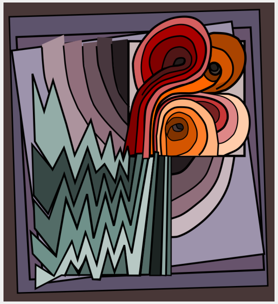
Maxwell Institute Probability and Stochastic Analysis Workshop · Edinburgh · March 13, 2024

Stochastic quantisation of the fractional Φ^4_3 Euclidean QFT in the full subcritical regime

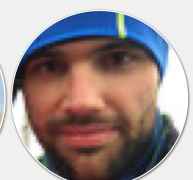


Massimiliano Gubinelli · University of Oxford

[made with TeXmacs]

Act I · Euclidean QFTs & stochastic analysis

Act II · the flow equation method for the fractional Φ^4_3

joint work with

Pawel Duch (Poznán) & **Paolo Rinaldi** (Bonn) act I

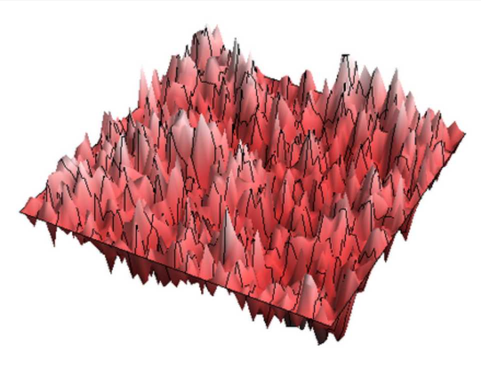
Euclidean QFTs & stochastic analysis

Euclidean quantum fields

Feynman–Kac formula / non-perturbative path integral formulation of QFTs

$$\nu(\mathrm{d}\varphi) = \frac{\exp[-S(\varphi)]\mathscr{D}\varphi}{Z}, \qquad \varphi \in \mathscr{S}'(\mathbb{R}^d)$$

$$S(\varphi) = \int_{\mathbb{R}^d} (|\nabla\varphi(x)|^2 + m^2\varphi(x)^2)\mathrm{d}x + \int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x$$



$$V(\varphi) = \lambda\varphi^4 + \cdots, \quad \lambda\cos(\beta\varphi), \quad \lambda\cosh(\beta\varphi), \quad \lambda\exp(\beta\varphi), \quad \cdots, \qquad d = 1,2,3$$

σ-models, gauge theories

(Markovian) probability measure ν on $\mathscr{S}'(\mathbb{R}^d)$ · natural generalisations of diffusion processes · introduced in the '70 in the context of the **constructive QFT program**

Gaussian free field

GFF · simplest example of EQF · Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ s.t.

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean · $m>0$ is the *mass* · $G(0)=+\infty$ if $d\geqslant 2$: not a function · distribution of regularity

$$\alpha < (2-d)/2$$

▷ can be used to construct a QFT but the theory is free: no interaction

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variation · fractional Laplacian covariance $s\in(0,1)$

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} (a-\Delta)^{-1}(x-y)\rho(\mathrm{d}a) = (m^2+(-\Delta)^s)^{-1}(x-y)$$

non-Gaussian Euclieand fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{Z}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side L

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi) e^{-\overbrace{\sum_{x\in\mathbb{Z}^d_{\varepsilon,L}} |\nabla_\varepsilon\varphi(x)|^2 + m^2\varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}}\mathrm{d}\varphi$$

ε is an <u>UV regularisation</u> and L the <u>IR regularisation</u>

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$ · e.g. take V_ε polynomial bounded below ($d = 2, 3$)

