Singular SPDEs and paracontrolled distributions

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Some problems in singular SPDEs /I

We want to define and solve (in a robust way) the following SPDEs:

- Stochastic differential equations (1+0): $u \in [0,T] \to \mathbb{R}^n$

$$\partial_t u(t) = \sum_i f_i(u(t))\xi^i(t)$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.

- Burgers equations (1+1): $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))Du(t,x) + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

- Generalized Parabolic Anderson model (1+2): $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

$$\partial_t u(t,x) = \Delta u(t,x) + \text{''} f(u(t,x))\xi(x) \text{''}$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.

Recall that

$$\xi \in \mathscr{C}^{-d/2-} \qquad \text{a.s.}$$

Some problems in singular SPDEs /II

Define and solve (in a robust way) the following SPDEs:

- Kardar-Parisi-Zhang equation (1+1)

$$\partial_t h(t,x) = \Delta h(t,x) + [(Du(t,x))^2 - \infty] + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

- Stochastic quantization equation (1+3)

$$\partial_t u(t,x) = \Delta u(t,x) + [u(t,x)^3 - \infty u(t,x)] + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.

- But not: Multiplicative SPDEs (1+1)

$$\partial_t u(t,x) = \Delta u(t,x) + [f(u(t,x))\xi(t,x)]$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise (only RS can deal with it)

Joint work with P. Imkeller and N. Perkowski.

Related approaches

- **Itô stochastic calculus.** Cannot be used when irregularity is in the space variable. When it works does not deliver a robust solution theory.
- **Malliavin calculus.** Not obvious how to deal with non-linear equations. Cannot set up fix point procedures.
- **Rough path theory.** (T. Lyons) Main source of inspiration. Delivers a robust solution theory for stochastic differential equations. Has limited applications to SPDEs.
- **Regularity Structures** (M. Hairer) Vast generalisation of Rough paths. Can deal with more equation than the present approach. Describe local behaviour of functions in terms of basic objects. Here instead we try to describe the same object in Fourier space.
- Related work on paracontrolled SPDEs by Chouk, Catellier, Mourrat, Weber, Zhu, Zhu.

Rough differential equation

Consider the simple controlled ODE (η smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^{m} f_i(u(t))\eta^i(t)$$

$u : \mathbb{R} \to \mathbb{R}^d$, $\eta : \mathbb{R} \to \mathbb{R}^d$ and smooth vectorfields $f_i : \mathbb{R}^d \to \mathbb{R}^d$.

Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for $\eta \in \mathscr{C}^{\gamma-1}$ with $\gamma < 1/2$.

Reason: $u \in \mathscr{C}^{\gamma}$ and $\eta \in \mathscr{C}^{\gamma-1}$ cannot be multiplied when $2\gamma - 1 \leq 0$. The r.h.s. of the equation is not well defined.

Here $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ is the Holder–Besov space (or a local version).

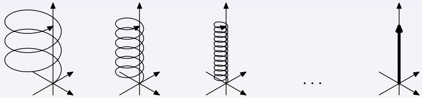
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2}$$



$$I(x^{n,1}, x^{n,2})(t) \not\to I(0,0)(t) = 0$$

The definite integral $I(\cdot,\cdot)(t)$ is not a continuous map $C^\gamma \times C^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Functional analysis is not enough

Consider the random functions $(X^n, Y^n) : \mathbb{R} \to \mathbb{R}^2$

$$X^N(t) = \sum_{1 \le n \le N} \frac{g_n}{n} \cos(2\pi n t) + \frac{g'_n}{n} \sin(2\pi n t)$$

$$Y^N(t) = \sum_{1 \le n \le N} \frac{g_n}{n} \sin(2\pi n t) - \frac{g'_n}{n} \cos(2\pi n t)$$

where $(g_n, g'_n)_{n \ge 1}$ are iid normal variables. Then

$$I(X^N, Y^N)(1) = \int_0^1 X^N(s) \partial_s Y^N(s) \mathrm{d}s = 2\pi \sum_{1 \le n \le N} \frac{g_n^2 + (g'_n)^2}{n} \to +\infty$$

almost surely as $N \to \infty$.

