Infinite dimensional rough dynamics

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Slusius ante annos duos tresve tibi communicavit, de quâ tu, suggerente Collinsio, rescripsisti eandem mihi ~~tibi inno~~ etiam innotuisse. Diversa ratione in eam incidimus. Nam res non eget demonstratione prout ego operor. Habito meo fundamento nemo potuit tangentes aliter ducere, nisi volens de rectâ viâ deviaret. Quinetiam non hic inhaeretur ad aequationes radicalibus unam vel utramque indefinitam quantitatem involventibus utcunque affectas, sed absque aliquâ talium aequationum reductione (quae opus plerumque redderet immensum) tangens confestim ducitur. Et eodem modo se res habet in quaestionibus de Maximis et minimis, alijsque quibusdam de quibus jam non loquor. ~~Fundamentum harum operationum~~ Fundamentum harum operationum satis obvium quidem, quoniam jam non possum explicationem ejus prosequi sic potius celavi. 6accdæ13eff7i3l9n4o4qrr4s8t12vx. Hoc fundamento conatus sum etiam reddere speculationes de Quadratura curvarum simpliciores, perveniq ad Theoremata quaedam generalia. Et ut candide agam ecce primum Theorema.

Ad Curvam aliquam sit $dz^{\theta} \times \overline{e+fz^{\eta}}|^{\lambda}$ ordinatim applicata termino diametri seu basis z normaliter insistens: ubi literae d, e, f denotant quaslibet quantitates datas, & θ, η, λ indices potestatum sive dignitatum quantitatum quibus affixae sunt. Fac $\frac{\theta+1}{\eta}=r$. $\lambda+r=s$. $\frac{d}{\eta f}\times\overline{e+fz^{\eta}}|^{\lambda+1}=Q$. & $r\eta-\eta=\pi$, & area Curvae erit Q in $\frac{z^{\pi}}{s}-\frac{r-1}{s-1}\times\frac{eA}{fz^{\eta}}+\frac{r-2}{s-2}\times\frac{eB}{fz^{\eta}}-\frac{r-3}{s-3}\times\frac{eC}{fz^{\eta}}+\frac{r-4}{s-4}\times\frac{eD}{fz^{\eta}}$ &c

literis A, B, C, D &c ~~deno~~ denotantibus terminos proximè antecedentes, nempe A terminum $\frac{z^{\pi}}{s}$, B terminum $-\frac{r-1}{s-1}\times\frac{eA}{fz^{\eta}}$ &c. Haec Series ubi r fractio est vel numerus negativus, continuatur in infinitum: ubi verò r integer est et affirmativus continuatur ad tot terminos tantùm quot sunt unitates in eodem r, et sic exhibet geometricam quadraturam Curvae. Rem exemplis illustro.

"Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa"

(I. Newton, letter to Henry Oldenburg, 24 October 1676)

Solving the controlled ODE in $\mathbb{R}^d$

$$\dot{y}(t) = V_\alpha(y(t))\dot{x}^\alpha(t), \qquad t \geqslant 0,$$

with $(V_\alpha)_\alpha$ family of vector fields and $y(0)$ given, is equivalent to asking for a function $y: \mathbb{R}_{\geqslant 0} \to \mathbb{R}^d$ such that

$$y(t) - y(s) = V_\alpha(y(s))(x^\alpha(t) - x^\alpha(s)) + o(|t-s|), \qquad 0 \leqslant s \leqslant t.$$

General references on RP/RS:

Lyons '98, Davie, Lyons-Qian, Friz–Victoir, Friz–Hairer, Hairer.

Talk based on joint work with I. Bailleul, A. Deya, M. Hofmanova, S. Tindel.

Goal: Replace differential/integral description with *non-infinitesimal* local one.

$$\delta y(s,t) := y(t) - y(s) = A(s,t) + R(s,t)$$

- A is a "germ" for the dynamics of y:

$$A(s,t) = V(y(s))X^1(s,t) + V_2(y(s))X^2(s,t) + \cdots$$

- the equation holds modulo error term $R(s,t)$ of order $o(|t-s|)$
- **Key insight**. this decomposition is rigid: to each given A there can correspond only one pair (y,R):

$$|\delta y(s,t) - \delta\tilde{y}(s,t)| = |R(s,t) - \tilde{R}(s,t)| = o(|t-s|)$$

Explicit bounds on R in terms of the "coherence" of A

$$\delta A(s,u,t) := A(s,t) - (A(s,u) + A(u,t)), \qquad s \leqslant u \leqslant t$$

