Reflected rough differential equations

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Slusius ante annos duos tresve tibi communicavit, de quâ tu, suggerente Collinsio, rescripsisti eandem mihi ~~tibi inno~~ etiam innotuisse. Diversa ratione in eam incidimus. Nam res non eget demonstratione prout ego operor. Habito meo fundamento nemo potuit tangentes aliter ducere, nisi volens de rectâ viâ deviaret. Quinetiam non hic hæretur ad æquationes radicalibus unam vel utramque indefinitam quantitatem involventibus utcunque affectas, sed absque aliquâ talium æquationum reductione (quæ opus plerumque redderet immensum) tangens confestim ducitur. Et eodem modo se res habet in quæstionibus de Maximis et minimis, alijsque quibusdam de quibus jam non loquor. ~~Fundamentum harum operationum~~ Fundamentum harum operationum satis obvium quidem, quoniam jam non possum explicationem ejus prosequi sic potius celavi. 6accdæ13eff7i3l9n4o4qrr4s8t12vx. Hoc fundamento conatus sum etiam reddere speculationes de Quadratura curvarum simpliciores, perveni ad Theoremata quædam generalia. et ut candide agam ecce primum Theorema.

Ad Curvam aliquam sit $dz^{\theta}\times\overline{e+fz^{\eta}}|^{\lambda}$ ordinatim applicata termino diametri seu basis z normaliter insistens: ubi literæ d, e, f denotant quaslibet quantitates datas, & θ, η, λ indices potestatum sive dignitatum quantitatum quibus affixæ sunt. Fac $\frac{\theta+1}{\eta}=r$. $\lambda+r=s$. $\frac{d}{\eta f}\times\overline{e+fz^{\eta}}|^{\lambda+1}=Q$. & $r\eta-\eta=\pi$, & area Curvæ erit Q in $\frac{z^{\pi}}{s}-\frac{r-1}{s-1}\times\frac{eA}{fz^{\eta}}+\frac{r-2}{s-2}\times\frac{eB}{fz^{\eta}}-\frac{r-3}{s-3}\times\frac{eC}{fz^{\eta}}+\frac{r-4}{s-4}\times\frac{eD}{fz^{\eta}}$ &c ~

literis A, B, C, D &c ~~&c~~ denotantibus terminos proxime antecedentes, nempe A terminum $\frac{z^{\pi}}{s}$, B terminum $-\frac{r-1}{s-1}\times\frac{eA}{fz^{\eta}}$ &c. Hæc Series ubi r fractio est vel numerus negativus, continuatur in infinitum: ubi verò r integer est et affirmativus continuatur ad tot terminos tantùm quot sunt unitates in eodem r, et sic exhibet geometricam quadraturam Curvæ. Rem exemplis illustro.

“Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa”

(I. Newton, letter to Henry Oldenburg, 24 October 1676)

Solving the controlled ODE in $\mathbb{R}^d$

$$\dot{y}(t) = V_\alpha(y(t))\dot{x}^\alpha(t), \qquad t \geqslant 0,$$

with $(V_\alpha)_\alpha$ family of vector fields and $y(0)$ given, is equivalent to asking for a function $y: \mathbb{R}_{\geqslant 0} \to \mathbb{R}^d$ such that

$$y(t) - y(s) = V_\alpha(y(s))(x^\alpha(t) - x^\alpha(s)) + o(|t-s|), \qquad 0 \leqslant s \leqslant t.$$

General references on RP/RS:

Lyons '98, Davie, Lyons-Qian, Friz–Victoir, Friz–Hairer, Hairer.

Talk based on joint work with A. Deya, M. Hofmanová, S. Tindel.

Goal: Replace differential/integral description with *non-infinitesimal* local one.

$$\delta y(s,t) := y(t) - y(s) = A(s,t) + R(s,t)$$

- A is a "germ" for the dynamics of y:

$$A(s,t) = V(y(s))X^1(s,t) + V_2(y(s))X^2(s,t) + \cdots$$

- the equation holds modulo error term $R(s,t)$ of order $o(|t-s|)$
- **Key insight**. this decomposition is *rigid*: to each given A there can correspond only one pair (y, R):

$$|\delta y(s,t) - \delta\tilde{y}(s,t)| = |R(s,t) - \tilde{R}(s,t)| = o(|t-s|)$$

Explicit bounds on R in terms of the “coherence” of A

$$\delta A(s,u,t) := A(s,t) - (A(s,u) + A(u,t)), \qquad s \leqslant u \leqslant t$$

