The regularizing effects of Irregular functions

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Regularization by noise in ODEs/PDEs:

Addition of noise has positive effects on the theory of the equation (in some pathwise sense)

→ ODEs:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + W_t$$

where (W_t) is a BM in $\mathbb{R}^d$ and b a less-than-Lipshitz vectorfield. Many results: Veretennikov, Davie, Krylov-Röckner, Flandoli, Attanasio, Fedrizzi, Proske, ... Essentially: bounded b: (in L^∞ or with some particular integrability: LPS condition).

→ Transport equation:

$$\mathrm{d}_t u(t,x) + b(x)\cdot\nabla u(t,x)\mathrm{d}t = \nabla u(t,x)\cdot \mathrm{d}W_t$$

good theory for L^∞ solutions and preservation of regularity. Flandoli–G.–Priola, Flandoli–Attanasio, Flandoli–Maurelli, Flandoli–Beck–G.–Maurelli

→ Some other PDE: Vlasov–Poisson, point vortices in 2d.

We want to provide a deterministic framework to discuss regularization.

- A notion of irregular functions.
- The averaging operator along irregular functions.
- Non-linear Young integral and ODEs.
- Regularization by irregular functions in the linear transport equation.
- Regularization by irregular functions in dispersive equations: NLS & KdV.

Given a function $w: [0,1] \to \mathbb{R}^d$ define the *averaging operator*

$$T_t^w f(x) = \int_0^t f(x + w_s) \mathrm{d}s, \qquad T_{t,s}^w f = T_t^w f - T_s^w f$$

acting on functions (or distributions) $f: \mathbb{R}^d \to \mathbb{R}$.

▷ $d = 1$, $w_t = t$. Then if $F'(x) = f(x)$ we have $T_t^w f(x) = \int_0^t F'(x+s)\mathrm{d}s = F(x+t) - F(x)$ and $T^w: L^\infty \to \mathrm{Lip}$:

$$|T_t^w f(x) - T_t^w f(y)| \leqslant \|f\|_\infty |x-y|, \qquad |T_{t,s}^w f(x)| \leqslant \|f\|_\infty |t-s|$$

▷ Tao–Wright: if w "wiggles enough" then T_t^w maps L^q into $L^{q'}$ with $q' > q$.

▷ Davie: if w is a sample of BM then a.s. (the exceptional set depends on f)

$$|T_{t,s}^w f(x) - T_{t,s}^w f(y)| \leqslant C_w \|f\|_\infty |x-y|^{1-} |t-s|^{1/2-}$$

Problem: study the mapping properties of T^w for w the sample path of a stochastic process.

Consider

$$Y_t^w(\xi) = \int_0^t e^{i\langle \xi, w_s \rangle} \mathrm{d}s$$

then $T_t^w f = \mathcal{F}^{-1}(Y_t^w \mathcal{F}(f))$.

Mapping properties of T^w in $(H^s)_{s\in\mathbb{R}}$ spaces can be discussed in terms of Y^w: $\|T_{t,s}^w f\|_{H^s} = \left\|(1+\xi^2)^{s/2} Y_{t,s}^w(\xi) \mathcal{F}f(\xi)\right\|_{H_\xi^s}$.

In our setting more convenient to look at the scale $(\mathcal{F}L^\alpha)_\alpha$.

$$\|f\|_{\mathcal{F}L^\alpha} = \int |f(\xi)|(1+\xi^2)^{\alpha/2} \mathrm{d}\xi$$

since $C^\alpha \subseteq \mathcal{F}L^\alpha$.

Definition 1. *We say that w is (ρ,γ)–irregular if there exists a constant K for which*

$$|Y_{t,s}^w(\xi)| \leqslant K(1+|\xi|)^{-\rho}|t-s|^\gamma$$

for $\xi \in \mathbb{R}^d$ and $0 \leqslant s \leqslant t \leqslant 1$.

Where we find irregularity?

▷ The fBM of Hurst index H is ρ–irregular for any $\rho < 1/2H$. (Catellier-G.)

$\Rightarrow$ there exists functions of arbitrarily high irregularity and arbitrarily L^∞-near any given continuous function.

▷ An irregular function cannot be too regular.

If $w \in C^\theta$ with $\alpha\theta + \gamma > 1$ and $\alpha \in [0,1]$, using the Young integral, we find

$$|t-s| = |e^{ia}(t-s)| = \left| \int_s^t \underbrace{e^{ia - iaw_r}}_{C^{\alpha\theta}} \mathrm{d}_r \underbrace{Y_r^w(a)}_{C^\gamma} \right|$$

$$\leqslant C\, K_w\, (|t-s|^\gamma + |t-s|^{\alpha\theta+\gamma} |a|^\alpha) \|w\|_\theta (1+|a|)^{-\rho} \to 0$$

if $t > s$ and $\alpha < \rho$. This implies that is not possible that $\theta > (1-\gamma)/\rho$.

▷ Not easy to say if a function is irregular.

