Universality and Singular SPDEs

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We are concerned here with large scale *effective* description of microscopic random phenomena.

White noise (CLT, Donsker's Invariance principle, ...)

- η: $\mathbb{R}^d \to \mathbb{R}$ a stationary random field under suitable assumptions (e.g. strong mixing, integrability) with law μ.
- Weak topology: $\eta(\varphi) = \int \mathrm{d}x \, \varphi(x) \eta(x)$ for a sufficiently large class of φ.
- Scaling transformation $\eta_\varepsilon(x) = \varepsilon^{-d/2} \eta(x / \varepsilon)$: keeps variance unchanged for $\eta(\varphi)$ but not mean.

Let $\mu_{\varepsilon,m}$ the law of $\varphi_\varepsilon - m$, $m_\varepsilon = \varepsilon^{-d/2} \mathbb{E}(\eta(x)) - \rho$, then

$$\mu_{\varepsilon, m_\varepsilon} \to \gamma_{\rho,c} \qquad \text{as } \varepsilon \to 0,$$

where $\gamma_{\rho,c}$ is the law of the white noise ξ with intensity c and mean ρ:

$$\mathbb{E}(\xi(\varphi)) = \rho \int \varphi(x) \mathrm{d}x, \qquad \mathrm{Var}(\xi(\varphi)) = c \int \varphi(x)^2 \mathrm{d}x.$$

The description of random non-gaussian scaling limits is less clear:

- Infinitely divisible distributions, Hierarchical models
- Ferromagnetic critical point in $d=2,3$ short range spin systems
- Large scale behaviour of $d=1,2,3,\dots$ interface models in equilibrium or not
- Interacting Euclidean quantum fields
- ....

There are a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects.

In simple cases the microscopic fluctuations average out when larger scales are considered, and the averaged quantities satisfy classical continuum equations. Hydrodynamics is a standard example of this, where atomic fluctuations average out and the classical hydrodynamic equations emerge. Unfortunately, there is a much more difficult class of problems where fluctuations persist out to macroscopic wavelengths, and fluctuations on all intermediate length scales are important too.

In this last category are the problems of fully developed turbulent fluid flow, critical phenomena, and elementary-particle physics. The problem of magnetic impurities in nonmagnetic metals (the Kondo problem) turns out also to be in this category.

A theoretical framework for the description of these more general scaling limits is provided by Wilson's RG

The renormalization group and critical phenomena*

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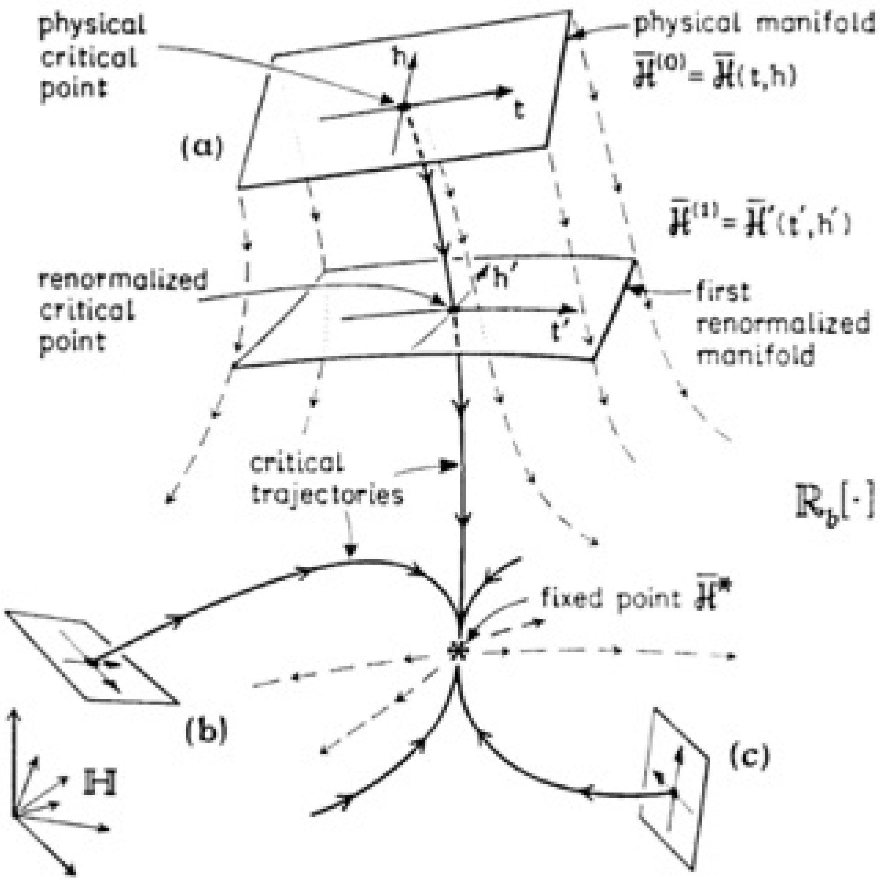
The possible types of cooperative behavior, in the renormalization group picture, are determined by the possible fixed points $\mathcal{H}^*$ of τ. Suppose for example that there are three fixed points $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, and $\mathcal{H}^*_C$. Then one would have three possible forms of cooperative behavior. If a particular system has an initial interaction $\mathcal{H}_0$, one has to construct the sequence $\mathcal{H}_1$, $\mathcal{H}_2$, etc. in order to find out which of $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, or $\mathcal{H}^*_C$ gives the limit of the sequence. If $\mathcal{H}^*_A$ is the limit of the sequence, then the cooperative behavior resulting from $\mathcal{H}_0$ will be the cooperative behavior determined by $\mathcal{H}^*_A$. In this example the set of all possible initial interactions $\mathcal{H}_0$ would divide into three subsets (called "domains"), one for each fixed point. Universality would now hold separately for each domain. See section 12 for further discussion.

