885–899, 2013. MR3112438.

[43] Dan Mikulincer and Yair Shenfeld. The Brownian transport map. *ArXiv:2111.11521 [math]*, nov 2021.

[44] Petru Mironescu and Haïm Brezis. Gagliardo–Nirenberg inequalities and non-inequalities: The full story. *Annales de l'Institut Henri Poincaré C*, 35(5):1355–1376, aug 2018.

[45] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, 2017. 10.1214/16-AOP1116.

[46] E. Nelson. A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles*, pages 69–73. M.I.T. Press, Cambridge, Mass., 1966.

[47] Edward Nelson. Construction of quantum fields from Markoff fields. *Journal of Functional Analysis*, 12(1):97–112, jan 1973. 10.1016/0022-1236(73)90091-8.

[48] Edward Nelson. The free Markoff field. *J. Functional Analysis*, 12:211–227, 1973.

[49] Tadahiro Oh, Mamoru Okamoto, and Leonardo Tolomeo. Focusing ϕ^4_3-model with a Hartree-type nonlinearity. *ArXiv:2009.03251 [math-ph]*, aug 2021. ArXiv: 2009.03251.

[50] G. Parisi and N. Sourlas. Random Magnetic Fields, Supersymmetry, and Negative Dimensions. *Physical Review Letters*, 43(11):744–745, sep 1979.

[51] Giorgio Parisi and Yong Shi Wu. Perturbation theory without gauge fixing. *Scientia Sinica. Zhongguo Kexue*, 24(4):483–496, 1981. MR626795.

[52] Doug Pickrell. P(phi)_2 quantum field theories and Segal's axioms. *Communications in Mathematical Physics*, 280(2):403–425, jun 2008.

[53] Hans Triebel. *Theory of Function Spaces II*. Springer, aug 2010.

[54] A. S. Üstünel. Variational calculation of Laplace transforms via entropy on Wiener space and applications. *Journal of Functional Analysis*, 267(8):3058–3083, 2014. 10.1016/j.jfa.2014.07.006.