Theorem (Lyons)

No continuous map on a Banach space of paths can represent the integral I and allow Brownian motion at the same time.

Concept of solution

Goal: Show that Ψ factorizes as

$$\eta \xrightarrow{J} (\eta, \theta \circ \eta) \xrightarrow{\Phi} u$$

(here $\partial_t \theta = \eta$ and $\theta \circ \eta = X^2(\eta)$ will be described later)

$\triangleright$ *Analytic step:* show that when $\gamma > 1/3$:

$$\Phi : \mathcal{X} \to \mathscr{C}^{\gamma}$$

is continous. $\mathcal{X} = \overline{\mathrm{Im} J} \subseteq \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ is the space of *enhanced signals* (or rough paths, or models).
But in general J is not a continuous map $\mathscr{C}^{\gamma-1} \to \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$.

$\triangleright$ *Probabilistic step:* prove that there exists a "reasonable definition" of $J(\xi)$ when ξ is a white noise. $J(\xi)$ is an explicit polinomial in ξ so direct computations are possible.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $\mathscr{C}^\gamma = B^\gamma_{\infty,\infty}$.

$f \in \mathscr{C}^\gamma$, $\gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \leq \|f\|_\gamma 2^{-i\gamma}, \qquad i \geq -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi)\hat{f}(\xi)$$

where $\rho_i : \mathbb{R}^d \to \mathbb{R}_+$ are smooth functions with support in annuli $\simeq 2^i \mathscr{A}$ when $i \geq 0$ and form a partition of unity

$$\sum_{i \geq -1} \rho_i(\xi) = 1$$

for all $\xi \neq 0$ so that

$$f = \sum_{i \geq -1} \Delta_i f$$

in $\mathcal{S}'$.

Paraproducts

Deconstruction of a product: $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$

$$fg = \sum_{i,j \geq -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i < j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$\pi_<(f,g) \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$\pi_\circ(f,g) \in \mathscr{C}^{\gamma+\rho} \qquad \text{if } \gamma + \rho > 0$$

Proof. Recall $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$.

$$i \ll j \Rightarrow \operatorname{supp} \mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A}$$

$$i \sim j \Rightarrow \operatorname{supp} \mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i:i<j-1} O(2^{-i\rho - j\gamma}) = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^\gamma,$$

while if $\rho < 0$

$$\Delta_q(f \prec g) = \sum_{i:i<j-1} O(2^{-i\rho - j\gamma}) = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j \gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i \gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

The main commutator

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

which however "is" smoother than $f \prec g$ if f or g has positive regularity.

Commutator

The trilinear operator $C(f, g, h) = (f \prec g) \circ h - f(g \circ h)$ satisfies

$$\|C(f, g, h)\|_{\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

when $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, $\alpha < 1$.

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in C^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathscr{C}^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathscr{C}^{2\gamma}$$

Then

$$\varphi(B) \circ DB = (\varphi'(B) \prec B) \circ DB + \underbrace{\mathscr{C}^{2\gamma} \circ DB}_{\text{OK}}$$

$$= \varphi'(B)(B \circ DB) + \mathscr{C}^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B) \underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathscr{C}^{3\gamma-1}$$

The Besov area

The Besov area $B \circ DB$ can be defined and studied efficiently using Gaussian arguments:

$$B^\varepsilon \circ DB^\varepsilon \to B \circ DB$$

almost surely in $\mathscr{C}^{2\gamma-1}_{\mathrm{loc}}$ as $\varepsilon \to 0$.

Remark. If $d = 1$ (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2}D(B \circ B)$$

which is well defined and belongs indeed to $\mathscr{C}^{2\gamma-1}$.

Tools: Besov embeddings $L^p(\Omega; C^\theta) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Controlled paths/distributions

Controlled paths are paths which “looks like” a *given* path which often is random (but not necessarily).

A "good" quantification of this proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

A mix of functional analytic arguments and probabilistic ones.

Basic analogies

- Itô processes

$$dX_t = f_t dM_t + g_t dt$$

- Amplitude modulation

$$f(t) = g(t) \sin(\omega t)$$

with $|\mathrm{supp}\, \hat{g}| \ll \omega$.