This is how one derives a form of universality in the renormalization group picture. It is not so bold as previous formulations [9]. Experience with soluble examples of the renormalization group transformation for critical phenomena shows that it generally has a number of fixed points, so one has to define domains of initial Hamiltonians associated with each fixed point, and only within a given domain is the critical behavior independent of the initial interaction.

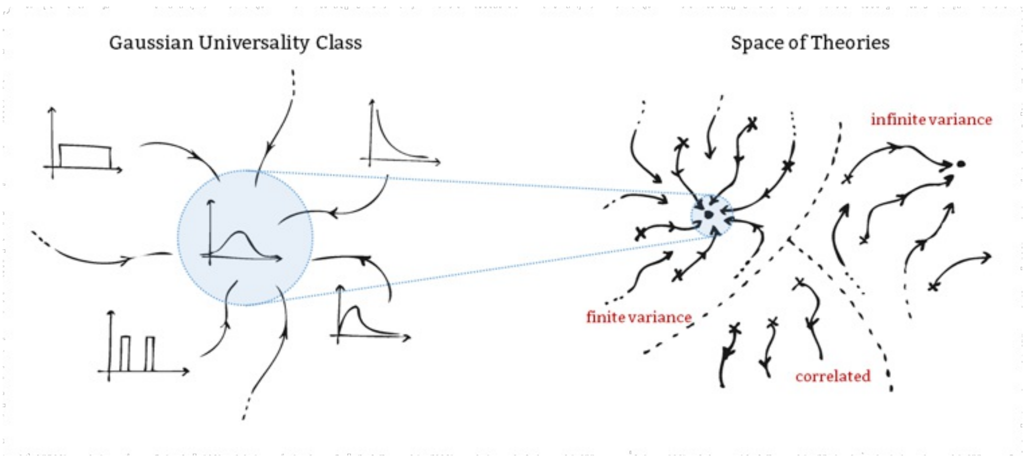
There is no *a priori* requirement that the sequence $\mathcal{H}_l$ approach a fixed point for $l \to \infty$. In

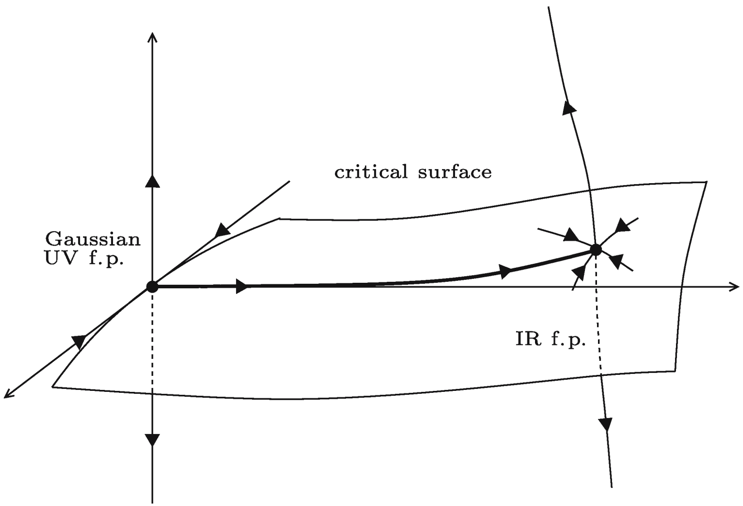
▷Rescaling, analysing how the theory changes from scale to scale, give rise to a dynamical system

▷Basins of attractions are universality classes, all the systems display similar large scale behaviour



CLT is a particular fixpoint with its own basin of attraction.

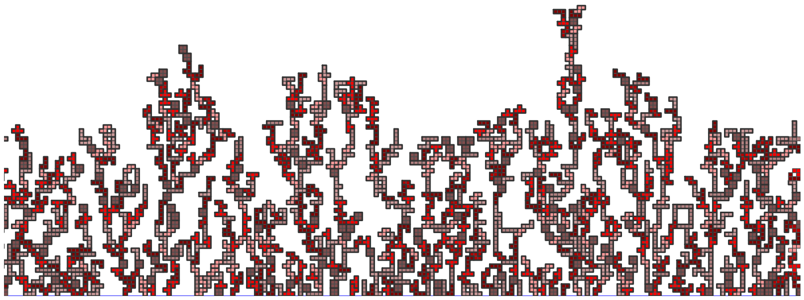


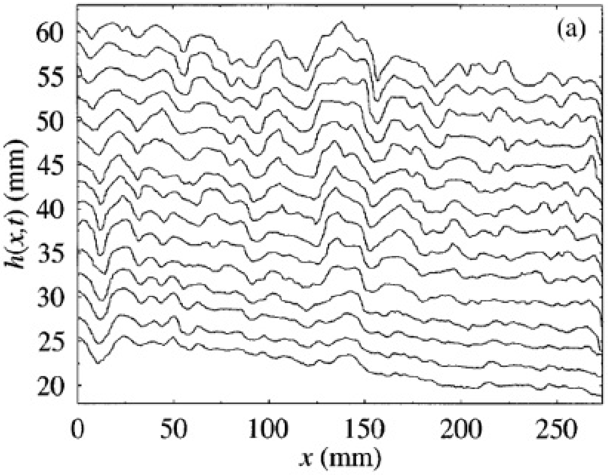


Unstable directions out of the Gaussian fixpoints (may) go to other (IR) fixpoints.