▷ In $d=1$ smooth functions are (ρ,γ) irregular for $\rho+\gamma=1$. In particular if we insist on $\gamma>1/2$ we have $\rho<1/2$.

▷ For $d>1$ smooth functions are not irregular: if $|t-s|\ll 1$

$$\int_s^t e^{i\langle a,w_r\rangle}\mathrm{d}r \simeq \int_s^t e^{i\langle a,w'_s\rangle(t-s)}\mathrm{d}r \simeq (1+|\langle a,w'_s\rangle|)^{-1} \not\lesssim (1+|a|)^{-\rho}.$$

▷ If w is ρ–irregular and φ is a C^1 perturbation then $w+\varphi$ is at least $\rho-(1-\gamma)$ irregular since:

$$Y^{w+\varphi}_{t,s}(\xi)=\int_s^t e^{i\langle\xi,w_r+\varphi_r\rangle}\mathrm{d}r=\int_s^t e^{i\langle\xi,\varphi_r\rangle}\mathrm{d}_r Y^w_{s,r}(\xi)$$

and we can use Young integral estimates.

▷ If W is a fBM and Φ an adapted smooth perturbation then $W+\Phi$ is as irregular as W (via Girsanov theorem).

If w is ρ–irregular then

$$T^w \colon H^s \rightarrow H^{s+\rho}$$

and

$$T^w \colon \mathcal{F}L^\alpha \rightarrow \mathcal{F}L^{\alpha+\rho}$$

Indeed

$$\|T^w_{t,s} f\|_{\mathcal{F}L^{\alpha+\rho}} = \int \mathrm{d}\xi \, (1+|\xi|)^{\alpha+\rho} |Y^w_{t,s}(\xi)(\mathcal{F}f)(\xi)|$$

$$\leqslant K_w |t-s|^\gamma \int \mathrm{d}\xi \, (1+|\xi|)^\alpha |(\mathcal{F}f)(\xi)| = K_w |t-s|^\gamma \|f\|_{\mathcal{F}L^\alpha}.$$

ODEs and the averaging operator

In order to exploit the averaging properties of w in the study of the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

we rewrite it in order to make the action of the averaging operator explicit: let $\theta_t = x_t - w_t$:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s)\mathrm{d}s = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

where $G_s(x) = T_s^w b(x)$ so that $\mathrm{d}_s G_s(x) = f(w_s + x)$.

If we assume that G is C^γ in time ($\gamma > 1/2$) with values in a space of regular enough functions we can study this equation as a Young type equation for $\theta \in C^\gamma$.

▷ **Non-linear Young integral:**

$$\int_0^t (\mathrm{d}_s G_s)(\theta_s) = \lim_{\Pi} \sum_i G_{t_{i+1},t_i}(\theta_{t_i})$$

This limit exists if $\theta \in C^\gamma$ and $G \in C_t^\gamma C_x^\nu$ with $\gamma(1+\nu) > 1$. The integral is in C_t^γ.

The integral equation

$$\theta_t = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

is well defined for $\theta \in C^\gamma$ and $G \in C_t^\gamma C_x^\nu$ with $(1+\nu)\gamma > 1$.

- Existence of global solutions.
- Uniqueness if $G \in C_t^\gamma C_x^{\nu+1}$ and differentiable flow.
- Smooth flow if $G \in C_t^\gamma C_x^{\nu+k}$.

▷ The equation

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

has a unique solution for w ρ–irregular and $b \in \mathcal{F}L^\alpha$ for $\alpha > 1-\rho$. In this case we can take $\theta \in C^1$ above and the condition for uniqueness is $G \in C_t^\gamma C_x^{1+}$.

Say that x is controlled by w if $\theta = x - w \in C^{\gamma}$. In this case we have

$$I_x(b) = \int_0^t b(x_s)\mathrm{d}s = \int_0^t (\mathrm{d}_s T_s^w b)(\theta_s)$$

and the r.h.s. is well defined as soon as $T^w b \in C_t^{\gamma} C_x^{\nu}$.

If w is ρ irregular and $b \in \mathcal{F}L^{\alpha}$ then $T^w b \in C_t^{\gamma} \mathcal{F}L_x^{\alpha+\rho}$ so if $\alpha + \rho \geqslant \nu$ we have $T^w b \in C_t^{\gamma} C_x^{\nu}$.

In this case $I_x(b)$ can be extended by continuity to all $b \in \mathcal{F}L^{\alpha}$ and in particular we have given a meaning to

$$\int_0^t b(x_s)\mathrm{d}s$$

when b is a distribution *provided* x is controlled by a ρ-irregular path.

For controlled paths the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

make sense even for certain distributions b as a Young equation for θ.

(Work in progress with Catellier)

We want to give a meaning to the transport equation

$$(\partial_t + b(x)\cdot\nabla + \partial_t w_t \cdot \nabla) u(t,x) = 0$$

for $u \in L^\infty$ and $w \in C^\sigma$ with $\sigma > 1/2$ (simplest case, possible to remove this condition).

