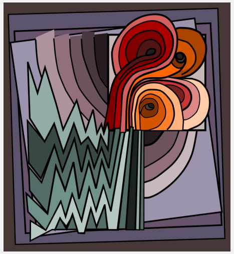
DIMS · Leipzig · June 18th, 2025

Elliptic equations and supersymmetry



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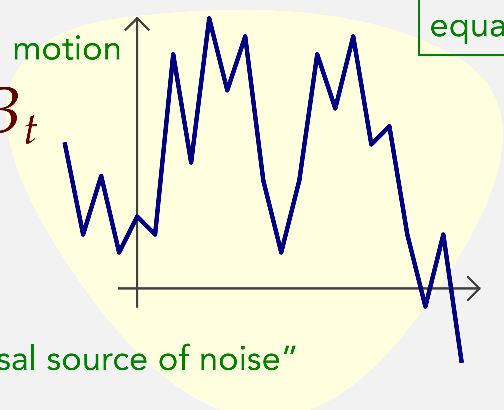
[made with TeXmacs]

Ito's brillant idea

Brownian motion

B_t



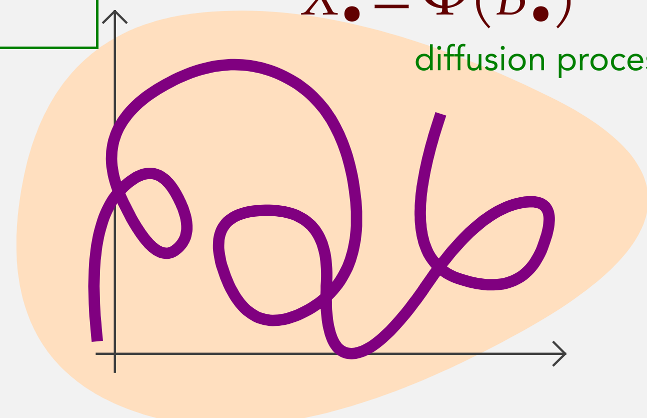
“universal source of noise”

stochastic differential equations

$$\overset{}{\underset{\Phi}{\Longrightarrow}}$$

$X_\bullet = \Phi(B_\bullet)$

diffusion process



$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

act I

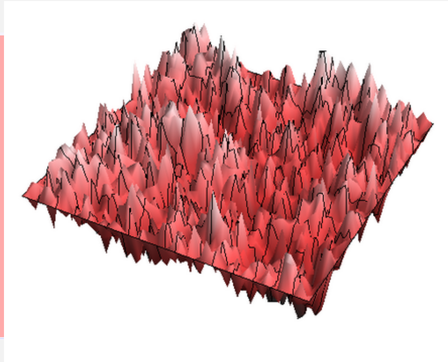
the quest for equations

Euclidean quantum fields

Feynman–Kac formula / non-perturbative path integral formulation of QFTs

$$\nu(\mathrm{d}\varphi) = \frac{\exp[-S(\varphi)]\mathscr{D}\varphi}{Z}, \qquad \varphi \in \mathscr{S}'(\mathbb{R}^d)$$

$$S(\varphi) = \frac{1}{2}\int_{\mathbb{R}^d} (|\nabla\varphi(x)|^2 + m^2\varphi(x)^2)\mathrm{d}x + \int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x$$



$$V(\varphi) = \lambda\varphi^4 + \cdots, \quad \lambda\cos(\beta\varphi), \quad \lambda\cosh(\beta\varphi), \quad \lambda\exp(\beta\varphi), \quad \cdots, \qquad d = 1,2,3$$

σ-models, gauge theories

(Markovian) probability measure ν on $\mathscr{S}'(\mathbb{R}^d)$ · natural generalisations of diffusion processes

introduced in the '70 in the context of the **constructive QFT program**

Gaussian free field

▷ GFF · simplest example of EQFT · Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ s.t.

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean · $m>0$ is the *mass* · $G(0)=+\infty$ if $d\geqslant 2$: not a function · distribution of regularity

$$\alpha < (2-d)/2$$

▷ can be used to construct a QFT but the theory is free: no interaction

the standard recipe for non-Gaussian Euclidean fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{Z}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side $2\pi L$

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}} \int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi) e^{-\overbrace{\frac{1}{2}\varepsilon^d \sum_{x\in\mathbb{Z}^d_{\varepsilon,L}} |\nabla_\varepsilon \varphi(x)|^2 + m^2\varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}} \mathrm{d}\varphi$$

ε is an UV regularisation and L the IR regularisation

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below. $d = 2,3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure.

