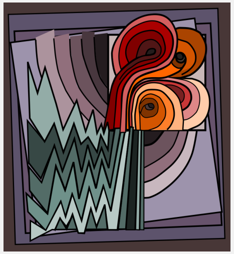
Constructive stochastic quantisation



(version 2 – Sept 12th)

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[made with TeXmacs]

the origins

Itô's original paper

(Japanese version 1942, M.A.M.S. 1951)

Differential Equations Determining a Markoff Process*

KIYOSI ITÔ

More generally, for a simple Markoff process with its states being represented by the real numbers and having continuous parameter, the problem of determining quantities corresponding to $p_{ij}^{(k)}$ mentioned above and of constructing the corresponding Markoff process once these quantities are given has been investigated systematically by Kolmogoroff[3], who reduced the problem to the study of differential equations or integro-differential equations satisfied by the transition probability function.

W. Feller[4] has proved under fairly strong assumptions that these equations possess a unique solution and furthermore that the solution exhibits the properties of transition probability function.

However, if we adopt more strict point of view such as the one J. Doob[5] has applied toward his investigation of stochastic processes, it seems to us that the aforementioned work done by Feller is not quite adequate. For example, even though the differential equation determining the transition probabil-

ity function of a continuous stochastic process was solved in §3 of that paper, no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is:

1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

§1. Definition of Differentiation of a Markoff Process

Let $\{y_t\}$ be a (simple) Markoff process and denote by F_{t_0t} the conditional probability distribution[1] of $y_t - y_{t_0}$ given that "y_{t_0} is determined". F_{t_0t} is clearly a P_{y_t}-measurable (ρ) function[2] of y_t, where ρ denotes the Lévy distance among probability distributions.

Definition 1.1.[3]
If

(1) $$F_{t_0t}^{*[1/t-t_0]}$$

(here $[a]$ is the integer part of the number a, and "$*k$" denotes the k-fold convolution) converges in probability with respect to the Lévy distance ρ as $t-t_0+0$, then we call the limit random variable (taking values in the space of probability distributions) the derivative of $\{y_t\}$ at t_0 and denote it by

(2) $$D_{t_0}\{y_t\} \text{ or } Dy_{t_0}.$$

Corollary 1.1. *Dy_{t_0} is an infinitely divisible probability distribution.*[4]

bability distribution.

Dy_{t_0} obtained above is a function of t_0 as well as of y_{t_0}, and so, we denote it by $L(t_0, y_{t_0})$ corresponds precisely to the "basic transition probability" discussed in the Introduction.

The precise formulation of the problem of Kolmogoroff, then, is to solve the equation

(4) $$Dy_t = L(t, y_t)$$

when the quantity $L(t,y)$ is given.

§2. A Comparison Theorem

Let us prove a comparison theorem for Dy_{t_0} which we shall make use of later.

Theorem 2.1. *Let $\{y_t\}$, $\{z_t\}$ be simple Markoff processes satisfying the following conditions:*

(1) $y_{t_0} = z_{t_0}$.
(2) $E(y_t - z_t \mid y_{t_0}) = o(t - t_0)$, *where o is the Landau symbol.*
(3) $\sigma(y_t - z_t \mid y_{t_0}) = o(\sqrt{t - t_0})$.

(Here $E(x|y)$ denotes the conditional expectation of x given y and $\sigma(x|y)$ denotes the conditional standard deviation of x given y. Also, the quantity o may depend on t_0 or y_{t_0}). Then, whenever Dz_{t_0} exists, Dy_{t_0} exists also, and $Dy_{t_0} = Dz_{t_0}$ holds.

H. Föllmer, "On Kiyosi Itô's Work and its Impact" (Gauss prize laudatio 2006)

In 1987 Kiyosi Itô received the Wolf Prize in Mathematics. The laudatio states that "he has given us a full understanding of the infinitesimal development of Markov sample paths. This may be viewed as Newton's law in the stochastic realm, providing a direct translation between the governing partial differential equation and the underlying probabilistic mechanism. Its main ingredient is the differential and integral calculus of functions of Brownian motion. The resulting theory is a cornerstone of modern probability, both pure and applied".

yet...

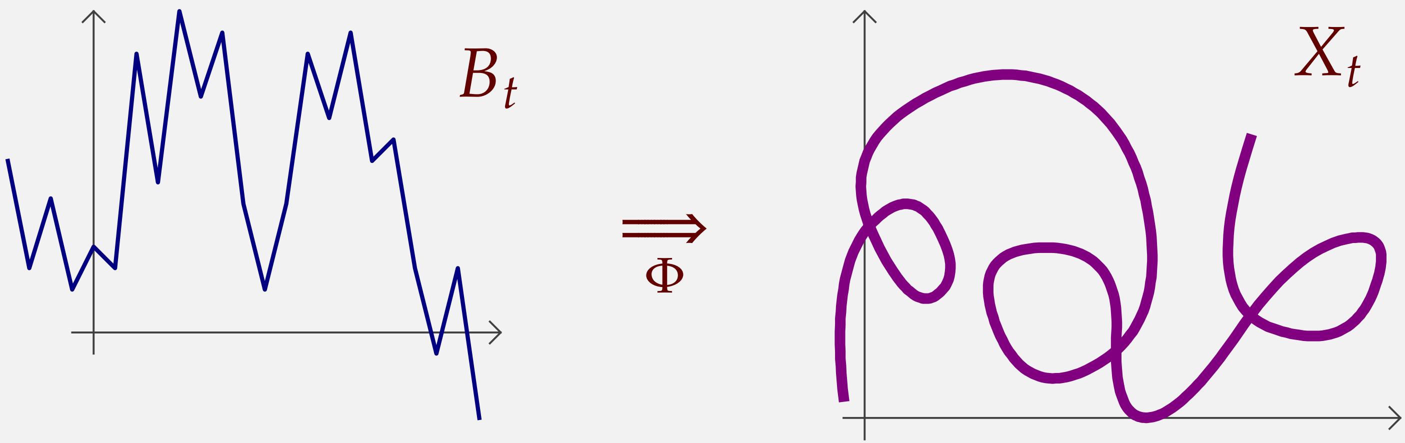
But when Kiyosi Itô came to Princeton in 1954, at that time a stronghold of probability theory with William Feller as the central figure, his new approach to diffusion theory did not attract much attention. Feller was mainly interested in the general structure of one-dimensional diffusions with local generator

$$F = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$$

motivated by his intuition that a "one-dimensional diffusion traveler makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m" [...]

Ito's brillant idea

Ito arrived to his calculus while trying to understand Feller's theory of diffusions an evolution in the space of probability measures and he introduced stochastic differential equations to define a map (**the Itô map**) which send Wiener measure to the law of a diffusion.



☞ useful byproduct: pathwise coupling between B and X

> [...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognised as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.
>
> [D. Stroock, "Elements of stochastic calculus and analysis ", Springer, 2018]

Nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…

the quest for equations

Euclidean (quantum) fields

conceptually: stationary Markovian d dimensional fields / Gibbsian continuous stochastic fields
probability measures ν on $\mathscr{S}'(\mathbb{R}^d)$ · (Feynman–Kac) path integral formalism

$$\nu(\mathrm{d}\varphi) \approx \frac{e^{-S(\varphi)}}{Z}\mathscr{D}\varphi \approx \frac{e^{-\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}}{Z'}\mu(\mathrm{d}\varphi), \qquad \mu(\mathrm{d}\varphi) \approx \frac{e^{-S_0(\varphi)}}{Z}\mathscr{D}\varphi$$

$$S(\varphi) = \underbrace{\int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 + m^2\varphi(x)^2}_{S_0(\varphi)} + \underbrace{\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}_{\mathscr{V}(\varphi)}$$

natural probabilistic objects

heuristic description · large scale & small scale problems · need for renormalisation

How to set up stochastic analysis for Euclidean fields?

Gaussian free field

▷ GFF · simplest example of EQFT · Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ s.t.

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean · $m>0$ is the *mass* · $G(0)=+\infty$ if $d\geqslant 2$: not a function · distribution of regularity

$$\alpha < (2-d)/2$$

▷ can be used to construct a QFT but the theory is free: no interaction

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variation · fractional Laplacian covariance $s\in(0,1)$

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} (a-\Delta)^{-1}(x-y)\rho(\mathrm{d}a) = (m^2+(-\Delta)^s)^{-1}(x-y)$$

the standard recipe for non-Gaussian Euclidean fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{R}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side $2\pi L$

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{R}^d_{\varepsilon,L}}} F(\varphi) e^{\overbrace{-\frac{1}{2}\varepsilon^d \sum_{x\in\mathbb{Z}^d_{\varepsilon,L}} |\nabla_\varepsilon\varphi(x)|^2 + m^2\varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}} \mathrm{d}\varphi$$

ε is an UV regularisation and L the IR regularisation

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below. $d = 2, 3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure.

❸ study the possible limit points [the $\boldsymbol{\Phi^4_d}$ **measure**] · ask interesting questions: uniqueness? non-uniqueness? decay of correlations? intrinsic description?

some models

▷ $d=1$ · time-reversal symmetric, translation invariant, Markov diffusions. the generator is given by an implicit expression involving the ground state Ψ of the Hamiltonian H

$$\mathscr{L} = \nabla \log \Psi \cdot \nabla + \frac{1}{2}\Delta \qquad H = -\Delta + x^2 + V(x).$$

▷ $d=2$ · various choices $(a_\varepsilon \to +\infty)$

$$V_\varepsilon(\xi) = \lambda \xi^{2l} + \sum_{k=0}^{2l-1} a_{k,\varepsilon}\xi^k, \quad V_\varepsilon(\xi) = a_\varepsilon \cos(\beta\xi)$$

$$V_\varepsilon(\xi) = a_\varepsilon \cosh(\beta\xi), \qquad V_\varepsilon(\xi) = a_\varepsilon \exp(\beta\xi)$$

▷ $d=3$ · “only” 4th order (6th order is critical)

▷ $d=4$ · all the possible limits are Gaussian (see Aizenmann–Duminil Copin)

stochastic equations for the free Gaussian free field

Gaussian free field $\mu : \mathbb{E}[\varphi(x)\varphi(y)] = (m^2 - \Delta)^{-1}(x-y)$ · ξ white noise

❶ "Gaussian map":

$$\varphi(x) = (m^2 - \Delta)^{-1/2}\xi(x), \qquad (m^2 - \Delta)\varphi(x) = (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❷ Stochastic mechanics (Nelson):

$$\partial_{x_0}\varphi(x_0, \bar{x}) = -(m^2 - \Delta_{\bar{x}})^{1/2}\varphi(x_0, \bar{x}) + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu):

$$\varphi(x) \sim \phi(t, x) \qquad \partial_t\phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) + c\xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x) \sim \phi(z, x) \qquad (-\Delta_z)\phi(z, x) = -(m^2 - \Delta_x)\phi(z, x) + c\xi(z, x), \qquad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

stochastic equations for non-Gaussian EQFTs $(V \neq 0)$

❶ Shifted Gaussian map (Albeverio/Yoshida) **[does not have the right properties!]**

$$(m^2-\Delta)\varphi(x)+V'(\varphi(x))=(m^2-\Delta)^{1/2}\xi(x), \qquad x\in\mathbb{R}^d$$

❷ Stochastic mechanics (Nelson): ground-state transformation **[implicit!]**

$$\partial_{x_0}\varphi(x_0,\bar{x})=[\nabla_{\varphi(x_0,\bar{x})}\log\Psi(\varphi)]+\xi(x_0,\bar{x}), \qquad x_0\in\mathbb{R},\bar{x}\in\mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu): Langevin diffusion

$$\varphi(x)\sim\phi(t,x) \qquad \partial_t\phi(t,x)=-(m^2-\Delta_x)\phi(t,x)-V'(\phi(t,x))+c\xi(t,x), \qquad t\in\mathbb{R},x\in\mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x)\sim\phi(z,x) \qquad (-\Delta_z)\phi(z,x)=-(m^2-\Delta_x)\phi(z,x)-V'(\phi(z,x))+c\xi(z,x), \quad z\in\mathbb{R}^2,x\in\mathbb{R}^d$$

an (pre)history of (Langevin) stochastic quantisation (personal & partial)