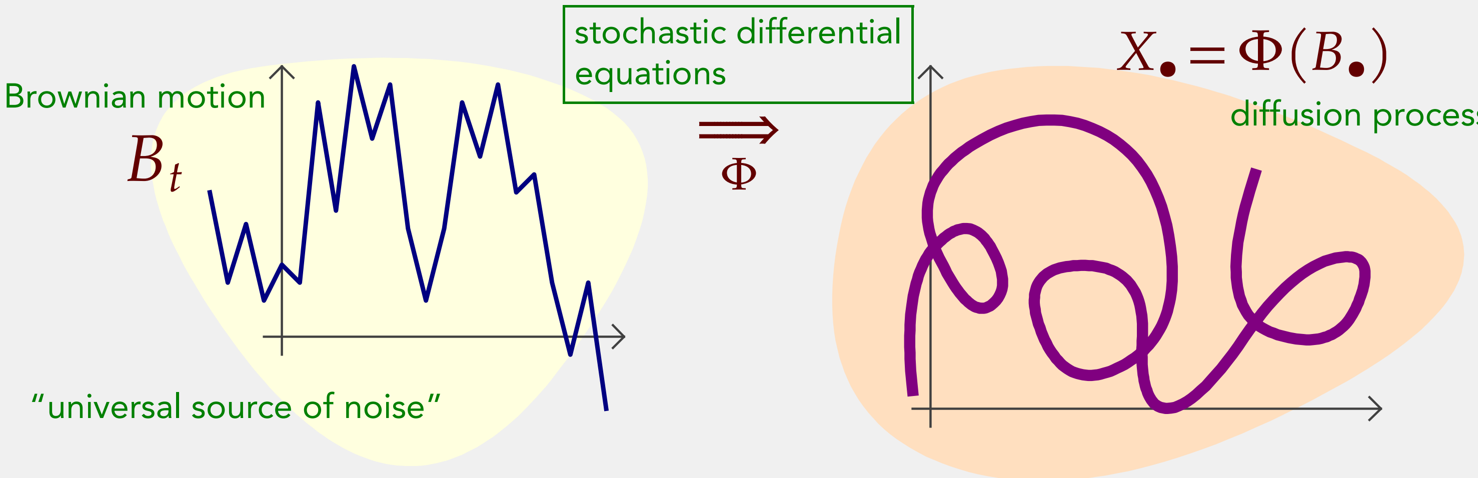
$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

the limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$

❸ study the possible limit points [the $\mathbf{\Phi}^4_d$ **measure**] · uniqueness? non-uniqueness? correlations? description?

what is **stochastic quantisation**?

Itô's fundamental intuition: stochastic analysis



$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

[...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.

(D. Stroock, "Elements of stochastic calculus and analysis ", Springer, 2018)

nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…

analysis vs. stochastic analysis

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y)\in\mathcal{O}\subseteq\mathbb{R}^2$	global description	$P_t(x,\mathrm{d}y)$
$\alpha(x-x_0)^2+\beta(y-y_0)^2=\gamma$	·	$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t)\approx x(t)+a\delta t+o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at+bt^2+\cdots$	building block	$(W_t)_t$
$(\ddot{x}(t),\ddot{y}(t))=F(x(t),y(t))$	local/global link	$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$

☞ other examples: rough paths, regularity structures, SLE,...

extend Itô fundamental insight to multi-dimensional random fields

stochastic quantisation map

$$e^{-\frac{1}{2}\int \xi C \xi} D\xi$$

Gaussian measure

source of noise

$$\varphi = \Phi(\xi)$$

$$\Longrightarrow$$

solution map of an SPDE

$$\frac{e^{-\int \varphi A \varphi + V(\varphi)}}{Z} D\varphi$$

Euclidean QFT

(S)PDE theory

(apriori estimates, geometry)

$$\partial_t \phi = -(m^2 - \Delta_x)\phi - V'(\phi) + c\xi,$$

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systematic use of tools from **PDE theory**, recent advances in **singular PDEs**, **renormalization group** ideas coupled with pathwise analysis

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stochastic quantisation as a stochastic analysis

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x,\mathrm{d}y)$	global description	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi$
$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$		$\left\langle F(\varphi)\frac{\delta S(\varphi)}{\delta\varphi}+\frac{\delta F(\varphi)}{\delta\varphi}\right\rangle=0$
t	change parameter	t
$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$	local description	$\phi(t+\delta t)\approx\alpha\phi(t)+\beta\delta X(t)+\cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X=\frac{1}{2}[(\Delta_x-m^2)X]+\xi$
$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$	local/global link	$\partial_t\phi=\frac{1}{2}[(\Delta_x-m^2)\phi-V'(\phi)]+\xi$

the exploration of a new paradigm…

❶ **parabolic** stochastic quantization

$$\varphi(x) \sim \phi(t,x) \qquad \partial_t \phi(t,x) = -(m^2 - \Delta_x)\phi(t,x) - V'(\phi(t,x)) + c\xi(t,x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