(Para)controlled structure

Idea

Use the paraproduct to *define* a controlled structure. We say $y \in \mathscr{D}_x^\rho$ if $x \in \mathscr{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with $y^x \in C^{\rho-\gamma}$ and $y^\sharp \in C^\rho$.

Paralinearization. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^\gamma$, $\gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ Another commutator: $f, g \in \mathscr{C}^{\rho-\gamma}$, $x \in \mathscr{C}^\gamma$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^\rho$$

Stability. ($\rho \leq 2\gamma$)

$$\varphi(y) = (\varphi'(y) y^x) \prec x + \mathscr{C}^\rho$$

so we can take $\varphi(y)^x = \varphi'(y) y^x$.

RDEs - I - the r.h.s.

$u : \mathbb{R} \to \mathbb{R}^d$, $\xi \in \mathscr{C}^{-1/2-}$ is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz. Take $\partial_t X = \xi$, $X \in \mathscr{C}^{1/2-}$ and assume that $u \in \mathscr{D}_X^{1-}$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{1-}$ and $u^X \in \mathscr{C}^{1/2-}$.

▷ Paralinearization:

$$f(u) = f'(u) \prec u + \mathscr{C}^{1-} = (f'(u)u^X) \prec X + \mathscr{C}^{1-}$$

▷ Commutator lemma:

$$f(u) \circ \xi = ((f'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{1-} \circ \xi$$

$$= \underbrace{(f'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{0-}} + \underbrace{C(f'(u)u^X, X, \xi) + \mathscr{C}^{1-} \circ \xi}_{\in \mathscr{C}^{1/2-}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{0-}$.

RDEs - II - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{0-}$ we have a control on the r.h.s. of the equation:

$$f(u)\xi = \underline{f(u) \prec \xi} + f'(u)u^X(X \circ \xi) + \mathscr{C}^{1/2-}$$

What about the l.h.s.?

$$\partial_t u = \partial_t u^X \prec X + \underline{u^X \prec \xi} + \partial_t u^\sharp$$

so letting $u^X = f(u)$ we have

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f'(u)f(u)(X \circ \xi) + \mathscr{C}^{1/2-}$$

RDEs - III - the paracontrolled fixed point.

The RDE

$$\partial_t u = f(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X &= \xi \\
\partial_t u^\sharp &= (f'(u)f(u))(X \circ \xi) - \underbrace{\partial_t f(u) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f,u,X,\xi)}_{\in \mathscr{C}^{1/2-}} \circ \xi \\
u &= f(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^{1-}$ if we assume that

$$X \in \mathscr{C}^{1/2-}, \qquad (X \circ \xi) \in \mathscr{C}^{0-}.$$

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = f(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1} \to \mathscr{C}^\gamma$$

is continuous for any $\gamma > 1/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by the unique solution in $\mathscr{D}_X^{2\gamma}$ to

$$\begin{cases} z = f(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (f'(z)f(z))\varphi - \underbrace{\partial_t f(z) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f, z, X, \xi) \circ \xi}_{\in \mathscr{C}^{1/2-}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then

$$u^n \to u = \Phi(u_0, \xi, \eta).$$

Relaxed form of the RDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-1}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - \mathrm{D}^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{-1-}(\mathbb{T}^2)$ space white noise.

$$\mathcal{L}u(t,x) = f(u(t,x))\xi(x), \qquad u(0,\cdot) = u_0(x)$$

▷ Paracontrolled ansatz $\mathcal{L}X = \xi$ so $X \in C([0,T], \mathscr{C}^{1-})$

$$u = f(u) \prec X + u^\sharp$$

▷ Paralinearization: $f(u) = (f'(u)f(u)) \prec X + R(f,u,X)$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f,u,X) \circ \xi$$

▷ A problem: if ξ is the white noise

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$

$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2 = c + \mathscr{C}^{0-}$$

with $c = +\infty$.

