Paracontrolled distributions (with applications to singular SPDEs)

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Controlled paths/distributions

Controlled paths are paths which “looks like” a *given* path which often is random (but not necessarily).

A "good" quantification of this proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

A mix of functional analytic arguments and probabilistic ones.

Basic analogies

- Itô processes
$$\mathrm{d}X_t = f_t \mathrm{d}M_t + g_t \mathrm{d}t$$
- Amplitude modulation
$$f(t) = g(t)\sin(\omega t)$$
with $|\mathrm{supp}\,\hat{g}| \ll \omega$.

Some applications (not covered by this talk)

▷ **Rough paths.** CP linearize of the space of rough paths. Separation of definition of the rough integral from the solution of the rough differential equation. (⇒ lecture notes of Friz and Hairer)

▷ **Regularization by noise in ODEs.** CP describes of the local behavior of paths which enjoy special regularization properties. (⇒ Catellier and Gubinelli, arXiv)

▷ **Stochastic Burgers equation with derivative white noise.** CP define a space of stochastic distributions for which some non-linear term can be defined and solutions to the equation found (but no uniqueness). (⇒ Gubinelli and Jara, SPDEs Analysis and Computation)

▷ **Modulated dispersive PDEs.** Dispersion in a non-linear PDE (Nonlinear Schrödinger, Kortevew–de-Vries) is modulated by an irregular signal. The space in which to solve the equation is a space of CP. (⇒ Chouk and Gubinelli, arXiv)

▷ **Korteweg–de Vries equation with distributional initial condition** Standard KdV in negative Sobolev spaces. CP provide a theory alternative to that of Bourgain spaces. (⇒ Gubinelli, CPAA)

Some problems in singular SPDEs /I

Define and solve (locally) the following SPDEs:

- Stochastic differential equations (1+0): $u \in [0,T] \to \mathbb{R}^n$

$$\partial_t u(t) = \sum_i f_i(u(t))\xi^i(t)$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.

- Burgers equations (1+1): $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))Du(t,x) + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

- Generalized Parabolic Anderson model (1+2): $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(x)$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.

Recall that

$$\xi \in \mathscr{C}^{-d/2-}$$

Some problems in singular SPDEs /II

Define and solve (locally) the following SPDEs:

- Kardar-Parisi-Zhang equation (1+1)

$$\partial_t h(t,x) = \Delta h(t,x) + "(Du(t,x))^2 - \infty" + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

- Stochastic quantization equation (1+3)

$$\partial_t u(t,x) = \Delta u(t,x) + "u(t,x)^3" + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.

- But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Joint work with P. Imkeller and N. Perkowski.
(Also K. Chouk and R. Catellier for $(\Phi)^4_3$).

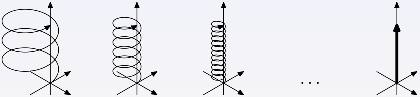
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2}$$



$$I(x^{n,1}, x^{n,2})(t) \not\to I(0,0)(t) = 0$$

The definite integral $I(\cdot,\cdot)(t)$ is not a continuous map $C^\gamma \times C^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Rough differential equation

Consider the simple controlled ODE (η smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^{m} f_i(u(t))\eta^i(t)$$

$u : \mathbb{R} \to \mathbb{R}^d$, $\eta : \mathbb{R} \to \mathbb{R}^d$ and smooth vectorfields $f_i : \mathbb{R}^d \to \mathbb{R}^d$.

Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for $\eta \in \mathscr{C}^{\gamma-1}$ with $\gamma < 1/2$.

Reason: $u \in \mathscr{C}^{\gamma}$ and $\eta \in \mathscr{C}^{\gamma-1}$ cannot be multiplied when $2\gamma - 1 < 0$. The r.h.s. of the equation is not well defined.

Here $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ is the Holder–Besov space (or a local version).

Concept of solution

Goal: Show that Ψ factorizes as

$$\eta \xrightarrow{J} (\eta, \theta \circ \eta) \xrightarrow{\Phi} u$$

(here $\partial_t \theta = \eta$ and $\theta \circ \eta = X^2(\eta)$ will be described later)

▷ *Analytic step:* show that when $\gamma > 1/3$:

$$\Phi : \mathcal{X} \to \mathscr{C}^\gamma$$

is continous. $\mathcal{X} = \overline{\mathrm{Im} J} \subseteq \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ is the space of *enhanced signals* (or rough paths, or models).
But in general J is not a continuous map $\mathscr{C}^{\gamma-1} \to \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$.