- 1981 · Parisi/Wu – stochastic quantisation for gauge theories (SQ)
- 1985 · Jona-Lasinio/Mitter · "On the stochastic quantization of field theory" (rigorous SQ for Φ^4_2 on bounded domain)
- 1988 · Damgaard/Hüffel · review book on SQ (theoretical physics)
- 1990 · Funaki · Control of correlations via SQ (smooth reversible dynamics)
- 1990–1994 · Kirillov · "Infinite-dimensional analysis and quantum theory as semimartingale calculus", "On the reconstruction of measures from their logarithmic derivatives", "Two mathematical problems of canonical quantization."
- 1993 · Ignatyuk/Malyshev/Sidoravicius · "Convergence of the Stochastic Quantization Method I,II" [Grassmann variables + cluster expansion]
- 2000 · Albeverio/Kondratiev/Röckner/Tsikalenko · "A Priori Estimates for Symmetrizing Measures..." [Gibbs measures via IbP formulas]
- 2003 · Da Prato/Debussche · "Strong solutions to the stochastic quantization equations"
- 2014 · Hairer – Regularity structures, local dynamics of Φ^4_3
- 2017 · Mourrat/Weber · global solutions for Φ^4_2, coming down from infinity for Φ^4_3
- 2018 · Albeverio/Kusuoka · "The invariant measure and the flow associated to Φ^4_3..."
- 2021 · Hofmanova/G. – Global space-time solutions for Φ^4_3 and verification of axioms (CMP)
- 2022 · Hairer/Steele – "optimal" tail estimates (JSP)
- 2020–2021 · Chandra/Chevyrev/Hairer/Shen · SQ for Yang–Mills 2d/3d (local theory) (arXiv)

the FBSDE approach

scale decomposition

goal: study the measure

$$\nu^V_{\varepsilon,L}(\mathrm{d}\varphi) := \frac{e^{-V(\varphi)}\mu_{\varepsilon,L}(\mathrm{d}\varphi)}{\int e^{-V(\varphi)}\mu_{\varepsilon,L}(\mathrm{d}\varphi)} \qquad V(\varphi) = \varepsilon^d \sum_{x\in\mathbb{R}^d_{\varepsilon,L}} v(\varphi(x))$$

where $\mu_{\varepsilon,L}$ is the GFF on $\mathbb{R}^d_{\varepsilon,L}$ of covariance $(m^2-\Delta_\varepsilon)^{-1}$ and $v:\mathbb{R}\to\mathbb{R}$ any nice bounded function.

Gaussian martingale : $(X_t)_{t\in[0,\infty]}$ such that $X_\infty \sim \mu_{\varepsilon,L}$ and

$$\mathrm{d}\langle X\rangle_t = \dot{G}_t \mathrm{d}t \qquad \mathbb{E}[X_t(x)X_s(y)] = G_{t\wedge s}(x-y)$$

where $\dot{G}_t = \partial_t G_t$, $G_\infty = (m^2-\Delta_\varepsilon)^{-1}$ and $G_0 = 0$. For example we can take

$$G_t = \int_0^t \dot{G}_s \mathrm{d}s, \qquad \dot{G}_s^{1/2} = \frac{1}{s^2} e^{-(m^2-\Delta_\varepsilon)/s}, \qquad t,s \geqslant 0,$$

but other choices are also possible, depending on the context. With a cyl. BM $(W_t)_{t\geqslant 0}$ we have

$$X_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}W_s. \qquad \mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)\varepsilon^d \delta_{x,y}$$

flow equation

For nice functions f, we can write

$$\int f(\varphi)\nu_{\varepsilon,L}(\mathrm{d}\varphi) = \frac{\mathbb{E}[f(X_\infty)\, e^{-V(X_\infty)}]}{\mathbb{E}[e^{-V(X_\infty)}]} = \mathbb{E}_{\mathbb{Q}}[f(X_\infty)]$$

where we defined

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{-V(X_\infty)}}{\mathbb{E}[e^{-V(X_\infty)}]}.$$

If we let

$$\mathscr{V}_t(\varphi) := -\log \mathbb{E}e^{-V(\varphi + X_\infty - X_t)}$$

then it is easy to prove that $\mathbb{E}[e^{-V(X_\infty)}|\mathscr{F}_t] = e^{-\mathscr{V}_t(X_t)}$ and that $\mathscr{V}_t$ is C^2 and satisfies the PDE (an Hamilton–Jacobi–Bellmann equation)

$$\partial_t \mathscr{V}_t(\varphi) = \frac{1}{2}\dot{G}_t \mathrm{D}^2 \mathscr{V}_t(\varphi) - \frac{1}{2}\dot{G}_t \mathrm{D}\mathscr{V}_t\, \mathrm{D}\mathscr{V}_t, \qquad \dot{G}_t \mathrm{D}\mathscr{V}_t\, \mathrm{D}\mathscr{V}_t = \sum_{x,y} \dot{G}_t(x-y)\mathrm{D}_{\varphi_x}\mathscr{V}_t(\varphi)\mathrm{D}_{\varphi_y}\mathscr{V}_t(\varphi) \tag{1}$$

FBSDE

Moreover by Itô's formula

$$d[f_t(X_t)e^{-\mathscr{V}_t(X_t)}] = \left[\partial_t f_t(X_t) - Df_t(X_t)\dot{G}_t D\mathscr{V}_t(X_t) + \frac{1}{2}\dot{G}_t D^2 f(X_t)\right] e^{-\mathscr{V}_t(X_t)} dt + d[\text{martingale}_{\mathbb{P}}]$$

so under the measure $\mathbb{Q}$ the process $(X_t)_{t\geqslant 0}$ is a weak solution to the SDE

$$dX_t(x) = -\sum_{y\in\mathbb{R}^d_{\varepsilon,L}} \dot{G}_t(x-y) D_{\varphi(y)}\mathscr{V}_t(X_t) + d\hat{X}_t(x), \qquad x \in \mathbb{R}^d_{\varepsilon,L}$$

where $(\hat{X}_t)_{t\geqslant 0}$ is a Brownian martingale with QV $d\langle\hat{X}\rangle_t = \dot{G}_t dt$. Taking $f_t = D\mathscr{V}_t$ we deduce also that $D\mathscr{V}_t(X_t)$ is a martingale and therefore that

$$D\mathscr{V}_t(X_t) = \mathbb{E}_{\mathbb{Q},t}[D\mathscr{V}_\infty(X_\infty)] = \mathbb{E}_{\mathbb{Q},t}[DV(X_\infty)], \qquad D_{\varphi(x)}V(X_\infty) = \varepsilon^d v'(X_\infty(x))$$

where $\mathbb{E}_{\mathbb{Q},t} = \mathbb{E}_{\mathbb{Q}}[\cdot|\mathscr{F}_t]$ and $\mathscr{F}_t = \sigma(X_s : s \in [0,t])$

Under the condition that the time-dependent SDE

$$\mathrm{d}\Phi_t = -\dot{G}_t \mathrm{D}\mathscr{V}_t(\Phi_t)\mathrm{d}t + \mathrm{d}X_t, \tag{2}$$

has a unique solution, the above argument give the following representation for the measure $\nu^V_{\varepsilon,L}$:

$$\int f(\varphi)\nu_{\varepsilon,L}(\mathrm{d}\varphi) = \mathbb{E}_{\mathbb{Q}} f(X_\infty) = \mathbb{E}[f(\Phi_\infty)]$$

which establish (2) as a stochastic quantisation formula.

This Markov process generated by (2) is called the Polchinski process (see Bodineau–Bauerschmidt–Dagailler).

Alternatively, the process $(\Phi_t)_t$ is described by the stochastic equation

$$\mathrm{d}\Phi_t = -\dot{G}_t \mathbb{E}_t[\mathrm{D}V(\Phi_\infty)] + \mathrm{d}X_t \tag{3}$$

as the conditional expectation $\mathbb{E}_t = \mathbb{E}[* \mid \mathscr{F}_t]$ is computed explicitly by the function $\mathrm{D}\mathscr{V}_t$ as $\mathrm{D}\mathscr{V}_t(\Phi_t) = \mathbb{E}_t[\mathrm{D}V(\Phi_\infty)]$.

Note that this equation is not a standard SDE as it depends on its “terminal” value Φ_∞. The computation of the conditional expectation is performed by a backwards differential equation, and in this sense the equation can be interpreted as a forward-backward SDE (FBSDE).

variational structure

Let

$$u_t^* := -\dot{G}_t^{1/2}\mathbb{E}_t[DV(\Phi_\infty)]$$

(we assumes $\dot{G}_t$ has a positive and symmetric square root $\dot{G}_s^{1/2}$) and observe that

$$\Phi_t = X_t + I_t(u^*), \qquad I_t(u^*) := \int_0^t \dot{G}_s^{1/2} u_s^* \mathrm{d}s.$$

Now let v be any L^2 adapted process and consider the scalar product

$$\mathbb{E}\int_0^\infty \langle v_s, u_s^*\rangle \mathrm{d}s = -\mathbb{E}\int_0^\infty \langle v_s, \dot{G}_s^{1/2}\mathbb{E}_s[DV(\Phi_\infty)]\rangle \mathrm{d}s$$

$$= -\mathbb{E}\int_0^\infty \langle \dot{G}_s^{1/2} v_s, \mathbb{E}_s[DV(\Phi_\infty)]\rangle \mathrm{d}s = -\mathbb{E}\int_0^\infty \langle \dot{G}_s^{1/2} v_s, DV(\Phi_\infty)\rangle \mathrm{d}s = -\mathbb{E}\langle I_\infty(v), DV(\Phi_\infty)\rangle$$

where we used the symmetry of $\dot{G}_s^{1/2}$. This computation implies that u^* is a critical point of the functional

$$\mathcal{J}^V(u) := \mathbb{E}\left[V(\Psi_\infty^u) + \frac{1}{2}\int_0^\infty \langle u_s, u_s\rangle \mathrm{d}s\right], \qquad \Psi_t^u := X_t + I_t(u)$$

using the function $(\mathscr{V}_t)_t$ defined before, we have by Ito's formula

$$V(\Psi^u_\infty) = \mathscr{V}_0(0) + \int_0^\infty \mathrm{D}\mathscr{V}_t(\Psi^u_t)\mathrm{d}X_t + \int_0^\infty \left[\partial_t \mathscr{V}_t(\Psi^u_t) + \frac{1}{2}\dot{\mathrm{G}}_t \mathrm{D}^2 \mathscr{V}_t(\Psi^u_t) + \langle u_t, \dot{\mathrm{G}}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\rangle\right]\mathrm{d}t$$

so taking expectations and using that $\mathscr{V}$ solves an HJB equation (1)

$$\mathbb{E}V(\Psi^u_\infty) = \mathscr{V}_0(0) + \mathbb{E}\int_0^\infty \left\langle \left(u_t + \frac{1}{2}\dot{\mathrm{G}}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\right), \dot{\mathrm{G}}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\right\rangle \mathrm{d}t$$

therefore

$$\begin{aligned}\mathscr{J}^V(u) &= \mathscr{V}_0(0) + \mathbb{E}\left[\int_0^\infty \left(u_t + \frac{1}{2}\dot{\mathrm{G}}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\right)\dot{\mathrm{G}}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\mathrm{d}t + \frac{1}{2}\int_0^\infty \langle u_t, u_t\rangle \mathrm{d}t\right] \\ &= \mathscr{V}_0(0) + \frac{1}{2}\mathbb{E}\left[\int_0^\infty \left\|u_t + \dot{G}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t)\right\|^2 \mathrm{d}t\right]\end{aligned}$$

and the equation

$$u_t = -\dot{G}_t^{1/2}\mathrm{D}\mathscr{V}_t(\Psi^u_t) \tag{4}$$

characterize an optimal feedback control for the minimization of the functional $\mathscr{J}$.

For any control $u \in \mathbb{H}_a$ (where $\mathbb{H}_a$ is a space of L^2 in ω, t, x and predictable processes) satisfying (4) we have that

$$\Psi_t^u = X_t - \int_0^t \dot{G}_s^{1/2} \mathrm{D}\mathscr{V}_s(\Psi_s^u) \mathrm{d}s,$$

so by the Lipshitzianity of $\mathrm{D}\mathscr{V}_s$ for all $s \in [0, \infty]$ we can infer that this equation has a unique solution and that u is (a.s.) unique.