❸ study the possible limit points [the $\boldsymbol{\Phi^4_d}$ **measure**] · ask interesting questions: uniqueness? non-uniqueness? decay of correlations? intrinsic description?

some models

▷ $d=1$ · time-reversal symmetric, translation invariant, Markov diffusions. the generator is given by an implicit expression involving the ground state Ψ of the Hamiltonian H

$$\mathscr{L} = \nabla \log \Psi \cdot \nabla + \frac{1}{2}\Delta \qquad H = -\Delta + x^2 + V(x).$$

▷ $d=2$ · various choices $(a_\varepsilon \to +\infty)$

$$V_\varepsilon(\xi) = \lambda \xi^{2l} + \sum_{k=0}^{2l-1} a_{k,\varepsilon} \xi^k, \quad V_\varepsilon(\xi) = a_\varepsilon \cos(\beta\xi)$$

$$V_\varepsilon(\xi) = a_\varepsilon \cosh(\beta\xi), \qquad V_\varepsilon(\xi) = a_\varepsilon \exp(\beta\xi)$$

▷ $d=3$ · “only” 4th order (6th order is critical)

▷ $d=4$ · all the possible limits are Gaussian (see Aizenmann–Duminil Copin)

stochastic quantisation map

$$e^{-\frac{1}{2}\int \xi C \xi} \mathrm{D}\xi$$

Gaussian measure

source of noise

$$\varphi = \Phi(\xi)$$

$$\Longrightarrow$$

solution map of an SPDE

$$\frac{e^{-\int \varphi A \varphi + V(\varphi)}}{Z} \mathrm{D}\varphi$$

Euclidean QFT

(S)PDE theory

(apriori estimates, geometry)

$$\partial_t \phi = -(m^2 - \Delta_x)\phi - V'(\phi) + c\xi,$$

systematic use of tools from **PDE theory**, recent advances in **singular PDEs**, **renormalization group** ideas coupled with pathwise analysis

stochastic equations for the free Gaussian free field

Gaussian free field $\mu : \mathbb{E}[\varphi(x)\varphi(y)] = (m^2 - \Delta)^{-1}(x-y)$ · ξ white noise

❶ "Gaussian map":

$$\varphi(x) = (m^2 - \Delta)^{-1/2}\xi(x), \qquad (m^2 - \Delta)\varphi(x) = (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❷ Stochastic mechanics (Nelson):

$$\partial_{x_0}\varphi(x_0, \bar{x}) = -(m^2 - \Delta_{\bar{x}})^{1/2}\varphi(x_0, \bar{x}) + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu):

$$\varphi(x) \sim \phi(t, x) \qquad \partial_t\phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) + c\xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

stochastic equations for non-Gaussian EQFTs $(V \neq 0)$

❶ Shifted Gaussian map (Albeverio/Yoshida) **[does not have the right properties!]**

$$(m^2 - \Delta)\varphi(x) + V'(\varphi(x)) = (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❷ Stochastic mechanics (Nelson): ground-state transformation **[too implicit!]**

$$\partial_{x_0}\varphi(x_0, \bar{x}) = [\nabla_{\varphi(x_0,\bar{x})} \log \Psi(\varphi)] + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu): Langevin diffusion $\varphi(x) \sim \phi(t, x)$

$$\partial_t \phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) - V'(\phi(t, x)) + c\xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

an (pre)history of (Langevin) stochastic quantisation (personal & partial)

- 1981 · Parisi/Wu – stochastic quantisation for gauge theories (SQ)
- 1985 · Jona-Lasinio/Mitter · "On the stochastic quantization of field theory" (rigorous SQ for Φ^4_2 on bounded domain)
- 1988 · Damgaard/Hüffel · review book on SQ (theoretical physics)
- 1990 · Funaki · Control of correlations via SQ (smooth reversible dynamics)
- 2003 · Da Prato/Debussche · "Strong solutions to the stochastic quantization equations"
- 2014 · Hairer – Regularity structures, local dynamics of Φ^4_3
- 2017 · Mourrat/Weber · global solutions for Φ^4_2, coming down from infinity for Φ^4_3
- 2018 · Albeverio/Kusuoka · "The invariant measure and the flow associated to Φ^4_3..."
- 2021 · Hofmanova/G. – Global space-time solutions for Φ^4_3 and verification of axioms
- 2022 · Hairer/Steele – "optimal" tail estimates
- 2020–2021 · Chandra/Chevyrev/Hairer/Shen · SQ for Yang–Mills 2d/3d (local theory)

Elliptic equations & supersymmetry

in the '80 Parisi–Sourlas discovered that

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the invariant solution of the following *elliptic* SPDE

$$(-\Delta_x + m^2)\phi(x) + \partial V(\phi(x))) = \xi(x), \qquad x \in \mathbb{R}^2$$

where $x \in \mathbb{R}^2$, $\phi\colon \mathbb{R}^2 \to \mathbb{R}^n$ and $V\colon \mathbb{R}^n \to \mathbb{R}$ is a non-negative smooth function bounded from below, should satisfy

$$\phi(0) \sim \frac{1}{\mathscr{Z}} \exp\left[-4\pi\left(\frac{m^2|\eta|^2}{2} + V(\eta)\right)\right] \mathrm{d}\eta.$$

With space dependence, the SPDE

$$(-\Delta_x - \Delta_z + m^2)\phi(x,z) + \partial V(\phi(x,z)) = \xi(x,z), \qquad x \in \mathbb{R}^2, z \in \mathbb{R}^d$$

where $x \in \mathbb{R}^2$, $\phi\colon \mathbb{R}^{d+2} \to \mathbb{R}$ and V is a non-negative smooth function bounded from below, should satisfy

$$\phi(0,\cdot) \sim \frac{1}{\mathscr{Z}} \exp\left[-4\pi\left(\int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(z)|^2 + \frac{m^2}{2}|\varphi(z)|^2 + V(\varphi(z))\mathrm{d}z\right)\right]\mathscr{D}\varphi.$$

where on the r.h.s. $\varphi\colon \mathbb{R}^d \to \mathbb{R}$.