▷ *Probabilistic step:* prove that there exists a "reasonable definition" of $J(\xi)$ when ξ is a white noise. $J(\xi)$ is an explicit polinomial in ξ so direct computations are possible.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $\mathscr{C}^\gamma = B^\gamma_{\infty,\infty}$.

$f \in \mathscr{C}^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \leqslant \|f\|_\gamma 2^{-i\gamma}, \qquad i \geqslant -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi)\hat{f}(\xi)$$

where $\rho_i : \mathbb{R}^d \to \mathbb{R}_+$ are smooth functions with support in annuli $\simeq 2^i \mathscr{A}$ when $i \geqslant 0$ and form a partition of unity

$$\sum_{i \geqslant -1} \rho_i(\xi) = 1$$

for all $\xi \neq 0$ so that

$$f = \sum_{i \geqslant -1} \Delta_i f$$

in $\mathcal{S}'$.

Paraproducts

Deconstruction of a product: $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$\pi_<(f,g) \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$\pi_\circ(f,g) \in \mathscr{C}^{\gamma+\rho} \qquad \text{if } \gamma + \rho > 0$$

Proof. Recall $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$.

$$i \ll j \Rightarrow \operatorname{supp} \mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A}$$

$$i \sim j \Rightarrow \operatorname{supp} \mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i:i<j-1} O(2^{-i\rho - j\gamma}) = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^\gamma,$$

while if $\rho < 0$

$$\Delta_q(f \prec g) = \sum_{i:i<j-1} O(2^{-i\rho - j\gamma}) = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j \gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i \gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

Small detour : Young integral

Take $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$ with $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{f \prec Dg}_{\mathscr{C}^{\gamma-1}} + \underbrace{f \circ Dg + f \succ Dg}_{\mathscr{C}^{\gamma+\rho-1}}$$

then

$$\begin{aligned}\int fDg &= \underbrace{\int f \prec Dg}_{\mathscr{C}^{\gamma}} + \underbrace{\int (f \circ Dg + f \succ Dg)}_{\mathscr{C}^{\gamma+\rho}} \\ &= f \prec g + \mathscr{C}^{\gamma+\rho}.\end{aligned}$$

Compare with standard estimate for the Young integral in Hölder spaces (valid when $\gamma + \rho > 1$):

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}).$$

Expansion in smalleness of increments vs. Expansion in regularity

(Para)controlled structure

Idea

Use the paraproduct to *define* a controlled structure. We say $y \in \mathscr{D}_x^\rho$ if $x \in \mathscr{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with $y^x \in C^{\rho-\gamma}$ and $y^\sharp \in C^\rho$.

Paralinearization. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^\gamma, \gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ A first commutator: $f, g \in \mathscr{C}^{\rho-\gamma}, x \in \mathscr{C}^\gamma$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^\rho$$

Stability. $(\rho \leqslant 2\gamma)$

$$\varphi(y) = (\varphi'(y)y^x) \prec x + \mathscr{C}^\rho$$

so we can take $\varphi(y)^x = \varphi'(y)y^x$.

The main commutator

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$$

which however "is" smoother than $f \prec g$ if f or g has positive regularity.

Paraproducts decouple the problem from the source of the problem.

Commutator

The trilinear operator $C(f,g,h) = (f \prec g) \circ h - f(g \circ h)$ satisfies

$$\|C(f,g,h)\|_{\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

when $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$.

▷ The paracontrolled structure allows algebraic computations to simplify the form of the resonating terms.

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in C^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathscr{C}^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathscr{C}^{2\gamma}$$

Then

$$\varphi(B) \circ DB = (\varphi'(B) \prec B) \circ DB + \underbrace{\mathscr{C}^{2\gamma} \circ DB}_{\text{OK}}$$

$$= \varphi'(B)(B \circ DB) + \mathscr{C}^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B) \underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathscr{C}^{3\gamma-1}$$

The Besov area

The Besov area $B \circ DB$ can be defined and studied efficiently using Gaussian arguments:

$$B^{\varepsilon} \circ DB^{\varepsilon} \to B \circ DB$$

almost surely in $\mathscr{C}^{2\gamma-1}_{\text{loc}}$ as $\varepsilon \to 0$.