Note that at the minimum $\mathscr{J}^V(u) = \mathscr{V}_0(0)$ so by the martingale property of $e^{-\mathscr{V}_t(X_t)}$ we also have

$$e^{-\mathscr{J}^V(u)} = e^{-\mathscr{V}_0(0)} = \mathbb{E}[e^{-\mathscr{V}_t(X_t)}] = \mathbb{E}[e^{-V(X_\infty)}],$$

that is

$$-\log \mathbb{E}[e^{-V(X_\infty)}] = \inf_{u \in \mathbb{H}_a} \mathscr{J}^V(u) \tag{5}$$

we collect these findings in the following theorem

Theorem 1. *Let V be a bounded and C^2 potential, then the FBSDE (3) has a unique strong solution $(\Psi_t)_{t\in[0,\infty]}$. This solution satisfies*

$$\mathrm{Law}(\Psi_\infty) = \nu^V_{\varepsilon,L}$$

and the process

$$u^*_t = -\dot{G}_t^{1/2}\mathbb{E}_t[DV(\Psi^{u^*}_\infty)], \qquad t \geqslant 0, \qquad \Psi^u_t = X_t + \int_0^t \dot{G}_s^{1/2} u_s \mathrm{d}s$$

$$\Psi^{u^*}_t = X_t - \int_0^t \dot{G}_s \mathbb{E}_t[DV(\Psi^{u^*}_\infty)]\mathrm{d}s$$

is the unique optimiser of the stochastic control problem

$$\inf_{u\in\mathbb{H}_a} \mathcal{J}^V(u) = \mathcal{J}^V(u^*).$$

From (5) we have the following formula for the Laplace transform of $\nu^V_{\varepsilon,L}$:

$$\mathcal{W}(f) := -\log\int e^{-f(\varphi)}\nu^V_{\varepsilon,L}(\mathrm{d}\varphi) = \inf_{u\in\mathbb{H}_a} \mathcal{J}^{f+V}(u) - \inf_{u\in\mathbb{H}_a} \mathcal{J}^V(u).$$

the Sine–Gordon model

change of setting

For pedagogical reasons we focus on measures $\nu_{T,\rho}$ on $\mathscr{S}'(\mathbb{R}^d)$ defined by choosing $0 < T < \infty$ and $\rho: \mathbb{R}^d \to \mathbb{R}_+$ of bounded support and letting

$$\int f(\varphi)\nu_{T,\rho}(\mathrm{d}\varphi) = \frac{\mathbb{E}[f(X_T)e^{-V^\rho(X_T)}]}{\mathbb{E}[e^{-V^\rho(X_T)}]}, \qquad V^\rho(\varphi) = \int_{\mathbb{R}^d} \rho(x) v(\varphi(x))\mathrm{d}x$$

where $(X_t)_{t \geqslant 0}$ is a Brownian martingale closed by a Gaussian free field X_∞ of mass m on $\mathbb{R}^d$.

The infrared cutoff ρ plays the role of L while the finite regularization time $T < \infty$ that of ε^{-1}.

Unfortunately this measures are not reflection positive nor Euclidean invariant, but they provide a convenient setting to study the problem of the removal of the cutoffs and the associated renomalization.

The infinite volume limit will be taken by taking $\rho = \rho_L$ where $\rho_L: \mathbb{R}^d \to [0,1]$ such that $\rho_L = 1$ in a ball of radius L centered at the origin and then letting this $L \to \infty$.

Sine-Gordon

We let $d=2$ and take

$$v(\varphi(x)) = \lambda\cos(\beta\varphi(x))$$

for some fixed β such that $\beta^2 \in (0, 8\pi)$ and constant $\lambda = \lambda_T \to +\infty$ to be adjusted as $T \to \infty$ to obtain a nontrivial limit. This prescription defines a cutoff version of the Sine-Gordon model.

We have therefore to deal with the FBSDE

$$\mathrm{d}\Phi_t(x) = -\int \dot{G}_t(x-y)\mathbb{E}_t[\mathrm{D}_{\varphi(y)}V(\Phi_T)]\mathrm{d}y + \mathrm{d}X_t(x), \qquad t \in [0,T].$$

What happens after time T is not very important for us as the measure $\nu_{T,\rho}$ is given by the law of Φ_T.

As $T \to \infty$ the *force* $F := \mathrm{D}V$ given by

$$F(\Phi_T)(x) = \mathrm{D}_{\varphi(x)}V(\Phi_T) = \rho(x)\lambda\beta\sin(\beta\Phi_T(x))$$

become very singular.

preliminary computation

We substitute X_T for Φ_T as a first approximation (e.g. in a perturbative iteration). In average this would give

$$\mathbb{E}_t[\mathrm{D}V(X_T)] = \rho(x)\lambda\beta\,\mathbb{E}_t[\sin(\beta X_T(x))] = \rho(x)\lambda\beta\,\hat{\mathbb{E}}[\sin(\beta X_t(x) + \beta(\hat{X}_T - \hat{X}_t)(x))]$$

$$= \rho(x)\lambda\beta\sin(\beta X_t(x))\hat{\mathbb{E}}[\cos(\beta(\hat{X}_T - \hat{X}_t)(x))] = \rho(x)\lambda\beta\sin(\beta X_t(x))\exp\left(-\frac{\beta^2}{2}(G_T(0) - G_t(0))\right)$$

and with

$$G_t(x) = \int_0^t \frac{1}{4\pi s} e^{-m^2/s - \frac{s}{4}|x|^2} \mathrm{d}s$$

we have

$$G_T(0) \approx \int_0^T \frac{1}{4\pi s} e^{-m^2/s} \mathrm{d}s \approx \frac{1}{4\pi} \log(T) \to \infty$$

so we see that $\mathbb{E}_t[\mathrm{D}V(X_T)] \to 0$ and we can heuristically conclude that in this limit we converge to a GFF unless we “tune up” the interaction along the way, that is we counterbalance the factor $\exp\left(-\frac{\beta^2}{2}G_T(0)\right)$ by choosing $\lambda = \lambda_T = \hat{\lambda}\exp\left(\frac{\beta^2}{2}G_T(0)\right) \approx \hat{\lambda}T^{\beta^2/8\pi}$.

the running coupling

With the choice

$$\lambda = \lambda_T = \hat{\lambda} \exp\left(\frac{\beta^2}{2} G_T(0)\right).$$

we will have, at least in this approximation, a well defined limit

$$\lim_{T\to\infty} \mathbb{E}_t[\mathrm{D}V(X_T)] = \rho(x)\hat{\lambda}\exp\left(\frac{\beta^2}{2} G_t(0)\right)\beta \sin(\beta X_t(x)) = \rho(x)\lambda_t \beta \sin(\beta X_t(x))$$

in which, the "renormalization" we put in place allowed us to replace the diverging "bare coupling" λ_T with a scale dependent *running coupling* $\lambda_t = \hat{\lambda} \exp\left(\frac{\beta^2}{2} G_t(0)\right)$.

While this computation hints to a possible approach to a non-trivial limit, we still need to confirm it rigorously for the full FBSDE. We should however keep with us the idea of the running coupling and the possible non-existence of the limiting potential V in the ultraviolet limit.

preparing the FBSDE to the UV limit $(T \to \infty)$

For any smooth functional $(F_t)_{t\in[0,T]}$ $e.g. F_t(\Phi_t) = \lambda_t \beta \sin(\beta\Phi_t)$ for which $F_T = F = -DV$ we have, by Itô formula:

$$\mathbb{E}_t[F_T(\Phi_T)] = F_t(\Phi_t) + \int_t^T \mathbb{E}_t\left[\partial_s F_s(\Phi_s) + DF_s(\Phi_s)\dot{G}_s \mathbb{E}_s[F_T(\Phi_T)] + \frac{1}{2}\dot{G}_s D^2 F_s(\Phi_s)\right] ds$$

so if we introduce the new variable $R_t := \mathbb{E}_t[F_T(\Phi_T)] - F_t(\Phi_t)$ we cast the FBSDE into a system of two equations for the unknown (Φ, R):

$$\begin{cases} \Phi_t = X_t + \int_0^t \dot{G}_s F_s(\Phi_s) ds + \int_0^t \dot{G}_s R_s ds, \\ R_t = \int_t^T \mathbb{E}_t[H_s(\Phi_s)] ds + \int_t^T \mathbb{E}_t[DF_s(\Phi_s)\dot{G}_s R_s ds] ds \end{cases} \quad (6)$$

where

$$H_s(\Phi_s) := \partial_s F_s(\Phi_s) + DF_s(\Phi_s)\dot{G}_s F_s(\Phi_s) ds + \frac{1}{2}\dot{G}_s D^2 F_s(\Phi_s).$$

In this system the UV cutoff T appears only as bound of integration, so we can imagine that, provided we obtain sufficient integrability for $H_s(\Phi_s)$ as $s \to \infty$ and a nice behaviour of the function $(F_t)_t$ for finite t, this systems could be under control.

We are led to find approximate solutions to the **flow equation** $H_s = 0$.

solving the flow equation

We need *any* (approximate) solution with prescribed final condition

$$F_T(\varphi)(x) = -\mathrm{D}V(\varphi)(x) = \rho(x)\,\beta\lambda_T \sin(\beta\varphi(x)) = \frac{\rho(x)\,\beta\lambda_T}{2i}(e^{i\beta\varphi(x)} - e^{-i\beta\varphi(x)})$$

Note that $\mathrm{D}e^{i\beta\varphi(x)} = e^{i\beta\varphi(x)}(i\beta\delta(x))$, so we can look for solutions (or approximate solution), in the space of polynomials in $e^{\pm i\beta\varphi(x)}$. To ease the notation we let $\xi = (\sigma, x)$ and $\psi(\xi) = \psi^\sigma(x) := e^{\sigma i\beta\varphi(x)}$, moreover if we have multiindices $\xi_{1:n} = (\xi_1, \ldots, \xi_n)$, then $\psi(\xi_{1:n}) = \prod_{i=1}^n \psi(\xi_i)$. With these notation we can introduce the functionals $(F_t^{[\ell]})_{\ell=1,\ldots}$

$$F_t^{[\ell]}(\varphi)(x_1) = \sum_{\sigma_i \in \{-1,1\}^\ell} \sigma_1 \int_{(\mathbb{R}^2)^{\ell-1}} \mathrm{d}x_{2:\ell} f_t^{[\ell]}(\xi_{1:\ell})\, \mathrm{e}^{i\beta\sigma_1\varphi(x_1)} \ldots \mathrm{e}^{i\beta\sigma_\ell\varphi(x_\ell)}$$

$$= \sum_{\sigma_1 \in \{-1,1\}} \sigma_1 \int \mathrm{d}\xi_{2:\ell} f_t^{[\ell]}(\xi_{1:\ell}) \psi(\xi_{1:\ell})$$

where the kernels $f_t^{[\ell]}(\xi_{1:\ell})$ can be chosen to be symmetric wrt. ξs.

We define them via the equation

$$\partial_s F_s^{[\ell]} + \frac{1}{2} \sum_{\ell_1+\ell_2=\ell} \mathrm{D}[F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]}] + \frac{1}{2} \dot{G}_s \mathrm{D}^2 F_s^{[\ell]} = 0 \tag{7}$$

This is an almost triangular system which can be solved explicity via a variation of constants:

$$F_t^{[\ell]} = \exp\left[\frac{1}{2}(G_T - G_t)\mathrm{D}^2\right] F_T^{\ell} - \frac{1}{2} \int_t^T \exp\left[\frac{1}{2}(G_s - G_t)\mathrm{D}^2\right] \sum_{\ell_1+\ell_2=\ell} \mathrm{D}[F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]}] \mathrm{d}s.$$

A simple computation shows that the “Laplacian” acts on the kernel as

$$\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}(\varphi)(x_1) = \sum_{\sigma_1 \in \{-1,1\}} \sigma_1 \int \mathrm{d}\xi_{2:\ell} \left[-\beta^2 \sum_{k,l=1,\ldots \ell} \sigma_k \sigma_l \dot{G}_t (x_k - x_l)\right] f_t^{[\ell]}(\xi_{1:\ell}) \psi(\xi_{1:\ell})$$

which together with the mild formulation and introducing the function

$$W_{t,s}(\xi_{1:\ell}) := -\frac{\beta^2}{2} \sum_{i,j} \sigma_i \sigma_j (G_s - G_t)(x_i - x_j), \quad t \leqslant s. \tag{8}$$

gives us equation for the kernels:

These equations reads

$$f_t^{[1]}(\xi_1) = e^{W_{t,T}(\xi_1)} \lambda_T \frac{\beta}{2i} = \exp\left(-\frac{\beta^2}{2}(G_T(0) - G_t(0))\right) \lambda_T \frac{\beta}{2i} = \lambda_t \frac{\beta}{2i}$$

and for $\ell > 1$

$$f_t^{[\ell]}(\xi_{1:\ell}) = -\beta^2 \sum_{I_1 + I_2 = [\ell]} C(|I_1|, |I_2|) \int_t^T e^{W_{t,s}(\xi_{1:\ell})} f_s(\xi_{I_1}) \left[\sum_{i \in I_1} \sum_{j \in I_2} \sigma_i \sigma_j \dot{G}_s(x_i - x_j)\right] f_s(\xi_{I_2}) \mathrm{d}s. \qquad (9)$$

modulo positive combinatorial coefficients which we gather in $C(|I_1|, |I_2|)$.