This is called *elliptic stochastic quantisation.*

Initially perturbative computations via Feynman diagrams, in the context of field theoretic approximation of the random field Ising model.

Then Parisi and Sourlas gave a non-perturbative “proof” via a **supersymmetric approach** (susy).

> Dimensional reduction: a supersymmetric euclidean QFT of bosons and fermions in in $d+2$ dimensions is equivalent to a bosonic theory in d dimensions.

A first partial rigorous discussion of this topic was given in a paper by A. Klein, L. J. Landau, and J. F. Perez in the '80.

G. Parisi and N. Sourlas. Random Magnetic Fields, Supersymmetry, and Negative Dimensions. Physical Review Letters, 43(11):744–745, 1979.

G. Parisi and N. Sourlas. Supersymmetric field theories and stochastic differential equations. *Nuclear Physics B*, 206(2):321–332, 1982.

A. Klein, L. J. Landau, and J. F. Perez. Supersymmetry and the Parisi-Sourlas dimensional reduction: a rigorous proof. *Comm. Math. Phys.*, 94(4):459–482, 1984.

this talk

☞ In the first part of the talk we review the proof of the previous conjecture, for a special class of potentials V, which closely follows the original idea of Parisi and Sourlas stressing the role of supersymmetry.

☞ In the second part we give an overview of the extensions obtained for more general potentials.

Join work with S. Albeverio and F. C. De Vecchi:

- S. Albeverio, F. C. De Vecchi and M. G. Elliptic stochastic quantization. Annals of Probability 48(4):1693–1741, 2020.
- S. Albeverio, F. C. De Vecchi and M. G. The Elliptic Stochastic Quantization of Some Two Dimensional Euclidean QFTs. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 57, 2021. https://doi.org/10.1214/20-AIHP1145.
- F. C. De Vecchi and M. G. A Note on Supersymmetry and Stochastic Differential Equations'. In *Geometry and Invariance in Stochastic Dynamics*, 71–87. Springer Proceedings in Mathematics & Statistics, 2021. https://doi.org/10.1007/978-3-030-87432-2_5.

the result (particular case)

Let V be a smooth bounded function with $-|\partial^2 V| + m^2 > \epsilon$.

Consider the SPDE

$$(-\Delta + m^2)\phi(x) + f(x)\,\partial V(\phi(x)) = \xi(x), \qquad x \in \mathbb{R}^2,$$

where $f: \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ is a smooth, radial, *cutoff* function depending decaying exponentially at infinity with $f(0) = 1$.

Then, for any $h: \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{E}\left[h(\phi(0))\,\Upsilon_f(\phi)\right] = \frac{\mathcal{Z}_f}{\mathcal{Z}} \int_{\mathbb{R}^n} h(y) \exp\left[-4\pi\left(\frac{m^2|\eta|^2}{2} + V(\eta)\right)\right] \mathrm{d}\eta$$

where

$$\Upsilon_f(\phi) = \exp\left[\int_{\mathbb{R}^2} f'(x)\, V(\phi(x))\,\mathrm{d}x\right], \qquad \mathcal{Z}_f = \mathbb{E}[\Upsilon_f(\phi)].$$

shifts in Wiener space

Step ➊. Rewrite the expectation of the solution to the elliptic equation as an integral with respect to a Gaussian measure.

For this reason we introduce the map $U\colon \mathscr{W} \to \mathscr{W}$ (where $\mathscr{W}$ is a suitable Banach space) of the form

$$U(\varphi) = \mathscr{I}(f \partial V(\varphi))$$

with $\mathscr{I} = (-\Delta + m^2)^{-1}$. Let

$$T(\varphi) = \varphi + U(\varphi)$$

and rewrite the SPDE as

$$T(\phi) = \mathscr{I}\xi.$$

This means that if ν is the law of ϕ we have

$$T_\star \nu = \mu$$

where μ is the (Gaussian) law of $\mathcal{I}\xi$.

When V is a convex and bounded and f is a smooth non-negative function decaying exponentially to 0 at infinity we have:

- $U(\varphi) \in \mathcal{I}(L^2(\mathbb{R}^2))$
- U is differentiable and $\nabla U(\varphi) = \mathcal{I}(f\,\partial^2 V(\varphi))$
- $\nabla U(\varphi)$ is Hilbert-Schmidt on $\mathcal{I}(L^2(\mathbb{R}^2))$
- T is invertible

general Girsanov's transform

Step 1. Apply the theory of general shifts in abstract Wiener spaces and prove that ν is absolutely continuous with respect to μ and

$$\mathrm{d}\nu(\varphi)=\det{}_2\big(I+\nabla U(\varphi)\big)\exp\left[-\delta(U)-\frac{1}{2}\|U\|^2_{\mathcal{I}(L^2(\mathbb{R}^2))}\right]\mathrm{d}\mu(\varphi).$$