Parisi–Wu · Langevin diffusion

[**MG** and M. Hofmanová. Global Solutions to Elliptic and Parabolic ϕ^4 Models in Euclidean Space. *CMP* 2019]

[**MG** and M. Hofmanová. A PDE construction of the Euclidean Φ^4_3 quantum field theory. *CMP* 2021]

❷ **elliptic** stochastic quantization

$$\varphi(x) \sim \phi(z,x) \qquad (-\Delta_z)\phi(z,x) = -(m^2 - \Delta_x)\phi(z,x) - V'(\phi(z,x)) + c\xi(z,x), \quad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

Parisi–Sourlas supersymmetric reduction

[S. Albeverio, F. C. De Vecchi, and **MG**. Elliptic stochastic quantization. *Ann. Prob.* 2020]

the exploration of a new paradigm

❸ forward–backward SDEs, stochastic control & renormalization group

$$\varphi \sim \phi_\infty \qquad \phi_t = W_t + \int_0^t \dot{G}_s \mathbb{E}_s F_T(\phi_T) \mathrm{d}s \qquad t \in [0, T]$$

Wilson–Ito diffusions

[N. Barashkov and **MG**. A variational method for ϕ^4_3. *Duke Journal* 2020]

[I. Bailleul, I. Chevyrev and **MG**. Wilson–Ito diffusions. arXiv 2023]

❹ integration by parts formulas & Kolmogorov operators

$$\int_{\mathscr{S}'(\mathbb{R}^d)} \left[\frac{\delta}{\delta \varphi} - \frac{\delta S}{\delta \varphi}(\varphi) \right] F(\varphi) \nu(\mathrm{d}\varphi) = 0, \qquad \forall F \text{ in some nice class}$$

Dyson–Schwinger equations · uniqueness results for EQFTs

[F. C. De Vecchi, **MG** and M. Turra · A Singular Integration by Parts Formula for the Exponential Euclidean QFT on the Plane · arXiv 2022] act II

the flow equation method for the fractional Φ^4_3

[P. Duch, **MG**, P. Rinaldi · Parabolic stochastic quantisation of the fractional Φ^4_3 Euclidean quantum field in the full subcritical regime · arXiv:2303.18112]

fractional Φ^4_3 model

probability measure $\nu_{\varepsilon,M}$ on $\Omega_{\varepsilon,M} = \{\varphi\colon \mathbb{T}^d_{\varepsilon,M} \to \mathbb{R}\}$, $\mathbb{T}^d_{\varepsilon,M} = (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^d$, $\varepsilon > 0$, $M \in \mathbb{N}$

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{\exp(-S_{\varepsilon,M}(\varphi))}{Z_{\varepsilon,M}} \prod_{x \in \mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x)$$

$$S_{\varepsilon,M}(\varphi) = \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon,M}} \left[\frac{1}{2}\varphi(x)(-\Delta_\varepsilon)^s \varphi(x) + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4}\varphi(x)^4 - \frac{r_\varepsilon}{2}\varphi(x)^2\right]$$

- *$m > 0$ mass, $\lambda > 0$ coupling constant, $r_\varepsilon > 0$ mass renormalization*

Theorem. *Let $d = 3$ and $s \in (3/4, 1]$. There exists a choice of the sequence of mass renormalisations $(r_\varepsilon)_{\varepsilon > 0}$, with $r_\varepsilon = r_\varepsilon(\lambda) \to +\infty$ as $\varepsilon \to 0$, such that*

$\longrightarrow$ *$(\nu_{\varepsilon,M})_{\varepsilon,M}$ is a tight family of probability measure on $\mathscr{S}'(\mathbb{R}^3)$ as $M \to \infty$ and $\varepsilon \to 0$*

stochastic quantisation

finite system of SDEs for $\phi^{(\varepsilon,M)}\colon \Lambda_{\varepsilon,M} \to \mathbb{R}$, $\Lambda_{\varepsilon,M} := \mathbb{R} \times \mathbb{T}^d_{\varepsilon,M}$

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]\phi^{(\varepsilon,M)} + \lambda(\phi^{(\varepsilon,M)})^3 - r_\varepsilon \phi^{(\varepsilon,M)} = \xi^{(\varepsilon,M)}$$