Remark. If $d = 1$ (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2}D(B \circ B)$$

which is well defined and belongs indeed to $\mathscr{C}^{2\gamma-1}$.

Tools: Besov embeddings $L^p(\Omega; C^{\theta}) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

RDEs - I - the r.h.s.

$u : \mathbb{R} \to \mathbb{R}^d$, $\xi \in \mathscr{C}^{-1/2-}$ is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz. Take $\partial_t X = \xi$, $X \in \mathscr{C}^{1/2-}$ and assume that $u \in \mathscr{D}^{1-}_X$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{1-}$ and $u^X \in \mathscr{C}^{1/2-}$.

▷ Paralinearization:

$$f(u) = f'(u) \prec u + \mathscr{C}^{1-} = (f'(u)u^X) \prec X + \mathscr{C}^{1-}$$

▷ Commutator lemma:

$$f(u) \circ \xi = ((f'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{1-} \circ \xi$$

$$= \underbrace{(f'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{0-}} + \underbrace{C(f'(u)u^X, X, \xi) + \mathscr{C}^{1-} \circ \xi}_{\in \mathscr{C}^{1/2-}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{0-}$.

RDEs - II - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{0-}$ we have a control on the r.h.s. of the equation:

$$f(u)\xi = \underline{f(u) \prec \xi} + f'(u)u^X(X \circ \xi) + \mathscr{C}^{1/2-}$$

What about the l.h.s.?

$$\partial_t u = \partial_t u^X \prec X + \underline{u^X \prec \xi} + \partial_t u^\sharp$$

so letting $u^X = f(u)$ we have

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f'(u)f(u)(X \circ \xi) + \mathscr{C}^{1/2-}$$

RDEs - III - the paracontrolled fixed point.

The RDE

$$\partial_t u = f(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X &= \xi \\
\partial_t u^\sharp &= (f'(u)f(u))(X \circ \xi) - \underbrace{\partial_t f(u) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f,u,X,\xi)}_{\in \mathscr{C}^{1/2-}} \circ \xi \\
u &= f(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^{1-}$ if we assume that

$$X \in \mathscr{C}^{1/2-}, \qquad (X \circ \xi) \in \mathscr{C}^{0-}.$$

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = f(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1} \to \mathscr{C}^{\gamma}$$

is continuous for any $\gamma > 1/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by the unique solution in $\mathscr{D}_X^{2\gamma}$ to

$$\begin{cases} z = f(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (f'(z)f(z))\varphi - \underbrace{\partial_t f(z) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f, z, X, \xi) \circ \xi}_{\in \mathscr{C}^{1/2-}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then

$$u^n \to u = \Phi(u_0, \xi, \eta).$$

Relaxed form of the RDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-1}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

"Itô" form of the RDE

In the smooth setting

$$u = \Phi(\xi, X \circ \xi + \varphi)$$

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

If we choose $\varphi = -X \circ \xi$ then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\partial_t v = f(v)\xi - f'(v)f(v)X \circ \xi$$

and has the particular property of being a continuous map of $\xi \in \mathscr{C}^{\gamma-1}$ alone.

Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - \mathrm{D}^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{-1-}(\mathbb{T}^2)$ space white noise.

$$\mathcal{L}u = f(u)\xi$$

▷ Paracontrolled ansatz $\mathcal{L}X = \xi$ so $X \in C([0,T], \mathscr{C}^{1-})$

$$u = f(u) \prec X + u^\sharp$$

▷ Paralinearization: $f(u) = (f'(u)f(u)) \prec X + R(f,u,X)$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f,u,X) \circ \xi$$

▷ A problem: if ξ is the white noise

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$

$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2 = c + \mathscr{C}^{0-}$$

with $c = +\infty$.