[see e.g. A. S. Üstünel, Analysis on Wiener Space and Applications. ArXiv:1003.1649, 2010]

This implies that

$$\mathcal{Z}_f\,\mathfrak{I}_h:=\mathbb{E}\left[h(\phi(0))\,\Upsilon_f(\phi)\right]\;=\;\int_{\mathcal{W}}h(\varphi(0))\,\Upsilon_f(\varphi)\,\mathrm{d}\nu(\varphi)$$

$$=\int_{\mathcal{W}}h(\varphi(0))\,\Upsilon_f(\varphi)\det{}_2\big(I+\nabla U(\varphi)\big)\exp\left[-\delta(U)-\frac{1}{2}\|U\|^2_{\mathcal{I}(L^2(\mathbb{R}^2))}\right]\mathrm{d}\mu(\varphi).$$

superspace

formally the superspace $\mathfrak{S}$ is the set of points $(x,\theta,\bar{\theta})$ where $x\in\mathbb{R}^2$ and $\theta,\bar{\theta}$ are two additional anticommuting coordinates. It is denoted $\mathbb{R}^{2|2}$.

two bosonic and two fermionic coordinates. a non-commutative space.

concrete construction: understand $\mathfrak{S}$ via the algebra of smooth functions $F\colon\mathfrak{S}\to\mathbb{C}$ on it.

let $\mathfrak{G}(\theta,\bar{\theta})$ be the Grassmann algebra generated by symbols $\theta,\bar{\theta}$, with the relations

$$\theta\bar{\theta}=-\bar{\theta}\theta,\qquad \theta^2=\bar{\theta}^2=0,$$

then $\mathfrak{G}(\theta,\bar{\theta})=\mathrm{span}(1,\theta,\bar{\theta},\theta\bar{\theta})$ and a smooth function

$$F\colon\mathbb{R}^2\to\mathfrak{G}(\theta,\bar{\theta})$$

is just a quadruplet $(f_\emptyset,f_\theta,f_{\bar{\theta}},f_{\theta\bar{\theta}})\in(C^\infty(\mathbb{R}^2))^4$ with the identification

$$F(x,\theta,\bar{\theta})=f_\emptyset(x)+f_\theta(x)\,\theta+f_{\bar{\theta}}(x)\bar{\theta}+f_{\theta\bar{\theta}}(x)\,\theta\bar{\theta}.$$

analysis on superspace

the functions. the function F can be considered as a function $\boldsymbol{F}\colon \mathfrak{S} \to \mathbb{R}$ on the superspace $\mathfrak{S}$ by formally writing

$$\boldsymbol{F}(x,\theta,\bar\theta) := F(x) = f_{\emptyset}(x) + f_{\theta}(x)\,\theta + f_{\bar\theta}(x)\bar\theta + f_{\theta\bar\theta}(x)\,\theta\bar\theta.$$

the volume form. we can introduce a linear functional defined as

$$\boldsymbol{F} \mapsto \int \boldsymbol{F}(x,\theta,\bar\theta)\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta := -\int_{\mathbb{R}^2} f_{\theta\bar\theta}(x)\,\mathrm{d}x,$$

induced by the standard Berezin integral $\alpha \in \mathfrak{S} \mapsto \int \alpha \mathrm{d}\theta\mathrm{d}\bar\theta \in \mathbb{C}$ on $\mathfrak{S}$ satisfying

$$\int 1\mathrm{d}\theta\mathrm{d}\bar\theta = \int \theta\,\mathrm{d}\theta\mathrm{d}\bar\theta = \int \bar\theta\mathrm{d}\theta\mathrm{d}\bar\theta = 0, \qquad \int \theta\bar\theta\mathrm{d}\theta\mathrm{d}\bar\theta = -1.$$

algebra structure. given $r \in C^1(\mathbb{R};\mathbb{R})$ we define the composition $r \circ \boldsymbol{F}: \mathfrak{S} \to \mathbb{R}$ by

$$r(\boldsymbol{F}(x,\theta,\bar{\theta})) \ := \ r(f_{\emptyset}(x)) + r'(f_{\emptyset}(x))\, f_{\theta}(x)\,\theta + r'(f_{\emptyset}(x))\, f_{\bar{\theta}}(x)\,\bar{\theta} + r'(f_{\emptyset}(x))\, f_{\theta\bar{\theta}}(x)\,\theta\,\bar{\theta},$$

in accordance with the same expression one would get if r were a polynomial.

distributions. we can define similarly the space of Schwartz superfunctions $\mathscr{S}(\mathfrak{S})$ and the Schwartz superdistributions $T \in \mathscr{S}'(\mathfrak{S})$ where T can be written

$$T = T_{\emptyset} + T_{\theta}\,\theta + T_{\bar{\theta}}\,\bar{\theta} + T_{\theta\bar{\theta}}\,\theta\,\bar{\theta}$$

with $T_{\emptyset}, T_{\theta}, T_{\bar{\theta}}, T_{\theta\bar{\theta}} \in \mathscr{S}'(\mathbb{R}^2)$ and duality pairing

$$T(f) = -T_{\emptyset}(f_{\theta\bar{\theta}}) + T_{\theta}(f_{\bar{\theta}}) - T_{\bar{\theta}}(f_{\theta}) - T_{\theta\bar{\theta}}(f_{\emptyset}), \qquad f_{\emptyset}, f_{\theta}, f_{\bar{\theta}}, f_{\theta\bar{\theta}} \in \mathscr{S}(\mathbb{R}^2).$$

supersymmetric representation of the density

Step 2. the superspace formalism allows to rewrite the integral $\mathfrak{I}_h$ in a surprising form.