$\lambda > 0, r_\varepsilon > 0 \cdot \boldsymbol{\xi}^{(\varepsilon,M)}$ *space–time white noise*

$$\mathbb{E}[\xi^{(\varepsilon,M)}(t,x)\xi^{(\varepsilon,M)}(s,y)] = \delta(t-s)\varepsilon^d \mathbb{1}_{x=y}, \qquad (t,x),(s,y) \in \Lambda_{\varepsilon,M}$$

$\exists$ stationary solution $\phi^{(\varepsilon,M)}$ s.t. $\mathrm{Law}(\phi^{(\varepsilon,M)}(t)) = \nu_{\varepsilon,M}$ for all $t \in \mathbb{R}$

$\triangleright$ PDE estimates in weighted Sobolev spaces + positivity of the fractional Laplacian

$\Longrightarrow$ tightness of $(\phi^{(\varepsilon,M)})_M \Longrightarrow \exists$ of sub-sequential limits as $M \to \infty$

▷ any accum. point $\phi^{(\varepsilon)}$ is solution in $\Lambda_\varepsilon := \mathbb{R} \times (\varepsilon\mathbb{Z})^d$ of an ∞ system of SDEs

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]\phi^{(\varepsilon)} + \lambda(\phi^{(\varepsilon)})^3 - r_\varepsilon \phi^{(\varepsilon)} = \xi^{(\varepsilon)}$$

▷ **subcriticality** : at small scales the non-linear term is a perturbation of the linear part of the equation with the white-noise source · $\phi^{(\varepsilon)}$ behaves like the solution $X^{(\varepsilon)}$ of the linear equation

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]X^{(\varepsilon)} = \xi^{(\varepsilon)}$$

☞ as $\varepsilon \to 0$ $X^{(\varepsilon)}$ has distributional limit in Besov spaces of regularity $< s - d/2$.

scale decomposition

$\mathscr{L}_\varepsilon := \partial_t + m^2 + (-\Delta_\varepsilon)^s \cdot F^\varepsilon(\psi) := -\lambda\psi^3 + r_\varepsilon \psi + \xi^{(\varepsilon)} \cdot (J_\sigma)$ scale decomposition $J_1 = \mathrm{Id}$

$$\mathscr{L}_\varepsilon \phi^\varepsilon = F^\varepsilon(\phi^{(\varepsilon)}) \quad \Longrightarrow \quad \phi^\varepsilon_\sigma := J_\sigma \phi^{(\varepsilon)} \quad \Longrightarrow \quad \mathscr{L}_\varepsilon \phi^\varepsilon_\sigma = J_\sigma(F^\varepsilon(\phi^{(\varepsilon)}))$$

$(F^\varepsilon_\sigma)_{\sigma\in(0,1)} : C(\Lambda_\varepsilon) \to \mathscr{S}'(\Lambda_\varepsilon)$ family of smooth functionals s.t. $\boldsymbol{F^\varepsilon_1 = F^\varepsilon}$,

$$\boldsymbol{F^\varepsilon(\phi^\varepsilon) = F^\varepsilon_\mu(\phi^\varepsilon_\mu) + R^\varepsilon_\mu} \qquad \boldsymbol{R^\varepsilon_\mu := \int_\mu^1 [\partial_\sigma F^\varepsilon_\sigma(\phi^\varepsilon_\sigma) + \mathrm{D}F^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\cdot(\partial_\sigma \phi^\varepsilon_\sigma)]\mathrm{d}\sigma}$$

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for any choice of $(F^\varepsilon_\sigma)_{\sigma\in[0,1]}$, $(\phi^\varepsilon_\mu, R^\varepsilon_\mu)_{\mu\in(0,1)}$ satisfies

$$\begin{cases} \mathscr{L}_\varepsilon \phi^\varepsilon_\mu = J_\mu(F^\varepsilon_\mu(\phi^\varepsilon_\mu) + R^\varepsilon_\mu) \\ R^\varepsilon_\mu = \int_\mu^1 H^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\cdot \dot{G}_\sigma R^\varepsilon_\sigma]\mathrm{d}\sigma \end{cases}$$

with

$$H^\varepsilon_\sigma(\psi) := \partial_\sigma F^\varepsilon_\sigma(\psi) + \mathrm{D}F^\varepsilon_\sigma(\psi) \cdot \dot{G}_\sigma F^\varepsilon_\sigma(\psi)$$