Lemma *Assume that*

$$|\delta A(s,u,t)| \leqslant \|\delta A\|_z |t-s|^z$$

for some $z > 1$*, then there exists a unique* y *such that*

$$\delta y(s,t) = A(s,t) + R(s,t), \qquad |R(s,t)| = o(|t-s|)$$

and moreover

$$|R(s,t)| \leqslant C_z \|\delta A\|_z |t-s|^z.$$

This result holds for general regular controls $\omega(s,t)$ (replacing $|t-s|$):

$$\omega(s,u) + \omega(u,t) \leqslant \omega(s,t), \qquad |t-s| \to 0 \Rightarrow \omega(s,t) \to 0$$

Local expansion of ODEs:

$$y(t) = y(s) + V_\alpha(y(s)) \underbrace{\int_s^t \mathrm{d}x^\alpha(r)}_{X^{1,\alpha}(s,t)} + V_{2,\alpha\beta}(y(s)) \underbrace{\int_s^t \int_s^r \mathrm{d}x^\alpha(w) \mathrm{d}x^\beta(r)}_{X^{2,\alpha\beta}(s,t)} + \cdots$$

with $V_{2,\alpha\beta}(\xi) = V_\alpha(\xi) \cdot \nabla V_\beta(\xi)$.

Definition 1 *A (step-2) rough path* $\mathbb{X} = (X^1, X^2)$ *is a pair such that*

$$\delta X^1(s,u,t) = 0, \qquad \delta X^2(s,u,t) = X^1(s,u) X^1(u,t)$$

$$|X^1(s,t)| + |X^2(s,t)|^{1/2} \leqslant \|\mathbb{X}\|_\gamma |t-s|^\gamma$$

for some $\gamma \geqslant 1/3$.

[Lyons '98]

▷ Let $x \in C^\gamma$ and $(x^\varepsilon)_\varepsilon$ some family of smooth approximations $x^\varepsilon \to x$ in C^γ.

▷ Smooth approximations by ODEs

$$\dot{y}^\varepsilon = V(y^\varepsilon)\dot{x}^\varepsilon(t)$$

▷ Taylor expansion gives

$$\delta y^\varepsilon(s,t) = A^\varepsilon(s,t) + R^\varepsilon(s,t)$$

$$A^\varepsilon(s,t) = V(y^\varepsilon(s))X^{\varepsilon,1}(s,t) + V_2(y^\varepsilon(s))X^{\varepsilon,2}(s,t) \qquad |R^\varepsilon(s,t)| \leqslant \|\dot{x}^\varepsilon\|_\infty |t-s|^3$$

▷ **Problem:** estimates for the remainder are not uniform in $\varepsilon \to 0$.

▷ Uniform estimates for R^ε from the coherence of the germ A^ε itself

$$\delta A^\varepsilon(s,u,t) = -\delta V(y^\varepsilon)(s,u) X^{\varepsilon,1}(u,t) + V_2(y^\varepsilon(s))\delta X^{\varepsilon,2}(s,u,t) - \delta V_2(y^\varepsilon)(s,u) X^{\varepsilon,2}(u,t)$$

$$= -\underbrace{(\delta V(y^\varepsilon)(s,u) - V_2(y^\varepsilon(s)) X^{\varepsilon,1}(s,u))}_{O(|R^\varepsilon(s,t)|)+O(|t-s|^{2\gamma})} \underbrace{X^{\varepsilon,1}(u,t)}_{O(|t-s|^{\gamma})} - \underbrace{\delta V_2(y^\varepsilon)(s,u)}_{O(|R^\varepsilon(s,t)|)+O(|t-s|^{\gamma})} \underbrace{X^{\varepsilon,2}(u,t)}_{O(|t-s|^{2\gamma})}$$

$$\|R^\varepsilon\|_{2\gamma} := \sup_{s,t} \frac{|R^\varepsilon(s,t)|}{|t-s|^{2\gamma}}.$$

▷ If $3\gamma > 1$ the sewing lemma gives

$$|\delta A^\varepsilon(s,u,t)| \lesssim (\|R^\varepsilon\|_{2\gamma} + \|\mathbb{X}^\varepsilon\|_\gamma)\|\mathbb{X}^\varepsilon\|_\gamma |t-s|^{3\gamma}$$
$$\Downarrow$$
$$\|R^\varepsilon\|_{3\gamma} \lesssim (\|R^\varepsilon\|_{2\gamma} + \|\mathbb{X}\|_\gamma^\varepsilon)\|\mathbb{X}^\varepsilon\|_\gamma$$

$$\|R^\varepsilon\|_{3\gamma} \lesssim_{\|\mathbb{X}^\varepsilon\|_\gamma} 1 \qquad \text{uniformly in } \varepsilon > 0.$$