Note that here we are using the running coupling

$$\lambda_t = \hat{\lambda} \exp\left(\frac{\beta^2}{2} G_t(0)\right) \qquad (10)$$

At this point we can form the sum

$$F^{[\leqslant \ell^*]} = \sum_{\ell=1}^{\ell^*} F^{[\ell]}$$

and observe that it satisfies

$$\partial_s F^{[\leqslant \ell^*]} + \frac{1}{2}\mathrm{D}[F^{[\leqslant \ell^*]}\dot{G}_s F^{[\leqslant \ell^*]}] + \frac{1}{2}\dot{G}_s \mathrm{D}^2 F^{[\leqslant \ell^*]} = \frac{1}{2}\sum_{\substack{\ell_1+\ell_2>\ell^* \\ \ell_1,\ell_2\leqslant \ell^*}} \mathrm{D}[F_s^{[\ell_1]}\dot{G}_s F_s^{[\ell_2]}]$$

which will contribute as a source to the remainder term in the FBSDE.

What kind of estimates we could hope for in terms of these functionals? Since $F^{[1]}$ is explicit, this is easy:

$$\|\mathrm{D}F_t^{[1]}(\varphi)\|_{L^\infty} + \|F_t^{[1]}(\varphi)\|_{L^\infty} \lesssim \lambda_t \lesssim \langle t\rangle^{1-\delta}$$

where we let $\delta := 1 - \frac{\beta^2}{8\pi} > 0$.

The structure of the equation (7) and the fact that we have the bound

$$\|\dot{G}_s\|_{L^1} \lesssim \langle s \rangle^{-2}$$

for the smoothing operator, suggests that

$$\|\mathrm{D}F_t^{[\ell]}(\varphi)\|_{L^\infty} + \|F_t^{[\ell]}(\varphi)\|_{L^\infty} \lesssim \lambda_t^\ell \langle t \rangle^{-(\ell-1)} \lesssim \langle t \rangle^{1-\ell\delta}$$

could propagate inductively provided $\delta\ell > 1$, indeed in this case, for the non-linear term

$$\sum_{\ell_1+\ell_2=\ell} \int_t^T \|\mathrm{D}[F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]}]\|_{L^\infty} \mathrm{d}s \lesssim \sum_{\ell_1+\ell_2=\ell} \int_t^T \lambda_s^{\ell_1+\ell_2} \langle s \rangle^{-(\ell_1+\ell_2)} \mathrm{d}s \lesssim \int_t^T \langle s \rangle^{-\delta\ell} \mathrm{d}s \lesssim \langle s \rangle^{1-\delta\ell} \approx \lambda_s^\ell \langle s \rangle^{-(\ell-1)},$$

When $2\delta > 1$ we can propagate all the regularities, when $4\delta > 1$ $(\beta^2 < 6\pi)$ we can propagate only $\ell \geqslant 4$. So we need a different strategy for $f^{[2]}$ and $f^{[3]} \ldots$

estimates for the kernels

The relevant, translation invariant norm is:

$$|||f||| := \sup_{\xi_1} \int \mathrm{d}\xi_{2:\ell} |f(\xi_{1:\ell})|,$$

and the function W, using the positivity of $\dot{G}_s$ can be estimated by

$$W_{t,s}(\xi_{1:\ell}) = -\frac{\beta^2}{2} \sum_{i,j} \sigma_i \sigma_j (G_s - G_t)(x_i - x_j) \leqslant 0, \quad \text{for } t \leqslant s$$

so the exponential $\mathrm{e}^{W_{t,s}(\xi_{1:\ell})} \leqslant 1$ will not pose problems to us.

We call $q(\xi_{1:\ell}) := \sum_{k=1}^{\ell} \sigma_k$ the charge of the configuration. The bound above is sharp for neutral ($q = 0$) configurations however if $q \neq 0$:

Lemma 2. *Suppose that $\xi_{1:\ell}$ is charged. Then there is a constant $C > 0$ such that for all $s \geqslant t$,*

$$W_{t,s}(\xi_1, \ldots, \xi_\ell) \leqslant \frac{\beta^2}{8\pi} (G_t(0) - G_s(0)) + C, \tag{11}$$

and in particular

$$\mathrm{e}^{W_{t,s}(\xi_{1:\ell})} \lesssim \lambda_t \lambda_s^{-1}.$$

The point is that charged configuration have a better UV behaviour from the linear part of the equation which helps us with integrability, as can be easily shown. Recall the definition (9), that the kernel $f^{[2]}$ is given by

$$f_t^{[2]}(\xi_1, \xi_2) = C \int_t^T \mathrm{d}s \, \mathrm{e}^{W_{t,s}(\xi_1,\xi_2)} f_s^{[1]}(\xi_1) \, \sigma_1 \sigma_2 \dot{G}_s(x_1 - x_2) f_s^{[1]}(\xi_2).$$

If the pair is charged then we estimate as follows (with $\ell = 2$)

$$\left\| f_t^{[2](\mathrm{charged})} \right\| \lesssim \sup_{\xi_1} \lambda_t \int_t^T \mathrm{d}s \, \lambda_s^{-1} |||f_s^{[1]}|||^2 \|\dot{G}_s\|_{L^1} \lesssim \lambda_t \int_t^T \mathrm{d}s \lambda_s \langle s \rangle^{-2} \lesssim \lambda_t^\ell \langle t \rangle^{-(\ell-1)}.$$

It remains to deal with neutral contributions. The first thing to note is that in this case

$$W_{t,s}(\xi_1, \xi_2) + \beta^2 G_s(0) = \beta^2 G_t(0) - \beta^2 G_t(x_1 - x_2) + \beta^2 G_s(x_1 - x_2) \leqslant \beta^2 G_s(x_1 - x_2)$$

(...)

we can absorbe the divergence of λ_s^2 coming from the two factors $f^{[1]}$ as

$$
\begin{aligned}
|||k_t f_t^{[2](0)}||| &= C\sup_{\xi_1}\left|\int \mathrm{d}\xi_2 k_t(\xi_1,\xi_2)\int_t^T \mathrm{d}s\,\mathrm{e}^{W_{t,s}(\xi_1,\xi_2)} f_s^{[1]}(\xi_1)\, f_s^{[1]}(\xi_2)\,\dot{G}_s(x_1-x_2)\right| \\
&\lesssim \sup_{x_1}\left|\int_t^T \mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}x_2\,\dot{G}_s(x_1-x_2)|x_1-x_2|^{2\alpha} t^{\alpha}\mathrm{e}^{ct|x_1-x_2|^2}\mathrm{e}^{W_{t,s}(\xi_1,\xi_2)+\beta^2 G_s(0)}\right| \\
&\lesssim \mathrm{e}^{\beta^2 G_t(0)} t^{\alpha}\int_t^T \mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}x|x|^{2\alpha}\mathrm{e}^{ct|x|^2}\dot{G}_s(x)\mathrm{e}^{\beta^2 G_s(x)}.
\end{aligned}
$$

where we introduced in the norm the weight

$$
k_t(\xi_1,\xi_2) := \begin{cases} t^{\alpha}|\delta_{12}x|^{2\alpha}\mathrm{e}^{ct|\delta_{12}x|^2}, & q(\xi_1,\xi_2)=0 \\ 1, & \text{otherwise.}\end{cases}
$$

which allows us to propagate the bound for $\alpha > (1-2\delta)\vee 0$ since in this case a direct computation shows that

$$
\int_{\mathbb{R}^2}\mathrm{d}x|x|^{2\alpha}\mathrm{e}^{ct|x|^2}\dot{G}_s(x)\mathrm{e}^{\beta^2 G_s(x)} \lesssim \langle s\rangle^{-2-\alpha}
$$

so we get

$$
|||k_t f_t^{[2](0)}||| \lesssim \lambda_t^2 t^{\alpha}\int_t^T \mathrm{d}s\langle s\rangle^{-2-\alpha} \lesssim \lambda_t^2\langle t\rangle^{-1} = \lambda_t^{\ell}\langle t\rangle^{-(\ell-1)}.
$$

We skip the simpler analysis of $f^{[3]}$ which will lead to the fact that for $\alpha = 1/2$ and $\delta > 1/4$ ($\beta^2 < 6\pi$), it holds that

$$|||f_t^{[3]}||| \lesssim \lambda_t^3 \langle t \rangle^{-2} = \lambda_t^\ell \langle t \rangle^{-(\ell - 1)}$$

so also this kernel is under control. Recall that α is the exponent in the weight for $f^{[2]}$.

Proposition 3. *For any $\varphi \in \mathcal{S}'(\mathbb{R}^2)$, the following estimates apply uniformly in $\varphi \in \mathcal{S}'(\mathbb{R}^2), \rho \leqslant 1$ and $T < \infty$*

$$\begin{aligned} \left\| \dot{G}_t^{1/2} F_t^{[\ell]}(\varphi) \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1}, \\ \left\| \mathrm{D} F_t^{[\ell]}(\varphi)\, \dot{G}_t^{1/2} \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1}, \\ \| H_t(\varphi) \|_{L^\infty} &\lesssim (\lambda_t \langle t \rangle^{-1})^4, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \left\| \dot{G}_t^{1/2} F_t^{[\ell]}(\varphi) - \dot{G}_t^{1/2} F_t^{[\ell]}(\tilde{\varphi}) \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1} \| \varphi - \tilde{\varphi} \|_{L^\infty}, \\ \left\| \mathrm{D} F_t^{[\ell]}(\varphi)\, \dot{G}_t^{1/2} - \mathrm{D} F_t^{[\ell]}(\tilde{\varphi})\, \dot{G}_t^{1/2} \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1} \| \varphi - \tilde{\varphi} \|_{L^\infty}, \\ \| H_t(\varphi) - H_t(\tilde{\varphi}) \|_{L^\infty} &\lesssim (\lambda_t \langle t \rangle^{-1})^4 \| \varphi - \tilde{\varphi} \|_{L^\infty}. \end{aligned} \tag{13}$$

The Lipshitz bounds holds also in weighted norms.

For the dependence on ρ we need to observe the following proposition, introducing explicitly this dependence with the notation F^{ρ}, H^{ρ}.

Proposition 4. *For $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ and $\rho_1, \rho_2 \leqslant 1$,it holds that*

$$\begin{aligned} \|Q_t(F_t^{\rho_1} - F_t^{\rho_2})(\varphi)\|_{L^{2,-n}} &\lesssim \lambda_t \langle t\rangle^{-1} \|\rho_1 - \rho_2\|_{L^{2,-n}}, \\ \|(\mathrm{D}F_t^{\rho_1} - \mathrm{D}F_t^{\rho_2})(\varphi)\, Q_t\|_{L^{2,-n}} &\lesssim \lambda_t \langle t\rangle^{-1}, \|\rho_1 - \rho_2\|_{L^{2,-n}}, \\ \|(H_t^{\rho_1} - H_t^{\rho_2})(\varphi)\|_{L^{2,-n}} &\lesssim (\lambda_t \langle t\rangle^{-1})^4 \|\rho_1 - \rho_2\|_{L^{2,-n}}. \end{aligned} \tag{14}$$

where

$$\|f\|_{L^{2,-n}}^2 = \int_{\mathbb{R}^d} |\langle x\rangle^{-n} f(x)|^2 \mathrm{d}x$$

is a weighted norm which focuses on what happens locally around the origin and for which when n is large enough ($n > 2$) one has $\|\rho_L - \rho_{L'}\|_{L^{2,-n}} \to 0$ when $L > L' \to \infty$.