⊡ (fractional) PDE estimates

Theorem. *let*

$$f_\sigma := \partial_t \phi_\sigma + (-\Delta)^s \phi_\sigma + \lambda \phi_\sigma^3,$$

then for $\bar{\mu}$ *large enough, there exists an universal constant* $C > 0$,

$$\boldsymbol{|\!|\!|\phi|\!|\!|_{\bar{\mu}} \lesssim 1 + C\lambda^{-1/3} |\!|\!| f |\!|\!|_{\#,\bar{\mu}}^{1/3}}$$

where

$$|\!|\!|\phi|\!|\!|_{\bar{\mu}} \approx \sup_{\sigma \in (\bar{\mu},1)} [\![\sigma]\!]^{\gamma} \|\phi_\sigma\|, \qquad |\!|\!| f |\!|\!|_{\#,\bar{\mu}} \approx \sup_{\sigma \in (\bar{\mu},1)} [\![\sigma]\!]^{3\gamma} \| f_\sigma \|$$

⚁ estimates for the flow decomposition

Theorem. *Let* (ϕ, R) *be solution of*

$$\begin{cases} \mathscr{L}\phi_\mu = J_\mu(F_\mu(\phi_\mu) + R_\mu) \\ R_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F_\sigma(\phi_\sigma) \cdot \dot{G}_\sigma R_\sigma]\mathrm{d}\sigma \end{cases}$$

Let $(F_\sigma)_\sigma$ *be s.t.,*

$$|||\sigma \mapsto J_\sigma F_\sigma(\psi_\sigma) - (-\lambda\psi_\sigma^3)|||_\# \leqslant [\![\bar{\mu}]\!]^\zeta [C_F(1 + |||\psi|||)^M + (1 + |||\psi|||)^2 |||\mathscr{L}\psi|||_\#],$$

for some (non-random) $M, \zeta > 0$ *and random* C_F *almost surely finite together with some suitable estimates for* $H_\sigma(\phi_\sigma)$ *and* $\mathrm{D}F_\sigma(\phi_\sigma)$.

Then there exists a (random) $\bar{\mu} \in (0,1)$ *such that*

$$|||\phi|||_{\bar{\mu}} \lesssim 1, \qquad |||\mu \mapsto J_\mu R_\mu^\varepsilon|||_{\#,\bar{\mu}} \lesssim 1.$$

⚁ appoximate effective force

Theorem. *There exists a random functional* $(F^{\varepsilon}_{\mu})_{\mu \in (0,1)}$ *satisfying the boundary condition*

$$F^{\varepsilon}_{1}(\psi) = -\lambda\psi^3 - r_{\varepsilon}\,\psi + \xi^{(\varepsilon)}$$

and the bound

$$|||\sigma \mapsto J_{\sigma} F_{\sigma}(\psi_{\sigma}) - (-\lambda\psi^3_{\sigma})|||_{\#} \leqslant [\![\bar{\mu}]\!]^{\zeta}[C_F(1 + |||\psi|||)^M + (1 + |||\psi|||)^2 |||\mathscr{L}\psi|||_{\#}]$$

where $C_F = \|F^{\varepsilon,\mathfrak{A}}\|$ *is a finite random constant together with some suitable estimates for*

$$H_{\sigma}(\phi_{\sigma}) := \partial_{\sigma} F^{\varepsilon}_{\sigma} + \mathrm{D}F^{\varepsilon}_{\sigma} \cdot \dot{G}_{\sigma} F^{\varepsilon}_{\sigma}$$

and $\mathrm{D}F_{\sigma}(\phi_{\sigma})$ *such that, for any large* p,

$$\sup_{\varepsilon > 0} \mathbb{E}[\|\boldsymbol{F}^{\varepsilon,\mathfrak{A}}\|^p] < \infty$$

⚃ random flow equation

▷ look for approximate solutions to

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma \cdot (\dot{G}_\sigma F_\sigma) = 0, \qquad F_1 = F$$

in the space of polynomial in the fields

▷ write $F_\sigma = \sum_k F_\sigma^{(k)}$, where we denote with $F_\sigma^{(k)}$ the component of degree k

$$F^{(k)}(\Psi)(z) = \int_{\Lambda^k} F^{(k)}(z; z_1, \ldots, z_k)\Psi(z_1)\cdots\Psi(z_k)\mathrm{d}z_1\cdots\mathrm{d}z_k, \qquad z \in \Lambda.$$