Step 2.1. introduce an approximation $\mathfrak{I}_h^\chi$ of $\mathfrak{I}_h$ having the following expression

$$\mathscr{Z}_f^\chi \ \mathfrak{I}_h^\chi := \int h(\varphi(0)) \underbrace{\det(I_{\mathscr{H}} + \nabla U_\chi)}_{\text{Step 2.3}} \exp\left(-\delta \circ \underbrace{U_\chi - \frac{1}{2}\|U_\chi\|^2_{\mathscr{T}^\chi(L^2)}}_{\text{Step 2.2}} + 4 \int V(\varphi(x)) f'(x)\, \mathrm{d}x\right) \mathrm{d}\mu^\chi(\varphi)$$

where $U_\chi = \varpi \mathscr{K}^\chi(\partial V(\varphi))$, $\mathscr{T}^\chi = (-\Delta + m^2)^{-1-\chi}$, $\mathscr{K}^\chi = (-\Delta + m^2)^{-\chi}$, μ^χ is the law of $\mathscr{T}^\chi \xi$ and $\varpi = \frac{1}{1+2\chi}$ is a constant.

Step 2.2. introduce an auxiliary Gaussian field η distributed as the Gaussian noise $\mathscr{T}\xi$ with law μ, and independent with respect to φ. We can then write

$$\exp\left[-\frac{\varpi^2}{2}\left\|\mathscr{K}^\chi(f\partial V(\phi))\right\|^2_{\mathscr{H}}\right] = \int \exp\left[-i\varpi\left\langle f\partial V(\phi), \mathscr{K}^\chi \eta\right\rangle\right] \mu(\mathrm{d}\eta).$$

more auxiliary fields

Step 2.3. introduce a Grassmann Gaussian field $\psi, \bar{\psi}$ realized as bounded operators on a suitable Hilbert space $H_{\psi,\bar{\psi}}$ with a state $\langle \cdot \rangle_{\psi,\bar{\psi}}$ for which

$$\left\langle \bar{\psi}(x) \bar{\psi}(x') \right\rangle_{\psi,\bar{\psi}} = \left\langle \psi(x)\, \psi(x') \right\rangle_{\psi,\bar{\psi}} = 0, \quad \left\langle \psi(x) \bar{\psi}(x') \right\rangle_{\psi,\bar{\psi}} = \varpi\, \mathcal{G}_{1+2\chi}(|x-x'|^2),$$

where $\mathcal{G}_{1+2\chi}(|x|^2)$ is the Green function of the operator $(-\Delta + m^2)^{-1-2\chi}$. From which we have

$$\det\left(I_{\mathscr{H}} + \varpi\, f\, \partial^2 V(\varphi)\, \mathcal{T}^{\chi} \right) = \left\langle \exp\left[\int \psi(x)\, f(x)\, \partial^2 V(\varphi(x)) \bar{\psi}(x)\, \mathrm{d}x \right] \right\rangle_{\psi,\bar{\psi}}.$$

We take the generators $\theta, \bar{\theta}$ to anticommute with the the fermion fields $\psi, \bar{\psi}$.

Step 2.4. introduce the complex Gaussian field $\omega := -\varpi\,((m^2 - \Delta)\,\varphi + i\,\mathscr{K}^{\chi}\eta)$ so that

$$\int f(x) \partial V(\varphi(x)) \omega(x) \mathrm{d}x = \left[-\delta \circ (\varpi\, \mathcal{T}^{\chi}(\partial V(\varphi))) - i\varpi \left\langle f\, \partial\, V(\phi), \mathscr{K}^{\chi}\eta \right\rangle \right]$$

the superfield

☞ we put all our fields together in a single object defining the **superfield** Φ:

$$\Phi(x,\theta,\bar\theta) := \varphi(x) + \bar\psi(x)\,\theta + \psi(x)\,\bar\theta + \omega(x)\,\theta\,\bar\theta,$$

a non-commutative random field, define on a non-commutative space!

$$\boxed{\mathscr{Z}_f^\chi\,\mathfrak{I}_h^\chi = \left\langle \left(-\int h(\Phi(X))(\theta\bar\theta\delta_0(x))\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right)\exp\left[-\int V(\Phi(X))\,\tilde f(|X|^2)\,\mathrm{d}X \right] \right\rangle_\chi}$$

where $X=(x,\theta,\bar\theta)$, $\mathrm{d}X = \mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta$, $|X|^2 = |x|^2 + 4\,\theta\,\bar\theta$.