- The limit $y^\varepsilon \to y$ exists provided $\mathbb{X}^\varepsilon \to \mathbb{X} = (X^1, X^2)$ in *rough path topology*.
- It satisfies the RDE [Davie]

$$\delta y(s,t) = V(y(s))X^1(s,t) + V_2(y(s))X^2(s,t) + O(|t-s|^{3\gamma}).$$

- Is unique under sufficient regularity for V, V_2.
- The map $\mathbb{X} \mapsto y = \Phi(\mathbb{X})$ is continuous.
- Rough path limit $\mathbb{X}$ is **not unique** for given x. It holds

$$X^1(s,t) = \tilde{X}^1(s,t) = \delta x(s,t), \qquad \tilde{X}^2(s,t) - X^2(s,t) = \delta\varphi(s,t).$$

- The limit RDE is not an ODE (or not that one expects...).

Example Pure area RP: there exists $\mathbb{X}^\varepsilon \to (0, \delta\varphi)$ with $\varphi \in C^1$. Then

$$\dot{y}^\varepsilon(t) = V(y^\varepsilon(t))\dot{x}^\varepsilon(t) \qquad \Rightarrow \qquad \dot{y}(t) = V_2(y(t))\dot{\varphi}(t)$$

What about (S)PDEs?

- Works on (singular) parabolic SPDEs: [Hairer!, G.–Imkeller–Perkowski, Kupiainen]
- Works on flow transformations (viscosity solutions, conservation laws, transport equations) [Friz et al., Souganidis–Gess, Hofmanova]

Today: description of distributions satisfying (rough in time) PDEs.

Transport equation

$$\partial_t u(t,x) = \dot{X}^{\alpha}(t) V_{\alpha}(x) \nabla u(t,x), \qquad t \geqslant 0.$$

Weak formulation: $u \in L^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^1(\mathbb{R}_+, (W^{1,1}(\mathbb{R}^d))^*)$

$$u_t(\varphi) = \int_{\mathbb{R}^d} \varphi(x) u(t,x) \mathrm{d}x$$

$$\partial_t u_t(\varphi) = u_t(\dot{X}^{\alpha}(t)(V_{\alpha} \cdot \nabla)^* \varphi) = u_t(\dot{A}_t^* \varphi), \qquad t \geqslant 0.$$

Rough dynamics: $u \in L^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^{\gamma}(\mathbb{R}_+, (W^{1,1}(\mathbb{R}^d))^*)$

$$\delta u(\varphi)_{s,t} = u_s(\mathbb{A}_{s,t}^{1,*} \varphi) + u_s(\mathbb{A}_{s,t}^{2,*} \varphi) + o(|t-s|)$$

where (*rough driver*) $\mathbb{A}_{s,t}^1 = X^{1,\alpha}(s,t) V_{\alpha} \cdot \nabla$, $\mathbb{A}_{s,t}^2 = X^{2,\alpha\beta}(s,t)(V_{\alpha} \cdot \nabla)(V_{\beta} \cdot \nabla)$.

- Rough driver: $\delta \mathbb{A}^1_{s,u,t} = 0 \qquad \delta \mathbb{A}^2_{s,u,t} = \mathbb{A}^1_{u,t}\mathbb{A}^1_{s,u}$
- Germ: $A_{s,t}(\varphi) = u_s(\mathbb{A}^{1,*}_{s,t}\varphi) + u_s(\mathbb{A}^{2,*}_{s,t}\varphi)$

$$\delta A_{s,u,t}(\varphi) = -(\delta u_{s,u} - u_s\mathbb{A}^{1,*}_{s,u})(\mathbb{A}^{1,*}_{u,t}\varphi) - \delta u_{s,u}(\mathbb{A}^{2,*}_{u,t}\varphi)$$