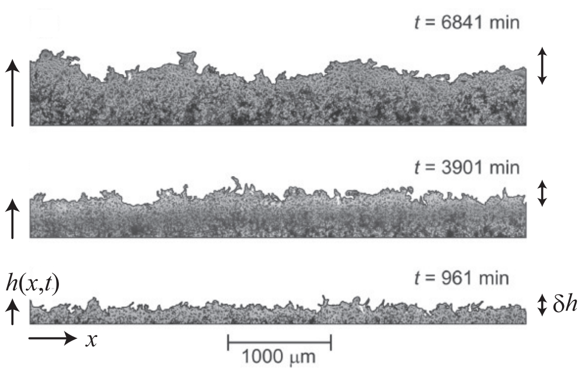
This hints to the possibility of introducing class of models which describe these fixpoints as (universal) perturbations of Gaussian models.

The trajectory describes *perfect* theories where rescaling implies only a change of parameters.

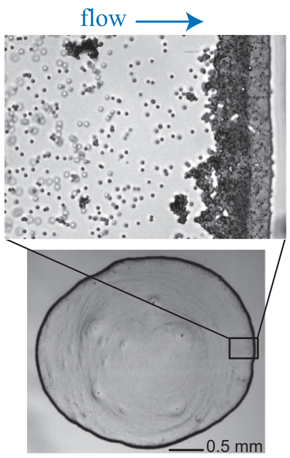




(a) proliferating cancer cells



(b) particle deposition in suspension droplet



KPZ universality class

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Dynamic Scaling of Growing Interfaces

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A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 81.15.Jj

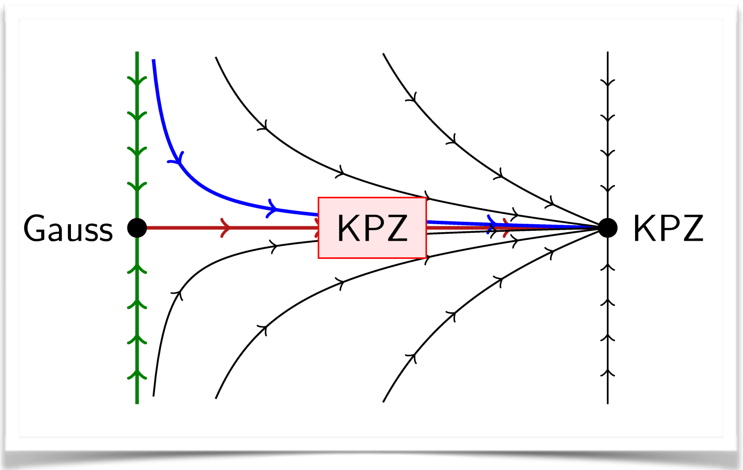
Many challenging problems are associated with growth patterns in clusters[1] and solidification fronts.[2] Several models have been proposed recently to describe the growth of smoke and colloid aggregates, flame fronts, tumors, etc.[1] It is generally recognized that the growth process occurs mainly at an "active" zone on the surface of the cluster, with interesting scaling properties.[3] However, a systematic *analytic* treatment of the static and dynamic fluctuations of the growing interface has been lacking so far.

In this paper we propose a model for the time evolution of the profile of a growing interface, and examine

The interface profile, suitably coarse-grained, is described by a height $h(\mathbf{x},t)$. As usual, it is convenient to ignore overhangs so that h is a single-valued function of $\mathbf{x}$. The simplest nonlinear Langevin equation for a local growth of the profile is given by[12]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \tag{1}$$

The first term on the right-hand side describes relaxation of the interface by a surface tension ν. The second term is the lowest-order nonlinear term that can appear in the interface growth equation, and is



The KPZ equation defines a one-parameter family of models

$$\partial_t h = \Delta h + \lambda[(\nabla h)^2 - \infty] + \xi$$

▷ Diffusive rescaling

$$h_\varepsilon(t,x) = \varepsilon^{1/2} h(t/\varepsilon^2, x/\varepsilon) - \varepsilon^{-1/2} m$$

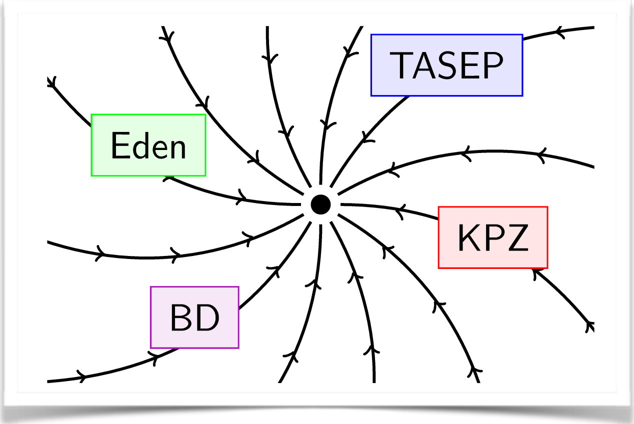
▷ $\lambda = 0$: Gaussian fixpoint

▷ λ grows under scaling (relevant direction)

$$\partial_t h_\varepsilon = \Delta h_\varepsilon + \lambda \varepsilon^{-1/2} (\nabla h_\varepsilon)^2 + \xi$$

▷ $\lambda \to \infty$: **KPZ fixpoint** equivalent to

$$\partial_t h_\delta = \delta \Delta h_\delta + \lambda (\nabla h_\delta)^2 + \sqrt{\delta}\, \xi_\delta, \qquad \delta \to 0.$$



▷ Recent results by Matetski, Quastel, Remenik on the law of the KPZ fixpoint as integrable system.

▷ The KPZ equation is the (unique?) critical trajectory exiting the Gaussian fp.