Lemma ***(Sewing lemma)*** *Assume that*

$$|\delta A(s,u,t)| \leqslant \|\delta A\|_z |t-s|^z$$

for some $z > 1$, then there exists a unique y such that

$$\delta y(s,t) = A(s,t) + R(s,t), \qquad |R(s,t)| = o(|t-s|)$$

and moreover

$$|R(s,t)| \leqslant C_z \|\delta A\|_z |t-s|^z.$$

This result holds for general regular controls $\omega(s,t)$ (replacing $|t-s|$):

$$\omega(s,u) + \omega(u,t) \leqslant \omega(s,t), \qquad |t-s| \to 0 \Rightarrow \omega(s,t) \to 0$$

Local expansion of ODEs:

$$y(t) = y(s) + V_\alpha(y(s)) \underbrace{\int_s^t \mathrm{d}x^\alpha(r)}_{X^{1,\alpha}(s,t)} + V_{2,\alpha\beta}(y(s)) \underbrace{\int_s^t \int_s^r \mathrm{d}x^\alpha(w)\mathrm{d}x^\beta(r)}_{X^{2,\alpha\beta}(s,t)} + \cdots$$

with $V_{2,\alpha\beta}(\xi) = V_\alpha(\xi) \cdot \nabla V_\beta(\xi)$.

Definition 1 *A (step-2) rough path* $\mathbb{X} = (X^1, X^2)$ *is a pair such that*

$$\delta X^1(s,u,t) = 0, \qquad \delta X^2(s,u,t) = X^1(s,u) X^1(u,t)$$

$$|X^1(s,t)| + |X^2(s,t)|^{1/2} \leqslant \|\mathbb{X}\|_\gamma |t-s|^\gamma$$

for some $\gamma \geqslant 1/3$.

[Lyons '98]

- Let $x \in C^\gamma$ and $(x^\varepsilon)_\varepsilon$ some family of smooth approximations $x^\varepsilon \to x$ in C^γ.

- Smooth approximations by ODEs

$$\dot{y}^\varepsilon = V(y^\varepsilon)\dot{x}^\varepsilon(t)$$

- Taylor expansion gives

$$\delta y^\varepsilon(s,t) = A^\varepsilon(s,t) + R^\varepsilon(s,t)$$

$$A^\varepsilon(s,t) = V(y^\varepsilon(s))X^{\varepsilon,1}(s,t) + V_2(y^\varepsilon(s))X^{\varepsilon,2}(s,t) \qquad |R^\varepsilon(s,t)| \leqslant \|\dot{x}^\varepsilon\|_\infty |t-s|^3$$

- **Problem:** estimates for the remainder are not uniform in $\varepsilon \to 0$.

- Uniform estimates for R^ε from the coherence of the germ A^ε itself

$$\delta A^\varepsilon(s,u,t) = -\delta V(y^\varepsilon)(s,u)X^{\varepsilon,1}(u,t) + V_2(y^\varepsilon(s))\delta X^{\varepsilon,2}(s,u,t) - \delta V_2(y^\varepsilon)(s,u)X^{\varepsilon,2}(u,t)$$

$$= -\underbrace{(\delta V(y^\varepsilon)(s,u) - V_2(y^\varepsilon(s))X^{\varepsilon,1}(s,u))}_{O(|R^\varepsilon(s,t)|)+O(|t-s|^{2\gamma})}\underbrace{X^{\varepsilon,1}(u,t)}_{O(|t-s|^{\gamma})} - \underbrace{\delta V_2(y^\varepsilon)(s,u)}_{O(|R^\varepsilon(s,t)|)+O(|t-s|^{\gamma})}\underbrace{X^{\varepsilon,2}(u,t)}_{O(|t-s|^{2\gamma})}$$

$$\|R^\varepsilon\|_{2\gamma} := \sup_{s,t} \frac{|R^\varepsilon(s,t)|}{|t-s|^{2\gamma}}.$$

- If $3\gamma > 1$ the sewing lemma gives

$$|\delta A^\varepsilon(s,u,t)| \lesssim (\|R^\varepsilon\|_{2\gamma} + \|\mathbb{X}^\varepsilon\|_\gamma)\|\mathbb{X}^\varepsilon\|_\gamma |t-s|^{3\gamma}$$
$$\Downarrow$$
$$\|R^\varepsilon\|_{3\gamma} \lesssim (\|R^\varepsilon\|_{2\gamma} + \|\mathbb{X}\|_\gamma^\varepsilon)\|\mathbb{X}^\varepsilon\|_\gamma$$

$$\|R^\varepsilon\|_{3\gamma} \lesssim_{\|\mathbb{X}^\varepsilon\|_\gamma} 1 \qquad \text{uniformly in } \varepsilon > 0.$$