(and using the equation $\delta u_{s,u} = u_s(\mathbb{A}^{1,*}_{s,u} + \mathbb{A}^{2,*}_{s,u}) + R_{s,u}$)

$$= -R_{s,u}(\mathbb{A}^{1,*}_{u,t}\varphi + \mathbb{A}^{2,*}_{u,t}\varphi) - u_s(\mathbb{A}^{2,*}_{s,u}\mathbb{A}^{1,*}_{u,t}\varphi + \mathbb{A}^{1,*}_{s,u}\mathbb{A}^{2,*}_{u,t}\varphi + \mathbb{A}^{2,*}_{s,u}\mathbb{A}^{2,*}_{u,t}\varphi)$$

$$|\delta A(s,u,t)|_{(W^{1,3})^*} \lesssim (\|R\|_{2\gamma,(W^{1,2})^*} + \|R\|_{\gamma,(W^{1,1})^*} + \|\mathbb{A}\|_\gamma)\|\mathbb{A}\|_\gamma |t-s|^{3\gamma}$$

$$\Downarrow$$

$$\|R\|_{3\gamma,(W^{1,3})^*} \lesssim (\|R\|_{2\gamma,(W^{1,2})^*} + \|R\|_{\gamma,(W^{1,1})^*} + \|\mathbb{A}\|_\gamma)\|\mathbb{A}\|_\gamma$$

Estimate cannot be closed.

▷ Loss of derivatives in the estimate can be compensated by the time regularity via a small interpolation argument, giving

$$\|R\|_{3\gamma,(W^{1,3})^*} \lesssim (\|R\|_{3\gamma,(W^{1,3})^*} + \|\mathbb{A}\|_{\gamma})\|\mathbb{A}\|_{\gamma}$$

and uniform apriori estimate:

$$\|R\|_{\gamma,(W^{1,1})^*} + \|R\|_{2\gamma,(W^{1,2})^*} + \|R\|_{3\gamma,(W^{1,3})^*} \lesssim_{\|\mathbb{A}\|_{\gamma}} 1.$$

▷ Existence of solutions via compactness of smooth approximations using this estimate.

Deterministic strategy for uniqueness of $\partial_t u = V \cdot \nabla u$:

Start with (commutator lemma!)

$$\partial_t u^2 = 2u\,\partial_t u = 2\,uV \cdot \nabla u = V \cdot \nabla u^2$$

and integrating over space we get

$$\partial_t \int u^2 = \int (V \cdot \nabla u^2) = -\int \mathrm{div}\, V\, u^2.$$

Gronwall

$$\int u_t^2 \leqslant \int u_0^2 \exp(t \|\mathrm{div}\, V\|_{L^\infty})$$

$$u_0 = 0 \quad \Longrightarrow \quad u_t = 0$$

Can we redo it roughly? We need “Ito formula” for the square and (rough) Gronwall...

Product of distributions is dangerous. Tensor product easier.

Consider the dynamics for $U(x,y)=(u\otimes u)(x,y)=u(x)u(y)$.

$$\delta U_{s,t}(\Phi)=U_s(\Gamma^{1,*}_{\mathbb{A},s,t}\Phi)+U_s(\Gamma^{2,*}_{\mathbb{A},s,t}\Phi)+o(|t-s|)$$

with

$$\Gamma^1_{\mathbb{A},s,t}=\mathbb{A}^1_{s,t}\otimes\mathbb{I}+\mathbb{I}\otimes\mathbb{A}^1_{s,t},\qquad \Gamma^2_{\mathbb{A},s,t}=\mathbb{A}^2_{s,t}\otimes\mathbb{I}+\mathbb{I}\otimes\mathbb{A}^2_{s,t}+\mathbb{A}^1_{s,t}\otimes\mathbb{A}^1_{s,t}$$

(we used the property $\mathbb{X}^{1,\alpha}_{s,t}\mathbb{X}^{1,\beta}_{s,t}=\mathbb{X}^{2,\alpha\beta}_{s,t}+\mathbb{X}^{2,\beta\alpha}_{s,t}$)

Another rough driver:

$$\delta\Gamma^1_{\mathbb{A},s,u,t}=0,\qquad \delta\Gamma^2_{\mathbb{A},s,u,t}=\Gamma^1_{\mathbb{A},u,t}\Gamma^1_{\mathbb{A},s,u}.$$

We want to use the fact that

$$u_t^2(\varphi) = \int u_t^2(x)\varphi(x)\mathrm{d}x = \lim_{\varepsilon\to 0} \iint u_t(x)u_t(y)\varepsilon^{-d}\psi\left(\frac{x-y}{\varepsilon}\right)\varphi\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y = U_t(\Phi_\varepsilon)$$

but $(\Phi_\varepsilon)_\varepsilon$ is a singular family of test functions.