▷ Precise mathematical description of this trajectory has been a longstanding mathematical problem moreover it is interesting to characterise models which can lead to KPZ_λ under scaling (weak–universality).

▷ Bertini and Giacomin (1996) provided a construction of this critical trajectory via a particular family of stochastic discrete models $(\mathrm{WASEP}_\alpha)_{\alpha\in\mathbb{R}}$ and a suitable rescaling transformation R_ε.

▷ α is a asymmetry parameter (inducing large scale flux of particles) whose influence "grows" under rescaling.

$$R_\varepsilon \mathrm{WASEP}_0 \to \text{Gaussian model}, \qquad R_\varepsilon \mathrm{WASEP}_{\varepsilon^{1/2}\lambda} \to \mathrm{KPZ}_\lambda$$

▷ KPZ_λ is identified via Hopf–Cole transformation:

$$h = \log Z\,, \qquad \partial_t Z = Z\,\xi$$

where the Stochastic Heat equation is interpreted in Ito sense (martingale theory).

▷ This trick does seldom work. Without more flexible description of KPZ_λ is it difficult to prove convergence.

▷ Hairer (2013, 2014) devised a successful approach to give an intrinsic meaning to the KPZ equation. This allows a rigorous description of the $(\mathrm{KPZ}_\lambda)_\lambda$ random fields solving

$$\partial_t h = \Delta h + \lambda[(\nabla h)^2 - \infty] + \xi.$$

The random field h is described in terms of the Gaussian fixpoint $\partial_t X = \Delta X + \xi$.

- Rough paths, regularity structures (Hairer)

$$h(x) - h(y) = X(x) - X(y) + Y(x, y) + h'(x) Z(x, y) + O(|x - y|^{3/2+})$$

- Paracontrolled distributions (G, Imkeller, Perkowski)

$$\Delta_i h = \Delta_i X + \Delta_i Y + (\Delta_{\leqslant i-1} h')\Delta_i Z + O(2^{-3/2i})$$

- Energy solutions/martingale problem (Jara, Gonçalves, G., Perkowski)

$$\mathrm{d}h(t) - \Delta h(t)\,\mathrm{d}t - \mathrm{d}\mathcal{B}(t) = \mathrm{d}M(t), \qquad \mathrm{d}\mathcal{B}(t) = \lim_{\sigma}\,[(\nabla \rho_\sigma * h)^2 - C_\sigma]\mathrm{d}t$$

- Other approaches: Renormalization group (Kupiainen), Otto & Weber approach...

▷ Hairer and Quastel proved (2015) that scaling limits of random fields $\mathrm{HQ}(F, \eta, L)$ solution to

$$\partial_t h = \Delta h + F(\nabla h) + \eta$$

on a periodic domain of size L, converges to KPZ:

$$R_\varepsilon \mathrm{HQ}(\varepsilon^{1/2} F, \eta, \varepsilon^{-1} L) \to \mathrm{KPZ}_\lambda$$

where λ is a function of F, whenever F is polynomial and η short range Gaussian field. (NB: proper recentering of the scaling transformation is needed.)

▷ Regularity structures/Paracontrolled distributions analysis of scaling limits of particle systems is still a difficult problem. The expansion requires a precise control of the dynamics (but see recent results by Matetski and Quastel)

▷ Gonçalves–Jara energy solutions allow to prove convergence to KPZ_λ for a large class of microscopic particle models, always in the same weak asymmetric regime.

▷ This and other results obtained via integrable models confirms the heuristic picture that there are no other relevant fixpoint for interface growth in 1d. The KPZ fixpoint describes the large scale dynamics of growing interfaces.

▷ Scalar fields in $d=3$ dimensions can be used to describe (mesoscopic) magnetization in ferromagnetic system or (Euclidean) scalar quantum fields in $2+1$ dimensions: we are looking for a non–gaussian fixpoint of the RG, the Wilson–Fisher fixed point.

▷ The relevant family $\Gamma(\mu)$ of centered Gaussian models has covariance

$$\mathbb{E}[X(x)X(y)] = (-\Delta+\mu)^{-1}(x,y)$$

▷ Under rescaling R_ε which fixes $\Gamma(0)$ the parameter μ grows: $R_\varepsilon\Gamma(\mu)=\Gamma(\varepsilon^{-2}\mu)$, leading to the *high temperature* fixpoint $\mu\to\infty$, where correlations are absent in the macroscopic scale.

▷ A class of perturbations of the models $\Gamma(\mu)$ is given in terms of a pathwise *dynamic* picture: promote $X(x)$ to a *time dependent* random field satisfying the Langevin equation

$$\partial_t X = -(-\Delta+\mu)X+\xi$$

and introduce the family of dynamic Ginzburg–Landau models $\mathrm{DGL}(V',\eta)$ of the form

$$\partial_t\varphi = \Delta\varphi - V'(\varphi)+\eta$$

where V' is an odd function (we want to preserve the $\varphi\leftrightarrow-\varphi$ symmetry).