Weak formulation: $u_t(\varphi) = \int \mathrm{d}x\, \varphi(x)\, u(t,x)$.

$$u_{t,s}(\varphi) = \int_s^t u_r(\nabla\cdot(b\varphi))\mathrm{d}r + \int_s^t u_r(\nabla\varphi)\mathrm{d}_r w_r$$

We assume that $u_t(\nabla\varphi) \in C_t^\sigma$ so that the last integral is a Young integral.

Using the flow of the Young equation it is possible to show existence and uniqueness of solutions to this equation for b which are not necessarily Lipshitz (for example just in $\mathcal{F}L^\alpha$ for $\alpha \in (0,1)$).

For the moment only in the case $\operatorname{div} b = 0$.

Two simple dispersive models with ρ-irregular modulation w:

- **Non-linear Schödinger equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$

$$\partial_t \varphi(t,x) = i\Delta\varphi(t,x)\partial_t w_t + i|\varphi(t,x)|^{p-2}\varphi(t,x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$

$$\partial_t u(t,x) = \partial_x^3 u(t,x)\partial_t w_t + \partial_x(u(t,x))^2.$$

To be compared to the non-modulated setting where $\partial_t w_t = 1$ and studied in the scale of $(H^s)_s$ spaces.

The equations are understood in the mild formulation

$$u(t) = \mathcal{U}_t^w u(0) + \int_0^t \mathcal{U}_t^w (\mathcal{U}_s^w)^{-1} \partial_x (u(s))^2 \mathrm{d}s.$$

with $\mathcal{U}_t^w = e^{i w_t \partial_x^3}$. (similarly for NLS). Here w can be an arbitrary continuous function.

Rewrite the mild formulation as

$$v(t) = (\mathcal{U}_t^w)^{-1} u(t) = u(0) + \int_0^t (\mathrm{d}_s X_s)(v(s))$$

where X is the bi-linear operator

$$X_t(\varphi) = X_t(\varphi, \varphi) = \int_0^t (\mathcal{U}_s^w)^{-1} \partial_x (\mathcal{U}_s^w \varphi)^2 \mathrm{d}s.$$

If w is ρ irregular then $X \in C^\gamma \mathrm{Lip}_{\mathrm{loc}}(H^\alpha)$ for $\alpha > -\rho$ and $\rho > 3/4$.

The above equation has local solutions for initial conditions in H^α with locally Lipshitz flow. Uniqueness in $C^\gamma H^\alpha$ (for v).

$\Rightarrow$ Regularization by modulation. In the non-modulated case it is known that there cannot be continous flow for $a \leqslant -1/2$ on $\mathbb{T}$ and $\alpha \leqslant -3/4$ on $\mathbb{R}$.

$\triangleright$ Global solutions thanks to the L^2 conservation and smoothing for $\alpha > 0$ or an adaptation of the I-method for $-3/2 \leqslant \alpha < 0$ and $\alpha > -\rho/(3 - 2\gamma)$.

$\triangleright$ **NLS:** global solutions for $\alpha \geqslant 0$ and $\rho > 1/2$.

A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same ρ-irregularity assumption.

Theorem 2. *Let* $T>0$*,* $p\in(2,5]$*,*$\rho>\min\left(\frac{3}{2}-\frac{2}{p},1\right)$ *then there exists a finite constant* $C_{w,T}>0$ *and* $\gamma^{\star}(p)>0$ *such that the following inequality holds:*

$$\left|\left|\int_0^{\cdot} U_{\cdot}(U_s)^{-1}\,\psi_s\,d\,s\right|\right|_{L^p([0,T],L^{2p}(\mathbb{R}))}\leq C_w\,T^{\gamma^{\star}(p)}||\,\psi\,||_{L^1([0,T],L^2(\mathbb{R}))}$$

for all $\psi\in L^1([0,T],L^2(\mathbb{R}))$.

As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity $i\, e{:}\mathcal{N}(\phi) = |\phi|^{\mu}\, \phi$: (Debussche–de Bouard, Debussche–Tsutsumi)

Theorem 3. *Let* $\mu \in (1, 4]$, $p = \mu + 1$, $\rho > \min(1, 3/2 - \frac{2}{p})$ *and* $u^0 \in L^2(\mathbb{R})$ *then there exists* $T^\star > 0$ *and a unique* $u \in L^p([0, T], L^{2p}(\mathbb{R}))$ *such that the following equality holds:*

$$u_t = U_t\, u^0 + i \int_0^t U_t (U_s)^{-1} (|u_s|^{\mu}\, u_s)\, d\, s$$

for all $t \in [0, T^\star]$*. Moreover we have that* $||\, u_t||_{L^2(\mathbb{R})} = ||\, u_0||_{L^2(\mathbb{R})}$ *and then we have a global unique solution* $u \in L^p_{loc}([0, +\infty), L^{2p}(\mathbb{R}))$ *and* $u \in C([0, +\infty), L^2(\mathbb{R}))$*. If* $u^0 \in H^1(\mathbb{R})$ *then* $u \in C([0, \infty), H^1(\mathbb{R}))$*.*

Thanks.