Renormalization

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denote by $\diamond$. Now

$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\mathcal{L}u^{\sharp} = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c)$$
$$+C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

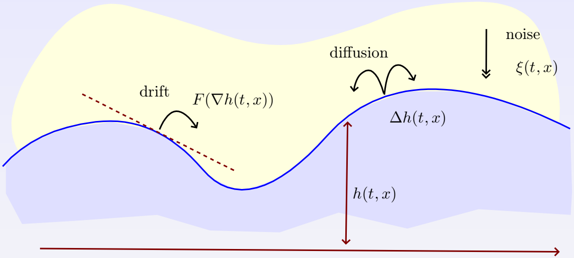
together with

$$u = f(u) \prec X + u^{\sharp}$$

and where

$$X \in \mathscr{C}^{1-}, \qquad (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^{\sharp} \in \mathscr{C}^{2-}.$$

The Kardar–Parisi–Zhang equation



Large scale dynamics of the height $h : [0,T] \times \mathbb{T} \to \mathbb{R}$ of an interface

$$\partial_t h \simeq \Delta h + F(Dh) + \xi$$

The universal limit should coincide with the large scale fluctuations of the KPZ equation

$$\partial_t h = \Delta h + [(Dh)^2 - \infty] + \xi$$

with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion /I

▷ Let $u = X + u_1$ then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▷ Let $X^{\vee}$ be the solution to

$$\mathcal{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

▷ Define $\mathcal{L}X^{\curlyvee} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\curlyvee} + u_3$ then $X^{\curlyvee} \in \mathscr{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\curlyvee}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\curlyvee} + U$$

$$\mathcal{L}U = 2\partial_x(UX) + 2\partial_x(X^{\curlyvee}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\curlyvee}+U)X^{\vee}) + \partial_x(2X^{\curlyvee}+U)^2$$

$$= {\color{red}2\partial_x(UX)} + \mathcal{L}(2X^{\Yup} + X^{\vee\vee}) + 2\partial_x((2X^{\curlyvee}+U)X^{\vee}) + \partial_x(2X^{\curlyvee}+U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	$X^{\curlyvee}$	$X^{\Yup}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\curlyvee}+U)X^{\vee}) + \partial_x(2X^{\curlyvee}+U)^2$$

are well defined.

The remaining problem is to deal with ${\color{red}2\partial_x(UX)}$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Y + U^\sharp$. Then

$$\mathcal{L}U = \mathcal{L}U' \prec Y + U' \prec \mathcal{L}Y - \partial_x U' \prec \partial_x Y + LU^\sharp$$

while

$$\mathcal{L}U = 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\vee\!\vee\!\vee} + X^{(\vee\vee)}) + 2\partial_x((2X^{\vee\!\vee} + U)X^{\vee}) + \partial_x(2X^{\vee\!\vee} + U)^2}_{Q(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

so we can set $U' = 2U$ and $\mathcal{L}Y = \partial_x X$ and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Y + \partial_x U' \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

▷ Observe that $Y, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Y) \circ X + U^{\sharp} \circ X = 2U(Y \circ X) + \underbrace{C(2U, Y, X)}_{1/2-} + \underbrace{U^{\sharp} \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Y \circ X \in \mathscr{C}^{0-}$

▷ The final fixed point equation reads

$$\mathcal{L} U^{\sharp} = 4\partial_x(U\underline{(Y \circ X)}) + 4\partial_x C(U, Y, X) + 2\partial_x(U^{\sharp} \circ X) - 2LU \prec Y$$
$$+ 2\partial_x U \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + Q(U)$$

▷ This equation has a (local in time) solution $U = \Phi(J(\xi))$ which is a continuous function of the data $J(\xi)$ given by a collection of multilinear functions of ξ:

$$J(\xi) = (X, X^{\vee}, X^{\Upsilon}, X^{\Upsilon_2}, X^{\vee\vee}, X \circ Y)$$

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3$: $\xi \in C^{-5/2-}$, $u \in C^{-1/2-}$,
$u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 -$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects: $(u_1^2 - c_1)$, $(u_1^3 - 3c_1 u_1)$, $(3\lambda(X \circ (u_1^2 - c_1)) - c_2)$, $(u_2 u_1)$, $(u_2^2 u_1)$

Thanks