estimates of thr FBSDE

With these informations we go back to the analysis of the FBSDE (6) which is now straighforward. In L^∞ we can indeed estimate, with Proposition 3, and recalling that

$$\|\dot{G}_s\|_{L^1} \lesssim \langle s \rangle^{-2},$$

$\|\dot{G}_s^{1/2}\|_{L^1} \lesssim \langle s \rangle^{-1}$, and letting

$$\Phi_t = X_t + Z_t$$

we have

$$\|Z_t\|_{L^\infty} \leqslant \int_0^t \|\dot{G}_s^{1/2}\|_{L^1} \|\dot{G}_s^{1/2} F_s(\Phi_s)\|_{L^\infty} \mathrm{d}s + \int_0^t \|\dot{G}_s\|_{L^1} \|R_s\|_{L^\infty} \mathrm{d}s$$

$$\leqslant C \int_0^t \lambda_s \langle s \rangle^{-2} \mathrm{d}s + \int_0^t \langle s \rangle^{-2} \|R_s\|_{L^\infty} \mathrm{d}s$$

$$\|R_t\|_{L^\infty} \leqslant \int_t^T \|H_s(\Phi_s)\|_{L^\infty} \mathrm{d}s + \int_t^T \|\mathrm{D}F_s(\Phi_s) \dot{G}_s^{1/2}\|_{L^\infty} \|\dot{G}_s^{1/2}\|_{L^1} \|R_t\|_{L^\infty} \mathrm{d}s$$

$$\lesssim \int_t^T (\lambda_s \langle s \rangle^{-1})^4 \mathrm{d}s + \int_t^T \lambda_s \langle s \rangle^{-2} \|R_t\|_{L^\infty} \mathrm{d}s$$

The inequality for $\|R_t\|_{L^\infty}$ gives via a (reverse) Grownwall (and $\delta > 1/4$)

$$\|R_t\|_{L^\infty} \lesssim \exp\left(\int_t^\infty \lambda_s \langle s\rangle^{-2} \mathrm{d}s\right) \int_t^\infty (\lambda_s \langle s\rangle^{-1})^4 \mathrm{d}s \lesssim \int_t^\infty (\lambda_s \langle s\rangle^{-1})^4 \mathrm{d}s \lesssim \langle t\rangle^{1-4\delta}$$

$$\|Z_t\|_{L^\infty} \lesssim C \int_0^\infty \lambda_s \langle s\rangle^{-2} \mathrm{d}s + \int_0^\infty \langle s\rangle^{-2} \|R_s\|_{L^\infty} \mathrm{d}s \lesssim C.$$

We have been careful not to estimate Φ_t in L^∞ because it wouldn't work.

Stationary Gaussian fields like X_t grow at infinitity, at least with some power of log.

The *interacting multiscale field* $(\Phi_t)_{t\in[0,T]}$ is an a.s. bounded perturbation $(Z_t)_{t\in[0,T]}$ of the multiscale GFF $(X_t)_{t\in[0,T]}$, a very powerful control which is available essentially on in this model and in some other Grassmann theory (maybe later).

We have a tight control of the interacting field $(\Phi_t^{\rho,T})_t$ where we introduced explicitly the dependence on the UV cutoff T coming from the initial condition of the flow equation for the effective force $(F_t^{\rho,T})_t$ and on the IR cutoff ρ which appears in the kernels of the effective force.

To prove $(\Phi_t^{\rho,T})_t$ has a limit when $T \to \infty$ and $\rho \to \infty$ we compare two solutions Φ^{ρ_1,T_1} and Φ^{ρ_2,T_2} via the equations they satisfy (together with the associated remainders) and the use of the information gathered on the stability of the flow equation in Prop. 3 and Prop. 4.

The estimates we performed in L^∞ could have been done also in weighted space $L^{p,-n}$ with $pn > d$ and $p \geqslant 1$. In particular the removal of the IR cutoff (below) requires to use weighted spaces as it can hold only in a local topology.

For small $\hat{\lambda}$ we can then prove that, for any $n > 0$,

$$\lim_{\substack{\rho \to 1 \\ T \to \infty}} \sup_t \{ \|Z_t^{\rho,T} - Z_t\|_{L^{\infty,-n}} + \|R_t^{\rho,T} - R_t\|_{L^{\infty,-n}} \} = 0$$

where (Z, R) is the unique solution of the FBSDE with the effective force $F^{1,\infty}$ obtained as the limit of $F^{\rho,T}$ as $T \to \infty$ and $\rho \to 1$.

Since

$$\mathrm{Law}(Z_T^{\rho,T} + X_T) = \nu^{\rho,T}$$

this implies the weak convergence of $(\nu^{\rho,T})_{\rho,T}$ to a limiting measure

$$\nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + X_\infty)$$

in the space of probability measures on $H^{-\varepsilon,-n}$ (for example, with $\varepsilon > 0$ and $n > d$) since it is in this space that $X_T \to X_\infty$.

The choice of $H^{-\varepsilon,-n}$ is a bit arbitrary, dictated by the properties of the GFF more than by the convergence of $Z_T^{\rho,T} \to Z_\infty$ which holds almost surely in the stronger space $L^{\infty,-n}$, as stated above.

The measure ν_{SG} is the sine-Gordon EQF.

coupling

Note that $Z_\infty \in L^\infty$, so this measure has at least the (negative) regularity of the GFF. It is interesting to obtain a more precise regularity for Z_∞, from the FBSDE we have for $\alpha > 0$ and $n > d$, and provided $\alpha < 2\delta$

$$\begin{aligned}
\|Z_\infty\|_{H^{\alpha,-n}} &\leqslant \left\| \int_0^\infty \mathrm{d}s\, \dot{G}_s(F_s(X_s) + R_s) \right\|_{H^{\alpha,-n}}^2 \\
&\leqslant \int_0^\infty \mathrm{d}s\, \langle s \rangle^{-1+\alpha} \|G_s^{1/2}(F_s(X_s) + R_s)\|_{L^{2,-n}}^2 \\
&\lesssim \int_0^\infty \mathrm{d}s\, \langle s \rangle^{-1+\alpha} \|G_s^{1/2}(F_s(X_s) + R_s)\|_{L^\infty}^2, \\
&\lesssim \int_0^\infty \mathrm{d}s\, \langle s \rangle^{-1+\alpha} (\langle s \rangle^{-2\delta} + \langle s \rangle^{-1}) < \infty.
\end{aligned}$$

Therefore $Z_\infty \in H^{\alpha,-n}$. (And a similar estimate would also show that $Z_\infty \in B^{\alpha}_{\infty,\infty}$).

As a byproduct of this construction we always obtain a *coupling* $\pi = \mathrm{Law}(X_\infty, X_\infty + Z_\infty)$ of the GFF μ with the EQF ν_{SG}, i.e. a measure on a product space for which the two marginals are the two given measures.

integrability

The existence of the coupling implies for example that there exists a small $\gamma > 0$ for which

$$\int_{\mathcal{S}'(\mathbb{R}^2)} e^{\gamma \|\varphi\|^2_{H^{-\varepsilon,-n}}} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) \lesssim \mathbb{E}[\exp(2\gamma \|X_\infty\|^2_{H^{-\varepsilon,-n}} + 2\gamma \|Z_\infty\|^2_{L^\infty})] < \infty$$

where the integrability of X_∞ follows from Fernique's theorem (or from Boué–Dupuis formula).

This integrability condition is well enough to satisfy the regularity axiom needed for OS reconstruction.

non-perturbative results

We note also that, if $\hat{\lambda}$ is not small, the FBSDE estimates still give good bounds on $(Z_T^{\rho,T})_{\rho,T}$ allowing to prove the tightness of the family $(\nu^{\rho,T})_{\rho,T}$ in $H^{-\varepsilon,-n}$, so some EQF exists for the sine-Gordon model but uniqueness then it is not clear (at least in this approach).

In general any limit will be described by a solution of the unregularized FBSDE where $(F_t)_t$ is the limiting approximate effective force. Note that there is no ambiguity in defining this effective force for any $\hat{\lambda}$ since it is specified by a triangular system which has always a unique solution.

The possibility to have more than limiting measure or more than one solution of the FBSDE is linked also to the phenomenon of phase transitions.

It is expected that uniqueness of the limit would hold in presence of the IR regularisation but at the moment I'm not able to prove it via the FBSDE, it can be an interesting research problem.

variational structure

Recall the variational problem is given by the minimization of the functional

$$\mathcal{J}(u) = \mathbb{E}\left[V(\Phi_T^u) + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s\right], \qquad \Phi_t^u = X_t + \int_0^t \dot{G}_s \mathbb{E}_s \mathrm{D}V(\Phi_T^u)\mathrm{d}s$$

to discuss the removal of the cutoff we employ Ito formula with an approximate effective potential $(V_s)_s$ obtained essentially as we have the force, i.e. $F_s = -\mathrm{D}V_s$, and $V_T = V$,

$$\mathcal{J}(u) = V_0(\Phi_0^u) + \mathbb{E}\left\{\int_0^T \left[\mathcal{H}_s(\Phi_s^u) + \mathrm{D}V_s(\Phi_s^u)\, G_s^{1/2} u_s + \frac{1}{2}(\mathrm{D}V_s \dot{G}_s \mathrm{D}V_s)(\Phi_s^u)\right]\mathrm{d}s + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s\right\}$$

$$= V_0(\Phi_0^u) + \mathbb{E}\left[\int_0^T \mathcal{H}_s(\Phi_s^u)\mathrm{d}s + \frac{1}{2}\int_0^T \left\|u_s + G_s^{1/2}\mathrm{D}V_s(\Phi_s^u)\right\|^2 \mathrm{d}s\right]$$

where

$$\mathcal{H}_t := \partial_t V_t + \frac{1}{2}\mathrm{Tr}\, \dot{G}_t \mathrm{D}^2 V_t - \frac{1}{2}\mathrm{D}V_t \dot{G}_t \mathrm{D}V_t.$$

In this new formulation we see that the running cost does not depends anymore on the L^2 norm of u but of

$$r_s := u_s + G_s^{1/2} \mathrm{D} V_s(\Phi_s^u).$$

Now we can invert this dependence and let u be a function of r, since the equation

$$\hat{Z}_t^r = -\int_0^t G_s \mathrm{D} V_s(X_s + \hat{Z}_s^r) \mathrm{d}s + \int_0^t G_s^{1/2} r_s \mathrm{d}s$$

has Lipshitz coefficients and unique global solutions, moreover the control

$$\hat{u} = r - G_s \mathrm{D} V_s(X_s + \hat{Z}_s^r)$$

is admissible for the original control problem and $\Phi_t^{\hat{u}} = X_t + \hat{Z}_t^r =: \hat{\Phi}_t^r$. So we can reformulate the control problem as

$$\hat{\mathcal{J}}(r) = V_0(\hat{\Phi}_0^r) + \mathbb{E}\left[\int_0^T \mathscr{H}_s(\hat{\Phi}_s^r) \mathrm{d}s + \frac{1}{2}\int_0^T \|r_s\|_{L^2}^2 \mathrm{d}s\right]$$

and

$$\inf_{u \in \mathbb{H}_a} \mathcal{J}(u) = \inf_{r \in \mathbb{H}_a} \hat{\mathcal{J}}(r)$$

Additionally we have the bound

$$\hat{J}_T(r) \geqslant -C + \frac{1}{2}\int_0^T \|r_s\|_{L^2}^2 \mathrm{d}s$$

uniformly in T. Moreover if we recall that the optimizer for $\mathcal{J}(u)$ is $u_t = -G_s^{1/2}\mathrm{D}V_t(\Phi_t^u) - G_s^{1/2}R_t$ we can deduce that the optimizer for $\hat{\mathcal{J}}(r)$ is $r = -G_s^{1/2}\mathrm{D}V_t(\Phi_t^u) - G_s^{1/2}R_t + G_s^{1/2}\mathrm{D}V_t(\Phi_t^u) = -G_s^{1/2}R_r$.

From these observations with some technical steps one deduces that

$$\lim_{T\to\infty} \inf_{u\in\mathbb{H}_a} \mathcal{J}^{\rho,T}(u) = \inf_{r\in\mathbb{H}_a} \hat{\mathcal{J}}^{\rho,\infty}(r)$$

this allows to extend the variational description to the UV limit.