▷ also derivatives of the field: $\Psi^{A_i}(z_i) := \partial^{A_i}\Psi(z_1)$ (jet bundle)

$$F^{(k)}(\Psi)(z) = \sum_{A_1,\ldots,A_k} \int_{\Lambda^k} F^{\langle A_1,\ldots,A_k\rangle}(z; z_1, \ldots, z_k)\Psi^{A_1}(z_1)\cdots\Psi^{A_k}(z_k)\mathrm{d}z_1\cdots\mathrm{d}z_k.$$

▷ example:

$$\Psi^3(z) = \int_{\Lambda^3} \delta^{(3)}(z; z_1, z_2, z_3)\Psi(z_1)\Psi(z_2)\Psi(z_3)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$$

norms

approximate the equation letting

$$F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F_\sigma^{[\ell]} \qquad \partial_\sigma F_\sigma^{[\ell+1]} + \sum_{\ell'=0}^{\ell} \mathrm{D}F_\sigma^{[\ell-\ell']} \cdot \dot{G}_\sigma F_\sigma^{(\ell')} = 0$$

$F^{\mathfrak{a}} \equiv F^{[\ell],\langle A_1,\ldots,A_k\rangle}$ for $\mathfrak{a} = (\ell, A_1, \ldots, A_k)$, $F^{\mathfrak{A}} := (F_\sigma^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A},\sigma\in(0,1)}$

$$\|F^{\mathfrak{a}}\| = \sup_{z\in\Lambda} \int_{\Lambda^k} |F^{[\ell],\langle A_1,\ldots,A_k\rangle}(z; z_1, \ldots, z_k)| \mathrm{d}z_1 \cdots \mathrm{d}z_k$$

$$\|F^{\mathfrak{A}}\| := \sup_{\mathfrak{a}\in\mathfrak{A}} \left[\sup_{\sigma \geqslant \mu \geqslant 0} [\![\sigma]\!]^{-[\mathfrak{a}]} \|F_\sigma^{\mathfrak{a}}\|_{\mu,\sigma} \right] \qquad [\mathfrak{a}] := -\alpha + \delta\ell(\mathfrak{a}) + \beta k(\mathfrak{a}) + |A(\mathfrak{a})|$$

(for suitable parameters α, β and δ to be fixed) · this means bounds of the form

$$\|F_\sigma^{\mathfrak{a}}\|_{\mu,\sigma} \lesssim \|F^{\mathfrak{A}}\| \, [\![\sigma]\!]^{[\mathfrak{a}]}$$

⚄ cumulants

cumulants of $(F^{\mathfrak{a}})_{\mathfrak{a}} \rightsquigarrow$ deterministic kernels $(\mathscr{F}^{\boldsymbol{a}})_{\boldsymbol{a}}$ defined by

$$\mathscr{F}^{\boldsymbol{a}} := \kappa_n(F^{\mathfrak{a}_1}, \cdots, F^{\mathfrak{a}_n}), \qquad \boldsymbol{a} \in \boldsymbol{A} = \{(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : \mathfrak{a}_k \in \mathfrak{A}, n \leqslant N\}$$

$$\|\mathscr{F}^{A}\| := \sup_{\boldsymbol{a} \in A} \left[\sup_{\sigma \geqslant \mu \geqslant 0} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_{\sigma}^{\boldsymbol{a}}\|_{\mu,\sigma} \right]^{1/n(\boldsymbol{a})}$$

$$[\boldsymbol{a}] = -\rho + \theta n(\boldsymbol{a}) + \delta L(\boldsymbol{a}) + \beta K(\boldsymbol{a}) + |A(\boldsymbol{a})|$$

Theorem. ***Kolmogorov-type argument:*** *from estimates on the norm $\|\mathscr{F}^{A}\|$ to those on $\|F^{\mathfrak{A}}\|$ as*

$$\{\mathbb{E}[\|\boldsymbol{F}^{\mathfrak{A}}\|^n]\}^{1/n} \lesssim \|\boldsymbol{\mathscr{F}}^{A}\|$$