$$V(\Phi(X)) = V(\varphi(x)) + \partial V(\varphi(x))(\bar\psi(x)\,\theta + \psi(x)\,\bar\theta + \omega(x)\,\theta\,\bar\theta) - \partial^2 V(\varphi(x))\bar\psi(x)\,\psi(x)\theta\,\bar\theta$$

$$\int V(\Phi(X))\,\tilde f(|X|^2)\,\mathrm{d}X = \int 4V(\varphi(x))\,f'(x) + f(x)\Big(\underbrace{\partial^2 V(\varphi(x))\bar\psi(x)\,\psi(x)}_{\text{det}} + \underbrace{\partial V(\varphi(x))\omega(x)}_{-\delta\circ U_\chi - \frac{1}{2}\|U_\chi\|^2_{\mathscr{T}\chi(L^2)}}\Big)\mathrm{d}x$$

supersymmetry

☞ **a supergroup** introduce the linear transformations $\tau(b,\bar{b}):\mathfrak{S}\to\mathfrak{S}$

$$\tau(b,\bar{b})\begin{pmatrix} x \\ \theta \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} x+2\bar{b}\theta\,\rho+2b\bar{\theta}\rho \\ \theta-(x\cdot b)\,\rho \\ \bar{\theta}+(x\cdot\bar{b})\,\rho \end{pmatrix} \in \mathfrak{G}(\theta,\bar{\theta},\rho)$$

for $b,\bar{b}\in\mathbb{R}^2$ and where ρ is a new odd variable anticommuting with $\theta,\bar{\theta}$ and we have

$$\tau(t\,b,t\,\bar{b})\,\tau(s\,b,s\,\bar{b})=\tau((t+s)\,b,(t+s)\,\bar{b}).$$

☞ a function F is supersymmetric iff it is invariant with respect to rotations in $\mathbb{R}^2$ and transformation of the form $\tau(b,\bar{b})$, i.e. if

$$F(X)=F(\tau(b,\bar{b})X)$$

supersymmetric distributions are defined by duality.

dimensional reduction / supersymmetric localisation

supersymmetric functions and distribution have the following property:

Theorem. *Let* $F \in \mathscr{S}(\mathfrak{S})$ *be a smooth supersymmetric function then we have that there exists a function* $f: \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$F(x, \theta, \bar{\theta}) = f(|x|^2 + 4\theta\bar{\theta}).$$

Suppose furthermore that $T \in \mathscr{S}'(\mathfrak{S})$ *is supersymmetric and such that* $T_\emptyset$ *is a continuous function, then we have the reduction formula*

$$\int T(x, \theta, \bar{\theta}) \cdot F(x, \theta, \bar{\theta}) \, \mathrm{d}x \mathrm{d}\theta \mathrm{d}\bar{\theta} = 4\pi T_\emptyset(0) F_\emptyset(0).$$

the supersymmetric GFF & localisation

by direct verification computation we have that

$$\langle \Phi(x,\theta,\bar\theta)\,\Phi(x',\theta',\bar\theta')\rangle = C_\Phi(x-x',\theta-\theta',\bar\theta-\bar\theta')$$

where

$$C_\Phi(x,\theta,\bar\theta) := \mathcal{G}_{2+2\chi}(|x|^2) - \varpi\,\mathcal{G}_{1+2\chi}(|x|^2)\,\theta\,\bar\theta = \mathcal{G}_{2+2\chi}\left(|x|^2 + 4\,\theta\,\bar\theta\right) \qquad [!]$$

where $\mathcal{G}_{2+2\chi}(|x|^2)$ is the Green function of the operator $(-\Delta+m^2)^{-2-2\chi}$.

Striking fact: The field $\Phi(x,\theta,\bar\theta)$ is supersymmetric "in law".

Theorem. *Let $V_1, V_2, \ldots, V_n$ be smooth functions such that they and their derivatives grow at most polynomially, and let $F_1, \ldots, F_n \in \mathcal{S}$ be supersymmetric functions then*

$$\left\langle \prod_{i=1}^{n}\left[\int V_i(\Phi(x,\theta,\bar\theta))F^i(x,\theta,\bar\theta)\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right]\right\rangle_\chi = \left\langle \prod_{i=1}^{n}\left[4\pi f^i_\emptyset(0)\,V_i(\varphi(0))\right]\right\rangle_\chi .$$

Step 3. using the previous theorem, it is simple to prove, by that if V is bounded with bounded first and second derivatives we have

$$\begin{aligned}
\mathscr{Z}_f^{\chi}\mathfrak{I}_h^{\chi} &= \left\langle \left(-\int h(\Phi(x,\theta,\bar{\theta}))\,(\theta\bar{\theta}\delta_0(x))\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta} \right) \times \right. \\
&\qquad \left. \times \exp\left[-\int V(\Phi(x,\theta,\bar{\theta}))\tilde{f}\,(|x|^2 + 4\,\theta\,\bar{\theta})\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta} \right] \right\rangle_{\chi} \\
&= \left\langle \int h(\varphi(0))\,e^{-4\pi V(\varphi(0))} \right\rangle_{\chi} \\
&= \int_{\mathbb{R}^n} h(\eta) \exp\left[-4\,\pi \left(\frac{(1+2\chi)}{2} m^{2(1+2\chi)}\,|\eta|^2 + V(\eta) \right) \right] \mathrm{d}\eta \,/\, \mathscr{Z}
\end{aligned}$$

extension to general potentials

there are two main problems in extending the previous results to more general potentials:

- when V is not convex the elliptic SPDE admits multiple solutions and so there is no clear relation between its solutions and the integral

$$\int h(\varphi(0))\, \Upsilon_f \Lambda_U \,\mathrm{d}\mu,$$

- even if we focus only on this integral, we cannot extend the proof involving the supersymmetry since

$$\Lambda_{U^\chi} \nrightarrow \Lambda_U$$

in L^p when V is a general smooth function.