By our estimates we have

$$\|G_s^{1/2}\mathrm{D}V_s(\Phi_s^u)\|_{L^2} \lesssim \langle t\rangle^{-2\delta}$$

which is in L_t^2 only when $2\delta > 1$ that is $\beta^2 < 4\pi$. However the renormalized control r still retain L^2 regularity beyond this range and it is expected to do so for all $\beta^2 < 8\pi$ provided suitable approximate solutions of the flow equations can be found.

the variational method for Φ^4_2 in infinite volume

[N. Barashkov, **MG** · On the variational method for Euclidean quantum fields in infinite volume · arXiv:2112.05562]

Boué–Dupuis formula

Theorem. *Let* $(B_t)_{t\geqslant 0}$ *be a Brownian motion on* $\mathbb{R}^n$*, then for any bounded* $F: C(\mathbb{R}_+;\mathbb{R}^n)\to\mathbb{R}$ *we have*

$$\log \mathbb{E}[e^{F(B_\bullet)}] = \sup_{u\in\mathbb{H}_a} \mathbb{E}\left[F(B_\bullet + I(u)_\bullet) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

with $u: \Omega\times\mathbb{R}_+\to\mathbb{R}^n$ *adapted to* B *and with*

$$I(u)_t := \int_0^t u_s \mathrm{d}s$$

$$\frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s \approx H(\mathrm{Law}(B_\bullet + I(u)_\bullet)|\mathrm{Law}(B_\bullet)).$$

[M. Boué and P. Dupuis, A Variational Representation for Certain Functionals of Brownian Motion, *Ann. Prob.* 26(4), 1641–59]

Boué–Dupuis for the $d=2$ GFF

$$\mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)(m^2-\Delta)^{-1}(x-y), \qquad t,s\in[0,1]$$

The BD formula gives

$$-\log\int e^{-F(\phi)}\mu(\mathrm{d}\phi) = -\log\mathbb{E}[e^{-F(W_1)}] = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[F(W_1+Z_1)+\frac{1}{2}\int_0^1\|u_s\|_{L^2}^2\mathrm{d}s\right]$$

where

$$Z_t = (m^2-\Delta)^{-1/2}\int_0^t u_s\mathrm{d}s, \qquad u_t = (m^2-\Delta)^{1/2}\dot{Z}_t$$

$$-\log\mathbb{E}[e^{-F(W_1)}] = \inf_{Z\in H^a}\mathbb{E}[F(W_1+Z_1)+\mathcal{E}(Z_\bullet)]$$

with

$$\mathcal{E}(Z_\bullet) := \frac{1}{2}\int_0^1\|(m^2-\Delta)^{1/2}\dot{Z}_s\|_{L^2}^2\mathrm{d}s = \frac{1}{2}\int_0^1(\|\nabla\dot{Z}_s\|_{L^2}^2+m^2\|\dot{Z}_s\|_{L^2}^2)\mathrm{d}s$$

Φ^4_2 in a bounded domain Λ

fix a compact region $\Lambda \Subset \mathbb{R}^2$ and consider the Φ^4_2 measure θ_Λ on $\mathscr{S}'(\mathbb{R}^2)$ with interaction in Λ and given by

$$\theta_\Lambda(\mathrm{d}\phi) := \frac{e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}{\int e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}, \qquad \phi \in \mathscr{S}'(\mathbb{R}^2)$$

with interaction potential $V_\Lambda(\phi) := \int_\Lambda \phi^4 - c\int_\Lambda \phi^2$. For any $f : \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let

$$e^{-\mathscr{W}_\Lambda(f)} := \int e^{-f(\phi)}\theta_\Lambda(\mathrm{d}\phi)$$

---

we have the variational representation, $Z = Z_1$, $Z_\bullet = (Z_t)_{t\in[0,1]}$:

$$\mathscr{W}_\Lambda(f) = \inf_{Z\in H^a} F^{f,\Lambda}(Z_\bullet) - \inf_{Z\in H^a} F^{0,\Lambda}(Z_\bullet)$$

where

$$F^{f,\Lambda}(Z_\bullet) := \mathbb{E}[f(W+Z) + \lambda V_\Lambda(W+Z) + \mathscr{E}(Z_\bullet)].$$

---

renormalized potential

$$V_\Lambda(W+Z)=\int_\Lambda \Big\{\underbrace{W^4-cW^2}_{\mathbb{W}^4}+4\underbrace{\Big[W^3-\frac{c}{4}W\Big]}_{\mathbb{W}^3}Z+6\underbrace{\Big[W^2-\frac{c}{6}\Big]}_{\mathbb{W}^2}Z^2+4WZ^3+Z^4\Big\}$$

take $c=12\mathbb{E}[W^2(x)]=+\infty$

$$V_\Lambda(W+Z)=\int_\Lambda \{4\mathbb{W}^3Z+6\mathbb{W}^2Z^2+4WZ^3+Z^4\}+\cdots$$

$$\mathbb{W}^n\in\mathscr{C}^{-n\kappa}(\Lambda)=B^{-n\kappa}_{\infty,\infty}(\Lambda)$$

here $B^{-\kappa}_{\infty,\infty}(\Lambda)$ is an Hölder–Besov space · a distribution $f\in\mathscr{S}'(\mathbb{T}^d)$ belongs to $B^{\alpha}_{\infty,\infty}(\Lambda)$ iff for any $n\geqslant 0$

$$\|\Delta_n f\|_{L^\infty}\leqslant(2^n)^{-\alpha}\|f\|_{B^{\alpha}_{\infty,\infty}(\Lambda)}$$

where $\Delta_n f=\mathscr{F}^{-1}(\varphi_n(\cdot)\mathscr{F}f)$ and φ_n is a function supported on an annulus of size $\approx 2^n$ · we have $f=\sum_{n\geqslant 0}\Delta_n f$ · if $\alpha>0$ $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d)$ is a space of functions otherwise they are only distributions

Euler–Lagrange equation for minimizers

Lemma. *there exists a minimizer $Z = Z^{f,\Lambda}$ of $F^{f,\Lambda}$. Any minimizer satisfies the Euler–Lagrange equations*

$$\begin{aligned} &\mathbb{E}\left(4\lambda\int_\Lambda Z^3 K + \int_0^1 \int_\Lambda (\dot{Z}_s(m^2-\Delta)\dot{K}_s)\mathrm{d}s\right) \\ =\ &\mathbb{E}\left(\int_\Lambda f'(W+Z)K + \lambda\int_\Lambda (\mathbb{W}^3 + \mathbb{W}^2 Z + 12 W Z^2) K\right) \end{aligned}$$

for any K adapted to the Brownian filtration and such that $K \in L^2(\mu, H)$.

▷ technically one really needs a relaxation to discuss minimizers, we ignore this all along this talk. the actualy object of study is the law of the pair $(\mathbb{W}, Z)$ and not the process Z. (similar as what happens in the Φ^4_3 paper)

apriori estimates

we use polynomial weights $\rho(x) = (1 + \ell|x|)^{-n}$ for large $n > 0$ and small $\ell > 0$.

Theorem. *There exists a constant C independent of $|\Lambda|$ such that, for any minimizer Z of $F^{f,\Lambda}(\mu)$ and any spatial weight $\rho: \Lambda \to [0,1]$ with $|\nabla\rho| \leqslant \varepsilon\rho$ for some $\varepsilon > 0$ small enough, we have*

$$\mathbb{E}\left[4\lambda \int_\Lambda \rho Z_1^4 + \int_0^1 \int_{\mathbb{R}^2} ((m^2 - \Delta)^{1/2} \rho^{1/2} \dot{Z}_s)^2 \mathrm{d}s\right] \leqslant C.$$

Proof. test the Euler–Lagrange equations with $K = \rho Z$ and then estimate the bad terms with the good terms and objects only depending on $\mathbb{W}$, e.g.

$$\left|\int_\Lambda \rho \mathbb{W}^3 Z\right| \leqslant C_\delta \|\mathbb{W}^3\|_{H^{-1}(\rho^{1/2})}^2 + \delta \|Z\|_{H^1(\rho^{1/2})}^2,$$

$$\left|\int_\Lambda \rho \mathbb{W}^2 Z^2\right| \leqslant C_\delta \|\rho^{1/8} \mathbb{W}^2\|_{C^{-\varepsilon}}^4 + \delta(\|\rho^{1/4} \bar{Z}\|_{L^4}^4 + \|\rho^{1/2} \bar{Z}\|_{H^{2\varepsilon}}^2), \cdots$$

tightness and bounds

$$\mathscr{W}_\Lambda(f) = \inf_Z F^{f,\Lambda}(Z) - \inf_Z F^{0,\Lambda}(Z) = F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

therefore

$$F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{f,\Lambda}) \leqslant \mathscr{W}_\Lambda(f) \leqslant F^{f,\Lambda}(Z^{0,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

and since, for any g,

$$F^{f,\Lambda}(Z^{g,\Lambda}) - F^{0,\Lambda}(Z^{g,\Lambda}) = \mathbb{E}[f(W + Z^{g,\Lambda}) + \lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})]$$

$$-\mathbb{E}[\lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})] = \mathbb{E}[f(W + Z^{g,\Lambda})]$$

$$\mathbb{E}[f(W + Z^{f,\Lambda})] \leqslant \mathscr{W}_\Lambda(f) \leqslant \mathbb{E}[f(W + Z^{0,\Lambda})]$$

---

consequences: tightness of $(\theta_\Lambda)_\Lambda$ in $\mathscr{S}'(\mathbb{R}^2)$ and optimal exponential bounds

$$\sup_\Lambda \int \exp(\delta \|\phi\|^4_{W^{-\kappa,4}(\rho)}) \theta_\Lambda(d\phi) < \infty$$

Euler–Lagrange equation in infinite volume

moreover

$$\int f(\phi)\,\theta_\Lambda(\mathrm{d}\phi) = \mathbb{E}[f(X+Z^{0,\Lambda})]$$

the family $(Z^{f,\Lambda})_\Lambda$ is converging (provided we look at the relaxed problem) and any limit point $Z = Z^f$ satisfies a EL equation:

$$\mathbb{E}\left\{\int_{\mathbb{R}^2} f'(W+Z)\,K + 4\lambda\int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,K + \int_0^1\int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\,\dot{K}_s \mathrm{d}s\right\} = 0$$

for any test process K (adapted to $\mathbb{W}$ and to Z).

<u>**a stochastic "elliptic" problem**</u>

the stochastic equation

rewrite the EL equation as

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \left(f'(W_1+Z_1)+4\lambda[\![(W_1+Z_1)^3]\!]+\dot{Z}_s(m^2-\Delta)\right)\dot{K}_s \mathrm{d}s\right\}=0$$

then

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \mathbb{E}\left[f'(W_1+Z_1)+4\lambda[\![(W_1+Z_1)^3]\!]+(m^2-\Delta)\dot{Z}_s \middle| \mathscr{F}_s\right]\dot{K}_s \mathrm{d}s\right\}=0$$

which implies that

$$(m^2-\Delta)\dot{Z}_s=-\mathbb{E}\left[f'(W_1+Z_1)+4\lambda[\![(W_1+Z_1)^3]\!]\middle|\mathscr{F}_s\right]$$

---

open questions

- uniqueness??
- Γ-convergence of the variational description of $\mathscr{W}_\Lambda(f)$?

not clear · we lack sufficient knowledge of the dependence on f of the solutions to the EL equations above

---

exponential interaction

we can study similarly the model with

$$V^{\xi}(\varphi)=\int_{\mathbb{R}^2}\xi(x)[\![\exp(\beta\varphi(x))]\!]\mathrm{d}x$$

for $\beta^2<8\pi$ and $\xi:\mathbb{R}^2\to[0,1]$ a smooth spatial cutoff function

$$V^{\xi}(W+Z)=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))\underbrace{[\![\exp(\beta W(x))]\!]\mathrm{d}x}_{M^{\beta}(\mathrm{d}x)}$$

$$=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))M^{\beta}(\mathrm{d}x),\qquad \text{[Gaussian multiplicative chaos]}$$

BD formula

$$\begin{aligned}\mathcal{W}^{\xi,\exp}(f) &= -\log\int\exp(-f(\phi))\mathrm{d}\nu^{\xi}\\ &= \inf_{Z\in\mathfrak{H}_a}\mathbb{E}\left[f(W+Z)+\int\xi\exp(\beta Z)\mathrm{d}M^{\beta}+\frac{1}{2}\int_0^1\int((m^2-\Delta)^{1/2}\dot{Z}_t)^2\mathrm{d}t\right]\end{aligned}$$

▷ the function $Z\mapsto V^{\xi}(W+Z)$ is convex!

variational description of the infinite volume limit

▷ thanks to convexity the EL equations have a unique limit Z in the ∞ volume limit

▷ moreover we have the Γ-convergence of the variational description:

$$\mathscr{W}_{\mathbb{R}^2}(f) = \lim_{n\to\infty}\left[-\log\int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp}\right]$$

$$= \lim_{n\to\infty}\,[\mathscr{W}_{\xi_n}(f) - \mathscr{W}_{\xi_n}(0)] = \inf_K G^{f,\infty,\exp}(K)$$

with functional

$$G^{f,\infty,\exp}(K) = \mathbb{E}\left[f(W+Z+K) + \underbrace{\int \exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^{\beta} + \mathcal{E}(K)}_{\geqslant 0}\right]$$

which depends via Z on the infinite volume measure for the exp interaction.

the FBSDE for Grassmann measures

Euclidean Fermions

Fermions: quantum particles satisfying Fermi–Dirac statistics

EQFT: Wick rotation of QFT. $t \to \tau = it$, $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ Euclidean space. Wightman functions $\to$ Schwinger functions.

$$\Psi, \Psi^* \to \psi, \bar{\psi}.$$

☞ K. Osterwalder and R. Schrader. Euclidean Fermi fields and a Feynman-Kac formula for Boson-Fermions models. *Helvetica Physica Acta*, 46:277–302, 1973.