▷ Scaling transformation

$$\varphi_\varepsilon(t,x) = \varepsilon^{-1/2}\varphi(t/\varepsilon^2, x/\varepsilon), \qquad \eta_\varepsilon(t,x) = \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon),$$

▷ Equation for $R_\varepsilon \mathrm{DGL}(V', \eta) = \mathrm{DGL}(\varepsilon^{-2}V'(\varepsilon^{1/2}\cdot\,), \eta_\varepsilon)$

$$\partial_t \varphi_\varepsilon = \Delta\varphi_\varepsilon - \varepsilon^{-5/2}V'(\varepsilon^{1/2}\varphi_\varepsilon) + \eta_\varepsilon$$

▷ If $V'(\varphi) = a_1\varphi + a_3\varphi^3 + \cdots$ then

$$\varepsilon^{-5/2}V'(\varepsilon^{1/2}\varphi_\varepsilon) = \varepsilon^{-2}a_1\varphi + \varepsilon^{-1}a_3\varphi^3 + \varepsilon^0 a_5\varphi^5 + \varepsilon^1 a_7\varphi^7 + \cdots$$

▷ Two relevant directions, associated to φ and φ^3:

- Direction φ points towards the high temperature (HT) fixpoint
- Direction φ^3 points in a new direction $\to$ Wilson–Fisher (WF) fixpoint

In order to construct the critical trajectory to WF we need to avoid to be attracted by HT.

▷ Allow for general family $(F_\varepsilon)_\varepsilon$ of interactions to be tuned while rescaling.

$$\mathscr{L} u_\varepsilon(t,x) = -\varepsilon^{-5/2} F_\varepsilon(\varepsilon^{1/2} u_\varepsilon(t,x)) + \eta_\varepsilon(t,x)$$

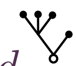
▷ Expand around the Gaussian model and parametrize F_ε via chaos expansion wrt. Y_ε

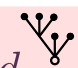
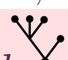
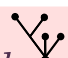
$$\mathscr{L} Y_\varepsilon = \eta_\varepsilon, \qquad v_\varepsilon = Y_\varepsilon + u_\varepsilon,$$

$$\tilde{F}_\varepsilon(x) := F_\varepsilon(x) - f_{0,\varepsilon} - f_{1,\varepsilon} x - f_{2,\varepsilon} H_2(x, \sigma^2_{Y,\varepsilon}) = \sum_{n \geqslant 3} f_{n,\varepsilon} H_n(x, \sigma^2_{Y,\varepsilon}),$$

▷ Introduce constants (with $\Phi^{(m)} = \varepsilon^{(m-5)/2} \tilde{F}^{(m)}_\varepsilon(\varepsilon^{1/2} Y_\varepsilon)$)

d_ε $:= \frac{1}{9} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(1)}_0 \Phi^{(1)}_{(s,x)}]$, $\tilde{d}_\varepsilon$ $:= 2\,\varepsilon^{-1/2} f_{3,\varepsilon} f_{2,\varepsilon} \int_{s,x} P_s(x) [C_{Y,\varepsilon}(s,x)]^2$,

d_ε $:= \frac{1}{6} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(0)}_0 \Phi^{(2)}_{(s,x)}]$, $\hat{d}_\varepsilon$ $:= \frac{1}{3} \int_{s,x} P_s(x) \mathbb{E}[\Phi^{(0)}_0 \Phi^{(1)}_{(s,x)}]$,

d_ε $:= 2\, d_\varepsilon$ $+ 3\, d_\varepsilon$.

▷ Assume

a) $(F_\varepsilon)_\varepsilon \subseteq C^9(\mathbb{R})$ and $\sup_{\varepsilon,x} \sum_{k=0}^{9} |\partial_x^k F_\varepsilon(x)| \leqslant C e^{c|x|}$,

b) the vector $\lambda_\varepsilon = (\lambda_\varepsilon^{(0)}, \lambda_\varepsilon^{(1)}, \lambda_\varepsilon^{(2)}, \lambda_\varepsilon^{(3)}) \in \mathbb{R}^4$

$$\lambda_\varepsilon^{(3)} = \varepsilon^{-1} f_{3,\varepsilon} \qquad \lambda_\varepsilon^{(1)} = \varepsilon^{-2} f_{1,\varepsilon} - 3\varepsilon^{-1} d_\varepsilon$$ 

$$\lambda_\varepsilon^{(2)} = \varepsilon^{-3/2} f_{2,\varepsilon} \qquad \lambda_\varepsilon^{(0)} = \varepsilon^{-5/2} f_{0,\varepsilon} - \varepsilon^{-3/2} f_{2,\varepsilon}\, d_\varepsilon$$  $$- 3\varepsilon^{-1} \tilde{d}_\varepsilon$$  $$- 3\varepsilon^{-1} \hat{d}_\varepsilon$$ 

has a finite limit $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \in \mathbb{R}^4$ as $\varepsilon \to 0$.

Theorem (Furlan, G, 2017) *The family of random fields $(u_\varepsilon)_\varepsilon$ converge in law and locally in time to a limiting random field $u(\lambda)$ in the space $C_T \mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3)$.*

The law of $u(\lambda)$ depends only on the value of λ and not on the other details of the nonlinearity or on the covariance of the noise term.

▷ The limit manifold $(u(\lambda))_\lambda$ contains the critical trajectory from $\Gamma(0)$ to WF. Called also the dynamic Φ_3^4 model with parameter vector $\lambda \in \mathbb{R}^4$.

▷ Proven for Pol/Gaussian by Hairer and Xu (2016), for Pol/Non-Gauss by Xu and Shen. Non-pol/Gaussian Furlan, G. (2017).

Hapern–Huang directions

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Fixed-Point Structure of Scalar Fields

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(Received 1 July 1994)

We search for alternatives to the trivial ϕ^4 field theory by considering nonpolynomial potentials. Such theories are renormalizable when the natural cutoff dependences of the coupling constants are taken into account. We find a continuum of fixed points, which includes the well-known Gaussian fixed point. The fixed-point density has a maximum at a location corresponding to a theory with a Higgs boson mass of approximately 2700 GeV. The Gaussian fixed point is UV stable in some directions in the extended parameter space. Along such directions we obtain nontrivial asymptotically free theories.