The first step in the extension of dimensional reduction consists in proving some results for the integral

$$\int h(\varphi(0))\,\Upsilon_f\,\Lambda_U\,\mathrm{d}\mu.$$

Theorem 1. (Klein, Landau, Perez) *Let V be an even polynomial convex at infinity then*

$$|\Lambda_U| \in L^p$$

and

$$\mathscr{Z}_f^{-1}\int_{\mathscr{W}} h(\varphi(0))\,\Upsilon_f(\varphi)\,\det{}_2\big(I+\nabla(U)(\varphi)\big)\exp\left[-\delta(U)-\frac{1}{2}\|U\|^2_{\mathscr{I}(L^2(\mathbb{R}^2))}\right]\mathrm{d}\mu(\varphi)$$

$$=\int_{\mathbb{R}^n} h(\eta)\exp\left[-4\pi\left(\frac{1}{2}m^2|\eta|^2+V(\eta)\right)\right]\mathrm{d}\eta/\mathscr{Z}$$

Hypothesis V_λ. We have the decomposition

$$V = V_B + \lambda V_U, \qquad V_U(y) = \sum_{i=1}^{n} (y^i)^4, \qquad y = (y^1, \dots, y^n) \in \mathbb{R}^n,$$

with $\lambda > 0$ and V_B a bounded function with all bounded derivatives on $\mathbb{R}^n$.

Theorem 2. *If V satisfies hypothesis V_λ then $\Lambda_U \in L^p$ and*

$$\int_{\mathscr{W}} h(\varphi(0))\, \Upsilon_f(\varphi)\, \Lambda_U(\varphi)\, \mathrm{d}\mu(\varphi) = \frac{\mathscr{Z}_f}{\mathscr{Z}} \int_{\mathbb{R}^n} h(\eta) \exp\left[-4\pi\left(\frac{1}{2}m^2|\eta|^2 + V(\eta)\right)\right] \mathrm{d}\eta$$

nonconvex potentials

the second step consists in extending the relation between the solutions to the elliptic SPDE and the integral $\int h(\varphi(0))\,\Upsilon_f \Lambda_U \mathrm{d}\mu$ when the potential is non-convex.

Definition 3. *The measure ν of ϕ is a weak solution to the elliptic SPDE if*

$$T_\star \nu = \mu$$

where $T(w) = w + \mathscr{I}\,(f\,\partial V(w))$, μ is the law of $\mathscr{I}\xi$ and $\mathscr{I} = (-\Delta + m^2)^{-1}$.

when V is non-convex, pathwise uniqueness is not clear, and there might be multiple weak solutions.

however, if V satisfies hypothesis $\boldsymbol{V}_\lambda$ it is possible to give a characterization of the generic weak solution.

Theorem 4. *The measure ν is a weak solution to the elliptic SPDE if and only if*

$$\frac{d\nu}{d\mu} = A|\Lambda_U| = A|\det{}_2(I+\nabla U)|\exp\left(-\delta(U) - \frac{1}{2}\|U\|^2_{\mathcal{I}(L^2)}\right),$$

where A is any non-negative measurable function such that

$$\sum_{\varphi' \in T^{-1}(\varphi)} A(\varphi') = 1$$

for μ-almost $\varphi \in \mathcal{W}$.

(via the general theory of Üstunel–Zakai)

Lemma 5. *If V satisfies hypothesis V_λ there exists at least a weak solution ν_0 such that*

$$\int_{\mathscr{W}} h(\varphi(0))\,\Upsilon_f(\varphi)\,d\nu_0(\varphi) = \frac{\mathscr{Z}_f}{\mathscr{Z}}\int_{\mathbb{R}^n} h(\eta)\exp\left(-4\pi\left(\frac{1}{2}m^2|\eta|^2 + V(\eta)\right)\right)\mathrm{d}\eta$$

Idea of the proof. We identify the measure ν_0 with the positive functional

$$L(g) = \int g(\varphi)\,\mathrm{d}\nu_0(\varphi)$$

defined on the Banach space $C_b^0(\mathscr{W})$ of continuous bounded functions on $\mathscr{W}$.