- The limit $y^\varepsilon \to y$ exists provided $\mathbb{X}^\varepsilon \to \mathbb{X} = (X^1, X^2)$ in *rough path topology*.
- It satisfies the RDE [Davie]

$$\delta y(s,t) = V(y(s))X^1(s,t) + V_2(y(s))X^2(s,t) + O(|t-s|^{3\gamma}).$$

- Is unique under sufficient regularity for V, V_2.
- The map $\mathbb{X} \mapsto y = \Phi(\mathbb{X})$ is continuous.
- Rough path limit $\mathbb{X}$ is **not unique** for given x. It holds

$$X^1(s,t) = \tilde{X}^1(s,t) = \delta x(s,t), \qquad \tilde{X}^2(s,t) - X^2(s,t) = \delta\varphi(s,t).$$

- The limit RDE is not an ODE (or not that one expects...).

Example Pure area RP: there exists $\mathbb{X}^\varepsilon \to (0, \delta\varphi)$ with $\varphi \in C^1$. Then

$$\dot{y}^\varepsilon(t) = V(y^\varepsilon(t))\dot{x}^\varepsilon(t) \qquad \Rightarrow \qquad \dot{y}(t) = V_2(y(t))\dot{\varphi}(t)$$

Rough differential equations on $\mathbb{R}_{\geqslant 0}$ reflected at 0.

We are interested in limits of "physical dynamics" of the form

$$\dot{y}^{\varepsilon}(t) = V(y^{\varepsilon}(t))\dot{X}^{1,\varepsilon}(t) + \frac{(y^{\varepsilon}(t))_-}{\varepsilon}.$$

Ideally $\int_s^t \frac{(y^{\varepsilon}(u))_-}{\varepsilon} \mathrm{d}u \to m(s,t)$ which is only a measure. Handled more effectively in p-variation spaces:

$$f \in \mathscr{V}^p \Leftrightarrow \exists \text{ control } \omega \quad s.t. \quad |f(t) - f(s)| \leqslant \omega(s,t)^{1/p}$$

Solution is a *pair* $(y,m) \in \mathscr{V}^p \times \mathscr{V}^1$ satisfying $(z > 1, 2 \leqslant p < 3)$

a) $\delta y(s,t) = V(y(s))X^1(s,t) + V_2(y(s))X^2(s,t) + \int_s^t \mathrm{d}m_u + \omega^{\natural}(s,t)^z$

b) $y(t) \geqslant 0, \qquad y(t)\mathrm{d}m(t) = 0.$

m is the **reflection measure**, supported on $\{t \geqslant 0 : y(t) = 0\} \subseteq \mathbb{R}$.

Skorokhod problem: given a path $w \in C([0,T];\mathbb{R})$ find (y,m) such that

$$y(t) = w(t) + m(t), \qquad y(t) \geqslant 0, \quad m(t) = \int_0^t \mathbf{1}_{y(s)=0}\, \mathrm{d}m(s).$$

- y *reflector* of w; m *regulator* of w ; $(y,m) = \Gamma(x) = (\Gamma_{\mathrm{ref}}(x), \Gamma_{\mathrm{reg}}(x))$,
- can be uniquely solved, contractive in C^0 [Saisho '87].
- the map Γ is **not** locally contractive from $C^\gamma \to C^\gamma$ [Ferrante, Rovira, '13]
- the map Γ is locally contractive from $\mathscr{V}^p \to \mathscr{V}^p$ [Falkowski, Slominski, '15]

▷ **Young DE** $(p < 2)$

- existence [Ferrante, Rovira, '13]
- uniqueness (cadlag setting) [Falkowski, Slominski, '15]

When $2 < p \leqslant 3$.

- existence (Schauder fixed point), controlled path [Aida '15, '16].
- no fixpoint method for uniqueness.
- not much information about the measure m.
- difference of two measures $m^1 - m^2$ cannot be controlled effectively.
- need for a suitable replacement for Gronwall Lemma.

Theorem

There is a unique solution (y, m) *with the initial condition* $y(0) > 0$.

Proof inspired by recent results on rough scalar conservation laws.