Euclidean fermion fields $\psi, \bar{\psi}$ form a Grassmann algebra

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \quad (\psi_\alpha^2 = 0).$$

Schwinger functions

- Schwinger functions are given by a Berezin integral on $\Lambda = \mathrm{GA}(\psi, \bar{\psi})$

$$\langle O(\psi,\bar{\psi})\rangle = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-S_E(\psi,\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi} e^{-S_E(\psi,\bar{\psi})}} = \frac{\left\langle O(\psi,\bar{\psi}) e^{-V(\psi,\bar{\psi})}\right\rangle_C}{\left\langle e^{-V(\psi,\bar{\psi})}\right\rangle_C}$$

$$S_E(\psi,\bar{\psi}) = \frac{1}{2}(\psi, C\,\bar{\psi}) + V(\psi,\bar{\psi}) \qquad \langle O(\psi,\bar{\psi})\rangle_C = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-\frac{1}{2}(\psi, C\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{-\frac{1}{2}(\psi, C\bar{\psi})}}$$

- Under $\langle\cdot\rangle_C$ the variables $\psi,\bar{\psi}$ are “Gaussian” (Wicks' rule):

$$\langle \psi(x_1)\cdots\psi(x_{2n})\rangle_C = \sum_{\sigma} (-1)^{\sigma} \langle \psi(x_{\sigma(1)})\psi(x_{\sigma(2)})\rangle_C \cdots \langle \psi(x_{\sigma(2n-1)})\psi(x_{\sigma(2n-1)})\rangle_C$$

algebraic probability

▷ a non-commutative probability space $(\mathscr{A},\omega)$ is given by a C^*-algebra $\mathscr{A}$ and a *state* ω, a linear normalized positive functional on $\mathscr{A}$ (i.e. $\omega(aa^*)\geqslant 0$).

▷ a random variable is an algebra homomorphism into $\mathscr{A}$

☞ L. Accardi, A. Frigerio, and J. T. Lewis. Quantum stochastic processes. *Kyoto University. Research Institute for Mathematical Sciences. Publications*, 18(1):97–133, 1982. 10.2977/prims/1195184017

example. (classical) random variable X with values on a manifold $\mathscr{M}$?

$$\Omega \xrightarrow{X} \mathscr{M} \xrightarrow{f} \mathbb{R}$$

$$f\in L^\infty(\mathscr{M};\mathbb{C})\to X(f)\in\mathscr{A}=L^\infty(\Omega;\mathbb{C}),\qquad X(fg)=X(f)X(g),\quad X(f^*)=X(f)^*.$$

algebraic data: $\mathscr{A}=L^\infty(\Omega;\mathbb{C})$, $\omega(a)=\int_\Omega a(\omega)\mathbb{P}(\mathrm{d}\omega)$, $X\in\mathrm{Hom}_*(L^\infty(\mathscr{M}),\mathscr{A})$.

Grassmann probability

- random variables with values in a Grassmann algebra Λ are algebra homomorphisms

$$\mathscr{G}(V) = \mathrm{Hom}(\Lambda V, \mathscr{A})$$

The embedding of ΛV into $\mathscr{A}$ allows to use the topology of $\mathscr{A}$ to do analysis on Grassmann algebras.

$$d_{\mathscr{G}(V)}(X, Y) := \|X - Y\|_{\mathscr{G}(V)} = \sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathscr{A}},$$

analogy. Gaussian processes in Hilbert space. Abstract Wiener space. “a convenient place where to hang our (analytic) hat on”.

back to QFT: IR & UV problems

QFT requires to consider the formula (Fermionic path integral)

$$\langle O(\psi,\bar{\psi})\rangle_{C,V} = \frac{\left\langle O(\psi,\bar{\psi})e^{-V(\psi,\bar{\psi})}\right\rangle_C}{\left\langle e^{-V(\psi,\bar{\psi})}\right\rangle_C}$$

with local interaction

$$V(\psi,\bar{\psi}) = \int_{\mathbb{R}^d} P(\psi(x),\bar{\psi}(x))\mathrm{d}x$$

and singular covariance kernel (due to reflection positivity)

$$\langle\bar{\psi}(x)\psi(y)\rangle \propto |x-y|^{-\alpha}$$

this gives an ill-defined representation

- **large scale (IR) problems**
- **small scale (UV) problems**

well understood in the constructive QFT literature (Gawedzki, Kupiainen, Lesniewski, Rivasseau, Seneor, Magnen, Feldman, Salmhofer, Mastropietro, Giuliani, ...)

what about stochastic quantisation for Grassmann measures?

☞ Ignatyuk/Malyshev/Sidoravicius | "Convergence of the Stochastic Quantization Method I,II", 1993. [Grassmann variables + cluster expansion]

weak topology + solution of equations in law + infinite volume limit but no removal of the UV cutoff

*

☞ "Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions" | joint work with Sergio Albeverio, Luigi Borasi, Francesco C. De Vecchi. arXiv:2004.09637 (PTRF)

algebraic probability viewpoint + strong solutions via Picard interation + infinite volume limit but no removal of the UV cutoff

☞ "A stochastic analysis of subcritical Euclidean fermionic field theories" | joint work with Francesco C. De Vecchi and Luca Fresta. arXiv:2210.15047

alg. prob. + forward-backward SDE + infinite volume limit & removal of IR cutoff in the whole subcritical regime

Grassmann stochastic analysis

▷ filtration $(\mathscr{A}_t)_{t\geqslant 0}$, conditional expectation $\omega_t\colon \mathscr{A}\to\mathscr{A}_t$,

$$\omega_t(ABC) = A\omega_t(B)C, \qquad A,C\in\mathscr{A}_t.$$

▷ Brownian motion $(B_t)_{t\geqslant 0}$ with $B_t\in\mathscr{G}(V)$

$$\omega(B_t(v)B_s(w)) = \langle v, Cw\rangle(t\wedge s), \qquad t,s\geqslant 0, v,w\in V.$$

$$\|B_t - B_s\| \lesssim |t-s|^{1/2}.$$

▷ Ito formula

$$\Psi_t = \Psi_0 + \int_0^t B_u(\Psi_u)\mathrm{d}u + X_t, \qquad \omega(X_t\otimes X_s) = C_{t\wedge s}$$

$$\omega_s(F_t(\Psi_t)) = \omega_s(F_s(\Psi_s)) + \int_s^t \omega_s[\partial_u F_u(\Psi_u) + \mathscr{L}F_u(\Psi_u)]\mathrm{d}u,$$

$$\mathscr{L}_u F_u = \frac{1}{2}\mathrm{D}^2_{\dot{C}_u}F_u + \langle B_u, \mathrm{D}F_u\rangle$$

the forward-backward SDE

[joint work with Francesco C. De Vecchi and Luca Fresta]

let Ψ be a solution of

$$\mathrm{d}\Psi_s = \dot{C}_s\, \omega_s(\mathrm{D}V(\Psi_T))\mathrm{d}s + \mathrm{d}X_s, \qquad s \in [0,T], \quad \Psi_0 = 0.$$

where $(X_t)_t$ is Gaussian martingale with covariance $\omega(X_t \otimes X_s) = C_{t \wedge s}$. Then

$$\omega(\mathrm{e}^{V(X_T)})\omega(\mathrm{e}^{-V(\Psi_T)}) = 1$$

and

$$\omega(O(\Psi_T)) = \frac{\omega(O(X_T)\mathrm{e}^{V(X_T)})}{\omega(\mathrm{e}^{V(X_T)})} = \frac{\langle O(\psi)e^{V(\psi)}\rangle_{C_T}}{\langle e^{V(\psi)}\rangle_{C_T}}$$

for any O.

$\triangleright$ this FBSDE provides a stochastic quantisation of the Grassmann Gibbs measure along the interpolation $(X_t)_t$ of its Gaussian component

the backwards step

let F_t be such that $F_T = \mathrm{D}V$. By Ito formula

$$B_s := \omega_s(\mathrm{D}V(\Psi_T)) = \omega_s(F_T(\Psi_T))$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

letting $R_t = B_t - F_s(\Psi_s)$ we have now the forwards-backwards system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

with

$$Q_u := \partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{DF}_u\rangle$$

solution theory

▷ standard interpolation for $C_\infty = (1+\Delta_{\mathbb{R}^d})^{\gamma - d/2}$, $\gamma \leqslant d/2$. $\chi \in C^\infty(\mathbb{R}_+)$, compactly supported around 0:

$$C_t := (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}\chi(2^{-2t}(-\Delta_{\mathbb{R}^d})), \qquad \|\dot{C}\|_{\mathscr{L}(L^\infty,L^\infty)} \lesssim 2^{2\gamma-d}, \|\dot{C}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim 2^{2\gamma}$$

▷ the system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s)+R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{D}F_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

can be solved by standard fixpoint methods for small interaction, uniformly in the volume since X stays bounded as long as $T < \infty$:

$$\|X_t\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{\gamma t}.$$

▷ decay of correlations can be proved by coupling different solutions (Funaki '96).

▷ limit $T \to \infty$ requires renormalization when $\gamma \in [0, d/2]$.

relation with the continuous RG

if we take F such that $Q=0$ we have $R=0$ and then

$$\Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s))\mathrm{d}s + X_t,$$

with

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{D} \mathrm{F}_u \rangle = 0, \qquad F_T = \mathrm{D}V.$$

define the effective potential V_t by the solution of the HJB equation

$$\partial_u V_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} V_u + \langle \mathrm{D}V_u, \dot{C}_u \mathrm{D}\mathrm{V}_u \rangle = 0, \qquad V_T = V.$$

then $F_t = \mathrm{D}V_t$ and the FBSDE computes the solution of the RG flow equation along the interacting field.

▷ so far a full control of the Fermionic HJB equation has not been achieved (work by Brydges, Disertori, Rivasseau, Salmhofer,...). Fermionic RG methods rely on a discrete version of the RG iteration.

approximate flow equation

thanks for the FBSDE we are not bound to solve exactly the flow equation and we can proceed to approximate it.

▷ **linear approximation.** take

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u = 0, \qquad F_T = \mathrm{D}V.$$

this corresponds to Wick renormalization of the potential V:

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \, (F_s(\Psi_s) + R_s) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

the key difficulty is to show uniform estimates for

$$\int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u$$

as $T \to \infty$. we cannot expect better than $\|\Psi_t\| \approx \|X_t\| \approx 2^{\gamma t}$.

polynomial truncation

a better approximation is to truncate the equation to a (large) finite polynomial degree

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \Pi_{\leqslant K}\langle F_u, \dot{C}_u \mathrm{D} F_u\rangle = 0$$

where $\Pi_{\leqslant K}$ denotes projection on Grassmann polynomials of degree $\leqslant K$ and take

$$F_t(\psi) = \sum_{k \leqslant K} F_t^{(k)} \psi^{\otimes k}.$$

With this approximation one can solve the flow equation and get estimates

$$\|F_t^{(k)}\| \leqslant \frac{2^{(\alpha - \beta k)t}}{(k+1)^2}, \qquad t \geqslant 0,$$

with $\alpha = 3\beta$, $\beta = d/2 - \gamma$, provided the initial condition $F_T = \mathrm{D}V$ is appropriately renormalized.

FBSDE in the full subcritical regime

with the truncation Π_K we have

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \, (F_s(\Psi_s) + R_s) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{D} F_u \rangle (\Psi_u)] \mathrm{d}u + \int_t^T \omega_t [\langle R_u, \dot{C}_u \mathrm{D} F_u (\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

but now observe that

$$\| \Psi_t \| \approx \| X_t \| \lesssim 2^{\gamma t} \qquad \| F_t^{(k)} \Psi_t^{\otimes k} \| \lesssim 2^{(\gamma k - \beta (k - 3)) t}$$

which is exponentially small for k large as long as $\gamma \leqslant d / 4$ (full subcrititcal regime).

now the term

$$\int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{D} F_u \rangle (\Psi_u)] \mathrm{d}u$$

can be controlled uniformly as $T \rightarrow \infty$ and also the full FBSDE system. (!)

a new class of equations

a new class of equations for Euclidean fields

Goal · identify a rigorous framework to analyse Euclidean fields

Let φ_∞ be a random field on $\mathbb{R}^d$, possibly distributional.