PACS numbers: 11.10.Hi, 11.10.Kk, 11.10.Lm

▷ Halpern and Huang theorized about possible non-polynomial relevant and asymptotically free directions at the Gaussian fp.

$$F(u) \propto \exp(c_d(d-2)u^2)$$

▷ The status of this proposal is not clear to me, some objection moved by Morris & C.

- Halpern, Kenneth, and Kerson Huang. “Halpern and Huang Reply:” *Physical Review Letters* 77, no. 8 (August 19, 1996): 1659–1659.
- Morris, Tim R. “Comment on “Fixed-Point Structure of Scalar Fields”.” *Physical Review Letters* 77, no. 8 (August 19, 1996): 1658–1658.
- Bridle, I. Hamzaan, and Tim R. Morris. “Fate of Nonpolynomial Interactions in Scalar Field Theory.” *Physical Review D* 94, no. 6 (September 28, 2016): 065040.

▷ Rigorous techniques can help to rule out such directions (my current guess).

▷ Taylor expansion

$$\begin{aligned}\mathscr{L}u_\varepsilon \;=\; & \eta_\varepsilon - \Phi^{(0)} - \Phi^{(1)}v_\varepsilon - \frac{1}{2}\Phi^{(2)}v_\varepsilon^2 - \frac{1}{6}\Phi^{(3)}v_\varepsilon^3 - R_\varepsilon(v_\varepsilon)\\ & -\varepsilon^{-3/2}f_{0,\varepsilon} - \varepsilon^{-1}f_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2}f_{2,\varepsilon}([\![Y_\varepsilon^2]\!] + 2v_\varepsilon Y_\varepsilon + v_\varepsilon^2).\end{aligned}$$

▷ Stochastic driving terms

$$\begin{aligned}
\mathscr{L}Y_\varepsilon^{\Psi} &:= \Phi^{(0)}, & \tilde{Y}_\varepsilon^{\vee} &:= \varepsilon^{-1/2}f_{2,\varepsilon}\,[\![Y_\varepsilon^2]\!],\\
Y_\varepsilon^{\vee} &:= \frac{1}{3}\Phi^{(1)} & Y_\varepsilon^{[\Psi,\mid]} &:= Y_\varepsilon^{\Psi}\circ Y_\varepsilon^{\mid} - d_\varepsilon^{[\Psi,\mid]},\\
\mathscr{L}Y_\varepsilon^{\mathrm{Y}} &:= Y_\varepsilon^{\vee}, & Y_\varepsilon^{[\Psi,\vee]} &:= Y_\varepsilon^{\Psi}\circ Y_\varepsilon^{\vee} - d_\varepsilon^{[\Psi,\vee]}Y_\varepsilon - \hat{d}_\varepsilon^{[\Psi,\vee]},\\
Y_\varepsilon^{\mid} &:= \frac{1}{6}\Phi^{(2)}, & Y_\varepsilon^{[\mathrm{Y},\vee]} &:= Y_\varepsilon^{\mathrm{Y}}\circ Y_\varepsilon^{\vee} - d_\varepsilon^{[\mathrm{Y},\vee]},\\
Y_\varepsilon^{\varnothing} &:= \frac{1}{6}\Phi^{(3)}, & \tilde{Y}_\varepsilon^{[\mathrm{Y},\vee]} &:= \tilde{Y}_\varepsilon^{\mathrm{Y}}\circ Y_\varepsilon^{\vee} - \tilde{d}_\varepsilon^{[\mathrm{Y},\vee]},
\end{aligned}$$

$Y_\varepsilon^\tau \in C_T\mathscr{C}^{\lvert\tau\rvert-\kappa}$		$Y_\varepsilon^{\varnothing}$	$Y_\varepsilon^{\mid}$	$Y_\varepsilon^{\vee}$	$\tilde{Y}_\varepsilon^{\vee}$	Y_ε^{Ψ}	$Y_\varepsilon^{[\Psi,\mid]}$	$Y_\varepsilon^{[\mathrm{Y},\vee]}$	$\tilde{Y}_\varepsilon^{[\mathrm{Y},\vee]}$	$Y_\varepsilon^{[\Psi,\vee]}$
$\lvert\tau\rvert$	=	0	$-1/2$	-1	-1	$1/2$	0	0	0	$-1/2$

$$\mathscr{L} v_\varepsilon = -Y_\varepsilon^{\Psi} - \tilde{Y}_\varepsilon^{\vee} - 3Y_\varepsilon^{\vee} v_\varepsilon - 3Y_\varepsilon^{\dot{|}} v_\varepsilon^2 - Y_\varepsilon^{\varnothing} v_\varepsilon^3$$
$$-\varepsilon^{-5/2} f_{0,\varepsilon} - \varepsilon^{-2} f_{1,\varepsilon}(Y_\varepsilon + v_\varepsilon) - \varepsilon^{-3/2} f_{2,\varepsilon}(2Y_\varepsilon v_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon)$$

- ▷ Paracontrolled Ansatz (a change of unknowns $v_\varepsilon \to v_\varepsilon^\sharp$)

$$v_\varepsilon = -Y_\varepsilon^{\Psi} - \tilde{Y}_\varepsilon^{\Upsilon} - 3v_\varepsilon \prec Y_\varepsilon^{\Upsilon} + v_\varepsilon^\sharp, \qquad \varphi_\varepsilon = v_\varepsilon + Y_\varepsilon^{\Psi}$$