Under appropriate conditions

$$A^\varepsilon_{s,t}(\varphi) = U_s(\Gamma^{1,*}_{\mathbb{A},s,t}\Phi_\varepsilon) + U_s(\Gamma^{2,*}_{\mathbb{A},s,t}\Phi_\varepsilon) \to u_s^2(\mathbb{A}^{1,*}_{s,t}\varphi) + u_s^2(\mathbb{A}^{2,*}_{s,t}\varphi)$$

and the coherence of this germ can be controlled uniformly in ε (similar to Di Perna–Lions commutator).

Result:

$$\delta u^2_{s,t}(\varphi) = u_s^2(\mathbb{A}^{1,*}_{s,t}\varphi) + u_s^2(\mathbb{A}^{2,*}_{s,t}\varphi) + o(|t-s|)$$

(Itô formula for the square).

Now take $\varphi_R(x) = (1 + |x/R|^2)^{-M}$, then

$$|(V \cdot \nabla)^* \varphi_R(x)| \lesssim \varphi_R(x), \ |(V \cdot \nabla)^*(V \cdot \nabla)^* \varphi_R(x)| \lesssim \varphi_R(x),$$

and from

$$\delta u^2_{s,t}(\varphi_R) = u^2_s(\mathbb{A}^{1,*}_{s,t} \varphi_R) + u^2_s(\mathbb{A}^{2,*}_{s,t} \varphi_R) + O(|t-s|^{3\gamma})$$

we get

$$h(t) \leqslant h(s) + h(s)|t-s|^{\gamma} + C\left(\sup_{r \leqslant t} h(r)\right)|t-s|^{3\gamma}$$

where $h(t) := u^2_t(\varphi_R)$. A small *rough Gronwall* lemma allows to conclude that

$$\sup_{t \leqslant T} h(t) \lesssim_T h(0)$$

and from there the uniqueness.

Scalar conservation law with time–dependent fluxes

$$\partial_t u(t,x) = \operatorname{div}(A_\alpha(x, u(t,x)))\dot{X}^\alpha(t)$$

Kinetic formulation : $f(t,x,\xi) := \mathbb{I}_{\xi < u(t,x)}$

$$\partial_t f(t,x,\xi) = V_\alpha(t,x,\xi) \cdot \nabla_{x,\xi} f(t,x,\xi)\dot{X}^\alpha(t) + \partial_\xi m(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\xi)$$

A solution is a *pair* (f, m) with $f \in L^\infty$ and m a measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

$$u(t,x) = \int (f(t,x,\xi) - \mathbb{I}_{\xi<0})\mathrm{d}\xi, \qquad -\partial_\xi f(t,x,\xi) = \delta_{u(t,x)}(\xi)\,(\text{Young measure})$$

Rough formulation:

$$\delta f_{s,t}(\varphi) = f_s(\mathbb{A}^{1,*}_{s,t}\varphi) + f_s(\mathbb{A}^{2,*}_{s,t}\varphi) + \delta m(\partial_\xi \varphi)$$

Existence via compactness, uniqueness via tensorization.

Scalar deterministic transport + Stratonovic Brownian perturbation

$$\mathrm{d}u_t = V \cdot \nabla u_t \mathrm{d}t + \nabla u_t \circ \mathrm{d}B_t.$$

The vectorfield V is not Lipshitz (no uniqueness classically).

Almost sure uniqueness if $V \in C^\varepsilon$ [Flandoli–G.–Priola]. A lot of work recently (Proske et al).

Change of variables: $v(t,x) = u(t, x - B_t)$

$$\mathrm{d}v_t(x) = V(x - B_t) \cdot \nabla v_t(x) \mathrm{d}t.$$

Rough (Young) dynamics

$$\delta v_{s,t} = \mathbb{A}^1_{s,t} v_s + o(|t-s|)$$

with $\mathbb{A}^1_{s,t} = \int_s^t V(x - B_r) \mathrm{d}r \cdot \nabla$

Lemma 2 **(Catellier–G.)** *For any $\varepsilon > 0$ there exists $\gamma > 1/2$ such that*

$$\left\| x \mapsto \int_s^t V(x - B_r) \mathrm{d}r \right\|_{C^{3/2+\varepsilon}} \lesssim |t - s|^{\gamma} \|V\|_{C^{1/2+2\varepsilon}}$$

for almost every Brownian path (exceptional set depends on V).

This property guarantees uniqueness of the RTE pathwise via a tensorization argument, no stochastic analysis involved.

[Catellier (via flow transformation), Maurelli (intrinsic)]

Thanks!