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flow equation for cumulants

Lemma. *Cumulants satisfy a flow equation*

$$\partial_\sigma \mathscr{F}_\sigma^{\boldsymbol{a}} = \sum_{\boldsymbol{b}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{\mathrm{G}}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) + \sum_{\boldsymbol{b},\boldsymbol{c}} \mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{\mathrm{G}}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}}),$$

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{\mathrm{G}}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}})\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}.$$

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{\mathrm{G}}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}})\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1} \|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma} \|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}.$$

[P. Duch · Renormalization of singular elliptic stochastic PDEs using flow equation · arXiv:2201.05031]

classification of cumulants

the structure of the flow equation for cumulants propagates estimates of the form

$$\sup_{\sigma \geqslant \mu} [\![\sigma]\!]^{-[a]} \|\mathscr{F}_\sigma^a\|_{\mu,\sigma} < \infty \qquad \Longrightarrow \qquad \|\mathscr{F}_\sigma^a\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{[a]}$$

- **irrelevant cumulants**$\rightsquigarrow [a] > 0$: the flow equation can be solved *backwards* starting from the final condition at $\sigma = 1$.
- **relevant cumulants**$\rightsquigarrow [a] < 0$: the flow equation cannot be integrated close to $\sigma = 1$;

  $\Longrightarrow$use the only freedom we have: fine tuning of the final condition

$$F(\psi) = -\lambda \psi^3 + r_\varepsilon \psi + \xi$$

- **marginal cumulants**$\rightsquigarrow [a =]0$: we shall handle them as the irrelevant.

relevant cumulants

♟ relevant cumulants · $\mathscr{F}^{\boldsymbol{a}}$ s.t.

$$[\boldsymbol{a}] = -\rho + \theta n(\boldsymbol{a}) + \delta L(\boldsymbol{a}) + \beta K(\boldsymbol{a}) + |A(\boldsymbol{a})| < 0$$

$\Longrightarrow$ the only relevant cumulant is $\boldsymbol{\kappa_1(F_\sigma^{[\ell](1)})}$

♞ $\kappa_1(F_\sigma^{[\ell](1)})$ is a <u>non-local</u> object **!** but the only freedom we have is the fine tuning a <u>constant</u> (r_ε)!

♝ Taylor expansion

$$\begin{aligned} F_\sigma^{[\ell](1)}(\psi)(z) &= \int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)\psi(z_1)\mathrm{d}z_1 \\ &= \sum_{A:0\leqslant|A|\leqslant 1} \partial^A\psi(z)\int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)(z_1-z)^A\mathrm{d}z_1 + \\ &\quad + \sum_{A:1<|A|\leqslant 2} c_A\int_0^1 \mathrm{d}t\int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)(z_1-z)^A\partial^A\psi(z+t(z_1-z))\mathrm{d}z_1 \end{aligned}$$

$\Longrightarrow$ we have **local terms** + some <u>non-local remainder</u> terms

localisation

♘ modify the flow equation by introducing the operators

$$(\boldsymbol{L}F_\sigma)(\Psi) = \sum_{\mathfrak{b}:k(\mathfrak{b})=1} \sum_{A:0\leqslant|A|\leqslant 1} \Psi^A(z)\int_\Lambda (z_1-z)^A F_\sigma^{[\ell(\mathfrak{b})]\langle 0\rangle}(z;z_1)\mathrm{d}z_1 + \sum_{\mathfrak{b}:k(\mathfrak{b})\neq 1} F_\sigma^{\mathfrak{b}}(\Psi)$$

$$(\boldsymbol{R}F_\sigma)(\Psi) = \sum_{\mathfrak{b}:k(\mathfrak{b})=1} \sum_{A:1<|A|\leqslant 2} c_A\int_0^1 \mathrm{d}t\int_\Lambda F_\sigma^{[\ell(\mathfrak{b})]\langle 0\rangle}(z;z_1)(z_1-z)^A\Psi^A(z+t(z_1-z))\mathrm{d}z_1$$

$$\partial_\sigma F_\sigma + (\boldsymbol{L}+\boldsymbol{R})[\mathrm{D}F_\sigma\cdot(\dot{G}_\sigma F_\sigma)] = 0 \Rightarrow \partial_\sigma \mathscr{F}_\sigma^{\boldsymbol{a}} = \textstyle\sum_{\boldsymbol{b}} \tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) + \sum_{\boldsymbol{b},\boldsymbol{c}} \tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}})$$