- ▷ Renormalized products

$$Y_\varepsilon^{\vee} \hat{\diamond} v_\varepsilon := v_\varepsilon Y_\varepsilon^{\vee} - v_\varepsilon \prec Y_\varepsilon^{\vee} + (3\, v_\varepsilon\, d_\varepsilon^{\tau_1} + d_\varepsilon^{\tau_2} Y_\varepsilon + \hat{d}_\varepsilon^{\tau_2} + \tilde{d}_\varepsilon^{\tau_1})$$
$$= v_\varepsilon \succ Y_\varepsilon^{\vee} - \tilde{Y}_\varepsilon^{\tau_1} - Y_\varepsilon^{\tau_2} - 3\, v_\varepsilon\, Y_\varepsilon^{\tau_1} + v_\varepsilon^\sharp \circ Y_\varepsilon^{\vee} - 3\, \overline{\mathrm{com}}_1(v_\varepsilon, Y_\varepsilon^{\Upsilon}, Y_\varepsilon^{\vee})$$
$$v_\varepsilon \diamond Y_\varepsilon := v_\varepsilon Y_\varepsilon + d_\varepsilon^{\tau_3} = \varphi_\varepsilon Y_\varepsilon - Y_\varepsilon^{\Psi} \prec Y_\varepsilon - Y_\varepsilon^{\Psi} \succ Y_\varepsilon - Y_\varepsilon^{\tau_3}$$
$$Y_\varepsilon^{\dot{|}} \diamond (Y_\varepsilon^{\Psi})^2 := Y_\varepsilon^{\dot{|}} (Y_\varepsilon^{\Psi})^2 - 2d_\varepsilon^{\tau_3} Y_\varepsilon^{\Psi}$$
$$Y_\varepsilon^{\dot{|}} \diamond v_\varepsilon^2 := Y_\varepsilon^{\dot{|}} v_\varepsilon^2 + 2d_\varepsilon^{\tau_3} v_\varepsilon = Y_\varepsilon^{\dot{|}} \diamond (Y_\varepsilon^{\Psi})^2 - 2\,(Y_\varepsilon^{\dot{|}} \diamond Y_\varepsilon^{\Psi})\,\varphi_\varepsilon + Y_\varepsilon^{\dot{|}} \varphi_\varepsilon^2$$

$$\mathbb{Y}_\varepsilon \to \mathbb{Y}(\lambda)$$

$$\mathbb{Y}_\varepsilon \;:=\; (Y_\varepsilon^{\varnothing}, Y_\varepsilon^{\bullet}, Y_\varepsilon^{\vee}, \tilde{Y}_\varepsilon^{\vee}, Y_\varepsilon^{\Psi}, Y_\varepsilon^{\Psi\bullet}, Y_\varepsilon^{\vee\vee}, \tilde{Y}_\varepsilon^{\vee\vee}, Y_\varepsilon^{\Psi\vee})$$

$$\mathbb{Y}(\lambda) := (\lambda^{(3)}, \lambda^{(3)}X, \lambda^{(3)}X^{\vee}, \lambda^{(2)}X^{\vee}, \lambda^{(3)}X^{\Psi}, (\lambda^{(3)})^2 X^{\Psi\bullet}, (\lambda^{(3)})^2 X^{\vee\vee}, \lambda^{(3)}\lambda^{(2)} X^{\vee\vee}, (\lambda^{(3)})^2 X^{\Psi\vee})$$

$$\begin{aligned}
\mathscr{L}X &:= \xi \\
X^{\Psi} &:= [\![X^3]\!], \\
X^{\vee} &:= [\![X^2]\!], \\
\Delta_q X^{\Psi\bullet} &:= \Delta_q(X^{\Psi} \circ X) = \int_{\zeta_1,\zeta_2} [\![X^3_{\zeta_1}]\!] X_{\zeta_2}\, \mu_{\zeta_1,\zeta_2}, \\
\Delta_q X^{\vee\vee} &:= \Delta_q(1-J_0)(X^{Y} \circ X^{\vee}) = \int_{\zeta_1,\zeta_2} (1-J_0)([\![X^2_{\zeta_1}]\!][\![X^2_{\zeta_2}]\!])\, \mu_{\zeta_1,\zeta_2}, \\
\Delta_q X^{\Psi\vee} &:= \int_{\zeta_1,\zeta_2} (1-J_1)([\![X^3_{\zeta_1}]\!][\![X^2_{\zeta_2}]\!])\, \mu_{\zeta_1,\zeta_2} + 6\int_{s,x} [\Delta_q X(t+s,\bar{x}-x) - \Delta_q X(t,\bar{x})] P_s(x)\, [C_X(s,x)]^2,
\end{aligned}$$

▷ Malliavin calculus $D, \delta, L = -\delta D$, $Q_1^n := \prod_{k=1}^n (k-L)^{-1}$:

$$\begin{aligned}\Phi_\zeta^{(m)} &= \sum_{k=0}^{n-1} \frac{\mathbb{E}\big(\Phi_\zeta^{(m+k)}\big)}{k!} [\![Y_{\varepsilon,\zeta}^k]\!] + \delta^n\big(Q_1^n \Phi_\zeta^{(m+n)} h_\zeta^{\otimes n}\big)\\ &= \sum_{k=0}^{n-1} \varepsilon^{(m+k-5)/2} \frac{(m+k)!}{k!} \tilde{f}_{m+k,\varepsilon} [\![Y_{\varepsilon,\zeta}^k]\!] + \delta^n\big(Q_1^n \Phi_\zeta^{(m+n)} h_\zeta^{\otimes n}\big)\end{aligned}$$