❶ We endow it with a decomposition over scales $(\varphi_a)_{a\geqslant 0}$ where φ_a is a description of φ_∞ including fluctuations at scales larger than $1/a$. $\varphi_a \to \varphi_\infty$ as $a \to \infty$ and $a \mapsto \varphi_a$ is continuous in some topology over smooth fields.

❷ Let $(\mathscr{F}_a)_a$ the filtration generated by φ_a. An **observable** is a martingale wrt. this filtration. The observable $(\mathscr{O}_a)_a$ is supported on a set $U \subseteq \mathbb{R}^d$ if

$$\mathscr{O}_a - \hat{\mathscr{O}}_a(\varphi_a, \nabla\varphi_a, \cdots) \to 0, \qquad \text{as } a \to \infty$$

where $\hat{\mathscr{O}}_a$ is a functional of φ_a which depends on the fields only on a $1/a$-enlargement of the set U. A field of observables $x \in \mathbb{R}^d \mapsto (\mathscr{O}_a(x))_a$ is **local** if $\mathscr{O}_a(x)$ is supported on $\{x\}$ for all x.

E.g. if φ_∞ is a function:

$$\mathscr{O}_a(x) = \mathbb{E}[\varphi_\infty(x) | \mathscr{F}_a]$$

❸ We assume that the **scale dynamics** of $(\varphi_a)_a$ is given by an Itô diffusion:

$$\mathrm{d}\varphi_a = B_a \mathrm{d}a + \mathrm{d}M_a, \qquad \mathrm{d}\langle M \otimes M \rangle_a = D_a^2 \mathrm{d}a$$

with adapted drift B_a and diffusivity "matrix" D_a^2. We want that the dynamics is specified only in terms of features of φ_∞ "brought back" to the scale a. So we postulate:

a) the existence of local observables for the "microscopic" drift $(f_a)_{a\geqslant 0}$ and for the "microscopic" diffusivity $(\Sigma_a^2)_{a\geqslant 0}$

b) that the characteristics B_a, D_a^2 of the diffusion at scale a are given by some spatial averaging of the microscopic characteristics:

$$B_a = \dot{C}_a f_a, \qquad D_a^2 = \dot{C}_a^{1/2} \Sigma_a^2 \dot{C}_a^{1/2}$$

where $(C_a)_a$ are spatial averaging operators at scale a and $\dot{C}_a = \partial_a C_a$. E.g.

$$(C_a f)(x) = \int_{\mathbb{R}^d} a^d \chi(a(x-y)) f(y) \mathrm{d}y$$

where $\chi: \mathbb{R}^d \to \mathbb{R}$ is a mass one, positive and positive definite function with support on the unit ball. Note that we could allow also random averaging: $C_a = C_a(\varphi_a)$.

Wilson–Ito diffusions

[I. Bailleul, I. Chevyrev, MG. Wilson–Ito diffusions. arXiv (July 2023)]

Definition. *A* ***Wilson–Ito diffusion*** $(\varphi_a)_a$ *is the solution of the SDE*

$$\mathrm{d}\varphi_a = \dot{C}_a f_a \mathrm{d}a + \dot{C}_a^{1/2} \Sigma_a \mathrm{d}W_a, \qquad a \geqslant 0$$

where W is a cylindrical Brownian motion, f_a, Σ_a^2 are local observables for the microscopic drift and diffusivity and C_a is a local averaging operator at scale a.

It describes the random field φ_∞.

Covariant under change of scales · $A = A(a)$, $\tilde{\varphi}_a := \varphi_{A(a)}$ then $\tilde{C}_a = C_{A(a)}$ and

$$\mathrm{d}\tilde{\varphi}_a = \dot{\tilde{C}}_a f_{A(a)} \mathrm{d}a + \dot{\tilde{C}}_a^{1/2} \Sigma_{A(a)} \mathrm{d}\tilde{W}_a,$$

where $\tilde{W}$ is a cylindrical Brownian motion.

Trivial example · $f_a = 0$, $\Sigma_a = 1$. Then $\varphi_\infty = \int_0^\infty \dot{C}_a^{1/2} \mathrm{d}W_a$ is a white noise, so in general the solutions are distributions.

are there non-trivial examples?

approximation strategy · fix some $A > 0$ and functional F_A and solve the forward-backward SDE:

$$d\phi_a = \dot{C}_a \mathbb{E}_a[F_A(\phi_A)]da + \dot{C}_a^{1/2} dW_a, \qquad a \in (0, A).$$

where $\mathbb{E}_a = \mathbb{E}[*|\mathscr{F}_a]$. Try to send $A \to \infty$ and at the same time make F_A more and more local.

coherent germs · another approach is to "guess" the drift $f_a \approx F_a(\varphi_a)$ where F_a is the "germ"

$$f_a = F_a(\varphi_a) + R_a$$

then we have a forward–backwards system for (φ_a, R_a):

$$\begin{cases} d\phi_a = \dot{C}_a(F_a(\varphi_a) + R_a)da + \dot{C}_a^{1/2} dW_a, \\ R_a = \mathbb{E}_a[\int_a^\infty \mathscr{L}_b F_b(\varphi_b) db + \int_a^\infty DF_b(\varphi_a)\dot{C}_b R_b db] \end{cases}$$

$$\mathscr{L}_b = \partial_b + \frac{1}{2}\Delta_{\dot{C}_b} + F_b \dot{C}_b D$$

Fully open problem in generality · I know very little about it · (some examples later)

a linear force

assume

$$f_a = \mathbb{E}_a[-A\phi_\infty] + \mathbb{E}_a[h(\phi_\infty)]$$

where A is a positive linear operator, e.g. $A = m^2 - \Delta$. Then, with $C_{\infty,a} := C_\infty - C_a$

$$\phi_\infty = \phi_a + \int_a^\infty \dot{C}_a(\mathbb{E}_a[-A\phi_\infty] + \mathbb{E}_a[h(\phi_\infty)])\mathrm{d}a + \int_a^\infty \dot{C}_a^{1/2}\mathrm{d}W_a$$

$$\mathbb{E}_a[\phi_\infty] = \phi_a - C_{\infty,a}A\mathbb{E}_a[\phi_\infty] + C_{\infty,a}\mathbb{E}_a[h(\phi_\infty)]$$

Let $\psi_a := (1 + C_{\infty,a}A)^{-1}\phi_a$ then $\psi_\infty = \phi_\infty$ and

$$\mathrm{d}\psi_a = \dot{Q}_a\mathbb{E}_a[h(\psi_\infty)]\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a, \qquad \dot{Q}_a := \partial_a(A^{-1}(1 + C_{\infty,a}A)^{-1})$$

the Gaussian field

$$X_a^Q := \int_0^a \dot{Q}_c^{1/2}\mathrm{d}W_c$$

has covariance $Q_\infty - Q_0 = (1 + A)^{-1}$, i.e. is a GFF when $A = m^2 - \Delta$.

gradient Wilson–Ito diffusions

Assume now also that $h(\psi_\infty) = -\mathrm{D}V_\infty(\psi_\infty)$, and let $(V_a)_{a\geqslant 0}$ be the solution to the *Polchinski equation* (HJB)

$$\partial_a V_a - \frac{1}{2}\mathrm{D}V_a \dot{Q}_a \mathrm{D}V_a + \frac{1}{2}\dot{Q}_a \mathrm{D}^2 V_a = 0$$

then one can prove that

$$\mathbb{E}_a[h(\psi_\infty)] = -\mathbb{E}_a[\mathrm{D}V_\infty(\psi_\infty)] = -\mathrm{D}V_a(\psi_a)$$

and by performing Doob's h-transform with $\mathrm{d}\mathbb{Q} = e^{-V_0(0)+V_a(\psi_a)}\mathrm{d}\mathbb{P}$ also that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[G(\psi_a)] &= \mathbb{E}_{\mathbb{Q}}[G(\psi_a)e^{V_0(0)-V_a(\psi_a)}] = \mathbb{E}_{\mathbb{P}}\left[G(X_a^Q)e^{V_0(0)-V_a(X_a^Q)}\right] \\ &= \frac{\mathbb{E}_{\mathbb{P}}\left[G(X_a^Q)e^{-V_\infty(X_\infty^Q)}\right]}{\mathbb{E}_{\mathbb{P}}\left[e^{-V_\infty(X_\infty^Q)}\right]}\end{aligned}$$

for any nice function G. In particular the law of ψ_∞ is given by the Gibbs measure

$$\nu_\infty(\mathrm{d}\psi) = \frac{e^{-V_\infty(\psi)}\mu^{Q_\infty}(\mathrm{d}\psi)}{\int e^{-V_\infty(\psi)}\mu^{Q_\infty}(\mathrm{d}\psi)}.$$

Euclidean fields as Wilson–Ito fields

The class of Wilson–Itô fields comprises as a particular case the Euclidean quantum fields constructed as perturbations of a Gaussian field. They are obtained by solving **Polchinski FBSDEs** of the form

$$\mathrm{d}\psi_a = -\dot{Q}_a \mathbb{E}_a[\mathrm{D}\, V_\infty(\psi_\infty)]\, \mathrm{d}a + \dot{Q}_a^{1/2} \mathrm{d}W_a.$$

Optimal control formulation · Let $u_a := -\dot{Q}_a^{1/2} \mathbb{E}_a[\mathrm{D}\, V_\infty(\psi_\infty)]$, test it with adapted $(v_a)_a$ and integrate:

$$\mathbb{E}\Big[\int_0^\infty \langle v_a, u_a\rangle \mathrm{d}a + \Big\langle \int_0^\infty \dot{Q}_a^{1/2} v_a \mathrm{d}a, \mathrm{D}V_\infty(\psi_\infty)\Big\rangle\Big] = 0.$$

It is the first-order condition for the minimisation of the functional

$$\Psi(u) := \mathbb{E}\left[V_\infty(\psi_\infty^u) + \frac{1}{2}\int_0^\infty \langle u_a, u_a\rangle \mathrm{d}a\right]$$

over all adapted controls $(u_a)_{a \geqslant 0}$, where

$$\psi_a^u := \int_0^a \dot{Q}_b^{1/2} u_b \mathrm{d}b + \int_0^a \dot{Q}_b^{1/2} \mathrm{d}W_b,$$

is the controlled process.

rigorous results

While Wilson–Ito fields are very young (less than one week) we have already established some results in the same flavour by looking at FBSDE or at the stochastic control formulation of Euclidean fields.

- N. Barashkov and MG · A Variational Method for Φ^4_3 · *Duke Mathematical Journal* (2020)
- N. Barashkov and MG · The Φ^4_3 Measure via Girsanov's Theorem · EJP (2021)
- N. Barashkov's PhD thesis · University of Bonn (2021)
- N. Barashkov and MG · On the Variational Method for Euclidean Quantum Fields in Infinite Volume, Prob. Math. Phys. (2023+)
- N. Barashkov · A Stochastic Control Approach to Sine Gordon EQFT · arXiv (2022)
- R. Bauerschmidt, M. Hofstetter · Maximum and Coupling of the Sine-Gordon Field · Ann. Prob. (2022)
- F. C. De Vecchi, L. Fresta, and MG · A stochastic analysis of subcritical Euclidean fermionic field theories · arXiv (2022)
- N. Barashkov, T. S. Gunaratnam, M. Hofstetter · Multiscale Coupling and the Maximum of φ^4_2 Models on the Torus · arXiv (2023)
- R. Bauerschmidt, T. Bodineau, B. Dagallier · Stochastic Dynamics and the Polchinski Equation: An Introduction · arXiv (2023)
- MG and S. J. Meyer · An FBSDE for Sine–Gordon up to 6π. In preparation.

some remarks on Wilson–Ito diffusions

- our the working hypothesis is that Wilson–Ito diffusions are natural mechanism to generate and analyse random fields
- they emerge from simple and natural assumptions and covers in principle much more than those theories that can be reached perturbatively from a Gaussian functional integral, e.g. from the path-integral picture
- they can be used for gauge theories and fields on manifolds and for Grassmann fields
- they allow for rigorous non-perturbative results in the whole space
- (hopefully) they provide a new framework for the stochastic analysis of Euclidean fields
- numerical simulations?
- still lot to understand: FBSDEs are non-trivial to analyse but PDE methods seems applicable similarly to Parisi–Wu style stochastic quantisation.

the end

(no human has been harmed with TEX/LATEX to produce this presentation)