☞ same estimates + relevant cumulant $\kappa_1(F_\sigma^{[\ell]\langle 0\rangle})$ now **local** by construction

$\rightsquigarrow\exists$ constants $(r_\sigma^\ell)_{\ell\geqslant 0}$ such that

$$\kappa_1(F^{[\ell]\langle 0\rangle}(z,z_1)) = \mathbb{E}(F^{[\ell]\langle 0\rangle}(z,z_1)) = r_\sigma^\ell\delta(z-z_1).$$

⚅ inductive procedure

solve and estimate the flow equation by induction over ℓ

$$M_\ell := \sup_{a:L(a)\leqslant \ell} \left[\sup_{\sigma\geqslant\mu} [\![\sigma]\!]^{-[a]} \|\mathscr{F}_\sigma^{a}\|_{\mu,\sigma}\right]^{1/n(a)} < \infty$$

start with $M_0 < \infty$ · assume $M_\ell < \infty$ and take $\mathscr{F}_\sigma^{a}$ with $L(a) = \ell + 1$

☞ for $\mathscr{F}_\sigma^{a}$ **irrelevant** ($[a] > 0$) ⇒ solve the equation **backwards** from the final condition $\mathscr{F}_1^{a} = 0$:

$$\mathscr{F}_\sigma^{a} = \int_\sigma^1 \left[\sum_b \tilde{\mathscr{A}}_b^{a}(\dot{G}_\eta, \mathscr{F}_\eta^{b}) + \sum_{b,c} \tilde{\mathscr{B}}_{b,c}^{a}(\dot{G}_\eta, \mathscr{F}_\eta^{b}, \mathscr{F}_\eta^{c})\right] \mathrm{d}\eta$$

$$\Rightarrow \qquad \|\mathscr{F}_\sigma^{a}\|_{\mu,\sigma} \leqslant C M_\ell^{n(a)+1} \int_\sigma^1 [\![\eta]\!]^{[a]-1}\, \mathrm{d}\eta \lesssim C [\![\sigma]\!]^{[a]} M_\ell^{n(a)+1}$$

this shows that we cannot do like that for $[a] < 0$...

☞ for $\mathscr{F}_\sigma^a$ **relevant** ($[a]<0$) ⇒ solve the equation **forwards** by fixing $r_{\mu_0}^{\ell+1}$ to be some arbitrary value at some reference scale $\mu_0<1$

$$\begin{aligned}
[\![\sigma]\!]^{-[a]}\|\mathscr{F}_\sigma^a\|_{\mu,\sigma} &\leqslant [\![\sigma]\!]^{-[a]}|r_{\mu_0}^{\ell+1}|+[\![\sigma]\!]^{-[a]}\int_{\mu_0}^{\sigma}\|\partial_{\sigma'}\mathscr{F}_{\sigma'}^a\|_{\mu,\sigma'}\mathrm{d}\sigma' \\
&\lesssim |r_{\mu_0}^{\ell+1}|+[\![\sigma]\!]^{-[a]}CM_\ell^{n(a)+1}\int_{\mu_0}^{\sigma}[\![\sigma']\!]^{[a]-1}\mathrm{d}\sigma' \\
&\lesssim |r_{\mu_0}^{\ell+1}|+CM_\ell^{n(a)+1}
\end{aligned}$$

Lemma.

$$\|\mathscr{F}^A\|=\sup_{a:L(a)\leqslant\bar{\ell},N(a)\leqslant N}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[a]}\|\mathscr{F}_\sigma^a\|_{\mu,\sigma}\right]^{1/n(a)}=M_{\bar{\ell}}\leqslant C(1+M_0)^{2^{\bar{\ell}}}<\infty$$

☞ this complete the proofs of uniform apriori estimates on the SPDE → tightness

summary

interplay of techniques

PDEs ⊕ Renormalization Group ⊕ Stochastic Analysis

- **PDEs**
  - coercive a priori estimates to deal with the so-called large field problem
- **Renormalization Group**
  - **flow equation** for the effective force + handling of renormalization;
- **Stochastic Analysis**
  - flow Equation for cumulants + Kolmogorov argument

☞ insensitive of how close we are to the critical case

☞ renormalisation conditions as good boundary conditions when solving the flow equation

☞ no need of requiring small coupling constant

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the end

(no human has been harmed by (La)TeX to produce this presentation)