▷ BDG–like estimates

$$\begin{aligned}&\Big\| \int_\zeta \hat{\Phi}_\zeta^{(m)} \mu_\zeta \Big\|_{L^p(\Omega)}\\ = \;& \Big\| \delta^{4-m} \int_\zeta Q_1^{4-m} \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m} \mu_\zeta \Big\|_{L^p(\Omega)} \leqslant \Big\| Q_1^{4-m} \int_\zeta \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m} \mu_\zeta \Big\|_{\mathbb{D}^{4-m,p}}\\ \lesssim \;& \sum_{k=0}^{4-m} \Big\| \mathrm{D}^k Q_1^{4-m} \int_\zeta \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m} \mu_\zeta \Big\|_{L^p(\Omega)} \lesssim \Big\| \Big\| \int_\zeta \Phi_\zeta^{(4)} h_\zeta^{\otimes 4-m} \mu_\zeta \Big\|_{H^{\otimes 4-m}}^2 \Big\|_{L^{p/2}(\Omega)}^{1/2}\\ \lesssim \;& \Big\| \int_\zeta \Phi_\zeta^{(4)} \Phi_{\zeta'}^{(4)} \langle h_\zeta^{\otimes 4-m}, h_{\zeta'}^{\otimes 4-m} \rangle_{H^{\otimes 4-m}} \mu_\zeta \mu_{\zeta'} \Big\|_{L^{p/2}(\Omega)}^{1/2}\\ \lesssim \;& \Big[\int_{\zeta,\zeta'} \big\| \Phi_\zeta^{(4)} \Phi_{\zeta'}^{(4)} \big\|_{L^{p/2}(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{4-m} |\mu_\zeta \mu_{\zeta'}| \Big]^{1/2}\\ \lesssim \;& \Big[\varepsilon \int_{\zeta,\zeta'} \Big\| \varepsilon^{-\frac{1}{2}} \Phi_\zeta^{(4)} \Big\|_{L^p(\Omega)} \Big\| \varepsilon^{-\frac{1}{2}} \Phi_{\zeta'}^{(4)} \Big\|_{L^p(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{4-m} |\mu_\zeta \mu_{\zeta'}| \Big]^{\frac{1}{2}}\\ \lesssim \;& \Big[\varepsilon^\delta \int_{\zeta,\zeta'} \Big\| \varepsilon^{-\frac{1}{2}} \Phi_\zeta^{(4)} \Big\|_{L^p(\Omega)} \Big\| \varepsilon^{-\frac{1}{2}} \Phi_{\zeta'}^{(4)} \Big\|_{L^p(\Omega)} |\langle h_\zeta, h_{\zeta'} \rangle|^{3-m+\delta} |\mu_\zeta \mu_{\zeta'}| \Big]^{\frac{1}{2}},\end{aligned}$$

▷ Partial contractions for products of local operators

$$\begin{aligned}
\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\big] + \delta Q_1 \mathrm{D}\,(\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}),\\
\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + \delta Q_1 \mathrm{D}\,(\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}),\\
\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2} &= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + \delta\big[J_0 \mathrm{D}\big(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big)\big] + \delta^2 Q_1^2 \mathrm{D}^2(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2})\\
&= \mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + Y_\varepsilon(\zeta_1)\mathbb{E}\big[\Phi^{(1)}_{\zeta_1}\Phi^{(1)}_{\zeta_2}\big] + Y_\varepsilon(\zeta_2)\mathbb{E}\big[\Phi^{(0)}_{\zeta_1}\Phi^{(2)}_{\zeta_2}\big] + \delta^2 Q_1^2 \mathrm{D}^2(\Phi^{(0)}_{\zeta_1}\Phi^{(1)}_{\zeta_2})
\end{aligned}$$

▷ Partial expansion for contractions

$$\begin{aligned}
\mathbb{E}\big[\Phi^{(m)}_{\zeta_1}\Phi^{(n)}_{\zeta_2}\big] &= \frac{3!^2}{(3-m)!(3-n)!}(\varepsilon^{-1} f_{3,\varepsilon})^2\,\mathbb{E}[[\![Y^{3-m}_{\varepsilon,\zeta_1}]\!][\![Y^{3-n}_{\varepsilon,\zeta_2}]\!]] + \frac{3!}{(3-m)!}\varepsilon^{-1} f_{3,\varepsilon}\,\mathbb{E}\big[[\![Y^{3-m}_{\varepsilon,\zeta_1}]\!]\hat{\Phi}^{(n)}_{\zeta_2}\big]\\
&\quad + \frac{3!}{(3-n)!}\varepsilon^{-1} f_{3,\varepsilon}\,\mathbb{E}\big[[\![Y^{3-n}_{\varepsilon,\zeta_2}]\!]\hat{\Phi}^{(m)}_{\zeta_1}\big] + \mathbb{E}\big[\hat{\Phi}^{(m)}_{\zeta_1}\hat{\Phi}^{(n)}_{\zeta_2}\big],
\end{aligned}$$

▷ Control of remainders

$$\begin{aligned}
&\hat{\Phi}^{(4-m)}_{\zeta_1}\hat{\Phi}^{(4-n)}_{\zeta_2}\\
&= \delta^m(Q_1^m \Phi^{(4)}_{\zeta_1} h^{\otimes m}_{\zeta_1})\delta^n\big(Q_1^n \Phi^{(4)}_{\zeta_2} h^{\otimes n}_{\zeta_2}\big)\\
&= \sum_{(q,r,i)\in I} C_{q,r,i}\,\varepsilon^{1+\frac{r+q}{2}-i}\,\delta^{m+n-q-r}(\langle \Theta^{m+r-i}_{1+r-i}(\zeta_1) h^{\otimes m+r-i}_{\zeta_1}, \Theta^{n+q-i}_{1+q-i}(\zeta_2) h^{\otimes n+q-i}_{\zeta_2}\rangle_{H^{\otimes q+r-i}})
\end{aligned}$$

Thanks.