Ideas

- consider two solutions (y,μ) and (z,ν)
- write the equation for $\delta(y-z)_{st}$
- in the stochastic setting: estimate $\mathbb{E}|y_t - z_t|^2$ - not possible here
- rather: estimate $|y_t - z_t|$
- but: $\varphi(\xi) = |\xi|$ not C^3 for the change of variable (Itô) formula

A scheme of proof

1. apply the Itô formula to $\varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\xi|^2}$
2. estimate the remainder uniformly in ε
3. send $\varepsilon \to 0$
4. estimate $|y_t - z_t|$
5. argue by contradiction

- Itô formula for $\varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\xi|^2}$

$$|\varphi_\varepsilon'(\xi)| \leqslant 1, \qquad |\varphi_\varepsilon''(\xi)| \leqslant \frac{1}{\sqrt{\varepsilon^2 + |\xi|^2}}, \qquad |\varphi_\varepsilon'''(\xi)| \lesssim \frac{1}{\varepsilon^2 + |\xi|^2}$$

- leads to

$$\delta\varphi_\varepsilon(y-z)_{st} = H_s^\varepsilon \mathbb{X}_{st}^1 + H_{2,s}^\varepsilon \mathbb{X}_{st}^2 + \int_s^t \varphi_\varepsilon'(y_r - z_r)(\mathrm{d}\mu_r - \mathrm{d}\nu_r) + h_{st}^{\varepsilon,\natural}$$

where

$$H^\varepsilon = \varphi_\varepsilon'(y-z)(F(y) - F(z))$$

$$H_2^\varepsilon = \varphi_\varepsilon'(y-z)(F_2(y) - F_2(z)) + \varphi_\varepsilon''(y-z)(F(y) - F(z))(F(y) - F(z))$$

- uniform estimate of the remainder $h^{\varepsilon,\natural}$ via

$$|||\varphi_\varepsilon||| = \sup_{y,z} (|\varphi_\varepsilon'(y-z)| + |y-z|\varphi_\varepsilon''(y-z) + |y-z|^2 \varphi_\varepsilon'''(y-z))$$

The choice of φ_ε is motivated by the fact that

$$\int_s^t \varphi_\varepsilon'(y_r - z_r)(\mathrm{d}\mu_r - \mathrm{d}\nu_r) \to -\int_s^t \mathbf{1}_{y_r \neq z_r} \mathrm{d}(\mu_r + \nu_r) = -\omega_M(s,t) + \int_s^t \mathbf{1}_{y_r = z_r} \mathrm{d}(\mu_r + \nu_r)$$

where $\omega_M(s,t) = \|\mu\|_{V_1^1([s,t])} + \|\nu\|_{V_1^1([s,t])}$.

▹ The test function $|y - z|$ allows to obain precious informations on the total variation ω_M of the two measures μ, ν.

So after $\varepsilon \to 0$ we get the formula a kind of Ito–Tanaka formula:

$$\begin{aligned} \delta|y-z|_{st} + \omega_M(s,t) \;=\; & \operatorname{sgn}(y_s - z_s)(F(y_s) - F(z_s))\mathbb{X}_{st}^1 \\ & + \operatorname{sgn}(y_s - z_s)(F_2(y_s) - F_2(z_s))\mathbb{X}_{st}^2 \\ & + \int_s^t \mathbf{1}_{y_r = z_r} \mathrm{d}(\mu_r + \nu_r) \\ & + \Phi_{st}^\natural \end{aligned}$$

From this formula

$$\begin{aligned}\delta|y-z|_{st}+\omega_M(s,t) &= \operatorname{sgn}(y_s-z_s)(F(y_s)-F(z_s))\mathbb{X}^1_{st}\\ &\quad+\operatorname{sgn}(y_s-z_s)(F_2(y_s)-F_2(z_s))\mathbb{X}^2_{st}\\ &\quad+\int_s^t \mathbf{1}_{y_r=z_r}\,\mathrm{d}(\mu_r+\nu_r)+\Phi^\natural_{st}\end{aligned}$$

- Sewing lemma: estimate $\Phi^\natural_{st}$ in terms of the LHS

$$|\Phi^\natural_{st}|\lesssim_{|t-s|}\|y-z\|_{L^\infty([s,t])}+\omega_M(s,t)$$

- this gives a “rough Gronwall” conclusion:

$$\sup_{r\in[s,t]}|y_r-z_r|+\omega_M(s,t)\lesssim|y_s-z_s|+\int_s^t \mathbf{1}_{y_r=z_r}\,\mathrm{d}(\mu_r+\nu_r)$$

- assume $y\neq z$ in (s,t) but $y_s=z_s$ then $0<\sup_{r\in[s,t]}|y_r-z_r|+\omega_M(s,t)\lesssim 0$.

$\Rightarrow$ **contradiction.**

Existence

- proved by approximating $\mathbb{X}$ by $\mathbb{X}^\varepsilon$ + uniform estimates + passage to the limit (like in [Aida '15, '16] but simpler proof)
- more general domains via an estimate of m by [Aida '16]

Open problems

- Smooth domains in $\mathbb{R}^d$ can be reduced to hyperplane case $\mathbb{R}^d \times \mathbb{R}_{\geqslant 0}$ by change of coordinates, but
- Proof does not work for hyperplanes. The limit argument fails and no substitute test functions. Trickier situation.

Thanks!