Renormalization

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denote by $\diamond$. Now

$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c)$$
$$+ C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

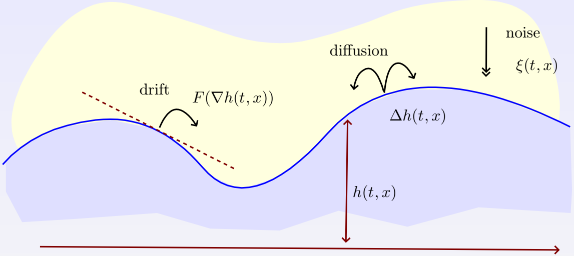
together with

$$u = f(u) \prec X + u^\sharp$$

and where

$$X \in \mathscr{C}^{1-}, \qquad (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^\sharp \in \mathscr{C}^{2-}.$$

The Kardar–Parisi–Zhang equation



Large scale dynamics of the height $h : [0,T] \times \mathbb{T} \to \mathbb{R}$ of an interface

$$\partial_t h \simeq \Delta h + F(Dh) + \xi$$

The universal limit should coincide with the large scale fluctuations of the KPZ equation

$$\partial_t h = \Delta h + [(Dh)^2 - \infty] + \xi$$

with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion /I

▹ Let $u = X + u_1$ then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▹ Let $X^{\vee}$ be the solution to

$$\mathcal{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

▹ Define $\mathcal{L}X^{\vee\!\!\vee} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\vee\!\!\vee} + u_3$ then $X^{\vee\!\!\vee} \in \mathscr{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\vee\!\!\vee}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\curlyvee} + U$$

$$\begin{aligned} \mathcal{L}U &= 2\partial_x(UX) + 2\partial_x(X^{\curlyvee}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\curlyvee} + U)X^{\vee}) + \partial_x(2X^{\curlyvee} + U)^2 \\ &= {\color{red}2\partial_x(UX)} + \mathcal{L}(2X^{\curlyvee} + X^{\vee\vee}) + 2\partial_x((2X^{\curlyvee} + U)X^{\vee}) + \partial_x(2X^{\curlyvee} + U)^2 \end{aligned}$$

and the regularities for the driving terms

X	$X^{\vee}$	$X^{\curlyvee}$	$X^{\curlyvee\!\curlyvee}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\curlyvee} + U)X^{\vee}) + \partial_x(2X^{\curlyvee} + U)^2$$

are well defined.

The remaining problem is to deal with ${\color{red}2\partial_x(UX)}$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Y + U^\sharp$. Then

$$\mathcal{L}U = \mathcal{L}U' \prec Y + U' \prec \mathcal{L}Y - \partial_x U' \prec \partial_x Y + LU^\sharp$$

while

$$\mathcal{L}U = 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\overset{\overset{\vee}{\vee}}{\vee}} + X^{\vee\vee}) + 2\partial_x((2X^{\overset{\vee}{\vee}} + U)X^{\vee}) + \partial_x(2X^{\overset{\vee}{\vee}} + U)^2}_{Q(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

so we can set $U' = 2U$ and $\mathcal{L}Y = \partial_x X$ and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Y + \partial_x U' \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

▷ Observe that $Y, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Y) \circ X + U^\sharp \circ X = 2U(Y \circ X) + \underbrace{C(2U, Y, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Y \circ X \in \mathscr{C}^{0-}$

▷ The final fixed point equation reads

$$\mathcal{L}U^\sharp = 4\partial_x(U\underline{(Y \circ X)}) + 4\partial_x C(U, Y, X) + 2\partial_x(U^\sharp \circ X) - 2LU \prec Y$$
$$+ 2\partial_x U \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + Q(U)$$

▷ This equation has a (local in time) solution $U = \Phi(J(\xi))$ which is a continuous function of the data $J(\xi)$ given by a collection of multilinear functions of ξ:

$$J(\xi) = (X, X^{\vee}, X^{\text{tree}_1}, X^{\text{tree}_2}, X^{\text{tree}_3}, X \circ Y)$$

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3$: $\xi \in C^{-5/2-}$, $u \in C^{-1/2-}$, $u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 - c_1))$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects: $(u_1^2 - c_1), (u_1^3 - 3c_1 u_1), (3\lambda(X \circ (u_1^2 - c_1)) - c_2), (u_2 u_1), (u_2^2 u_1)$

Thanks