The measure ν_0 satisfies the thesis of the theorem if and only if

$$\begin{aligned} L(\tilde{g}\circ T) &= \int_{\mathscr{W}} \tilde{g}(\varphi)\,\mathrm{d}\nu_0(\varphi) \\ L(h(\varphi(0))\,\Upsilon_f) &= \frac{\mathscr{Z}_f}{\mathscr{Z}}\int_{\mathbb{R}^n} h(y)\exp\left(-4\pi\left(\frac{1}{2}m^2|\eta|^2+V(\eta)\right)\right)\mathrm{d}\eta. \end{aligned}$$

It is possible to prove, using the previous theorem on

$$\int h(\varphi(0))\,\Upsilon_f\,\Lambda_U\,\mathrm{d}\mu,$$

that the two previous conditions on L are compatible.

Thus the thesis follows exploting the fact that $\Lambda_U \in L^p$ and using the extension theorem for positive functionals on Riesz spaces.

general potentials

The main idea of the third step consists in exploiting the properties of the elliptic SPDE.

Hypothesis QC. The potential $V:\mathbb{R}^n \to \mathbb{R}$ is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and such that there exists a function $H:\mathbb{R}^n \to \mathbb{R}$ with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V(y + r\hat{n})\rangle \leqslant H(y), \qquad \hat{n} \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with $\mathbb{S}^{n-1}$ is the $n-1$ dimensional sphere.

The following families of functions satisfy Hypothesis **QC**:

- Smooth convex functions,
- Smooth bounded functions,
- Smooth functions having the second derivative semidefinite positive outside a compact set,
- Any positive linear combinations of the previous functions.

Applying the maximum principle to the random equation

$$(-\Delta + m^2)\,\bar{\phi} + f\,\partial V(\mathscr{I}\,\xi + \bar{\phi}) = 0,$$

where $\bar{\phi} = \phi - \mathscr{I}\,\xi$, and using hypothesis **QC** on the potential V we obtain the apriori PDE bound:

$$\|\bar{\phi}\|_\infty \lesssim \|H\,(\mathscr{I}\,\xi)\,f\|_\infty.$$

It is important to note that the previous inequality depends on V only through the function H. The idea is to approximate the potential V with a sequence of potentials $(V_i)_i$.

It is possible to prove the existence of a sequence of potentials V_i converging, uniformly on compact sets of $\mathbb{R}^n$, to V and satisfying hypothesis V_λ and hypothesis **QC** with the same function H.

These facts imply that:

- the sequence T_i converges to T uniformly on compact sets of $\mathscr{W}$,
- any sequence of measures ν_i such that $T_{i,\star}(\nu_i) = \mu$ is tight,
- any weak limit ν of a subsequence ν_{i_n} is such that $T_\star(\nu) = \mu$.

IWe consider the weak limit ν_0 of a sequence of weak solutions $\nu_{i,0}$, satisfying the dimensional reduction theorem, then ν_0 will satisfy the dimensional reduction theorem.

Theorem 6. *Suppose that V satisfies the hypothesis **QC** and f satisfies the hypothesis CO, then there exists at least a weak solution ν_0 such that*

$$\mathbb{E}_{\nu_0}\left[h(\phi(0))\,\Upsilon_f(\phi)\right] = \frac{\mathscr{Z}_f}{\mathscr{Z}}\int_{\mathbb{R}^n} h(\eta)\exp\left[-4\pi\left(\frac{1}{2}m^2|\eta|^2 + V(\eta)\right)\right]\mathrm{d}\eta.$$

IR cutoff removal

The main difficulty in this operation is to prove the convergence $\mathscr{Z}_f^{-1} \Upsilon_f(\phi) \to 1$. Indeed this factor does not actually converge and what we can reliably expect is that

$$\lim_{f\to 1} \mathscr{Z}_f^{-1} \mathbb{E}[\Upsilon_f(\phi_f) \,|\, \sigma(\phi_f(0))] \to 1.$$

Theorem. *Suppose that* $\partial^2 V + m^2 > \epsilon$ *let* ϕ *be the solution to the elliptic equation with* $f \equiv 1$*. Then for any* $x \in \mathbb{R}^2$ *and any measurable and bounded function* h *defined on* $\mathbb{R}^n$ *we have*

$$\mathbb{E}[h(\phi(x))] = \mathscr{Z}^{-1} \int_{\mathbb{R}^n} h(\eta) \exp\left(-4\pi\left(\frac{1}{2} m^2 |\eta|^2 + V(\eta)\right)\right) \mathrm{d}\eta.$$

the idea of the proof is to consider the equation

$$(-\Delta+m^2)\phi_t(x)+t\,\partial V_T(\phi_t(x))=\xi(x),$$

when V_T is a trigonometric polynomial and $t\in\mathbb{R}$ such that

$$t\,\partial^2 V_T+m^2>0.$$

under this hypothesis, and the condition that h is a polynomial, it is possible to prove that $\phi_t(x)$ is real analytic with respect to t and so $\mathfrak{H}(t)=\mathbb{E}[h(\phi_t(0))]$ is real analytic with respect to t.

then we are able to prove that

$$\partial_t^k\mathfrak{H}(0)=\partial_t^k\left\{\mathscr{Z}_t^{-1}\int_{\mathbb{R}^n}h(\eta)\left[-4\pi\left(\frac{1}{2}m^2|\eta|^2+t\,V_T(\eta)\right)\right]\mathrm{d}\eta\right\}\Bigg|_{t=0}.$$

since it is possible to approximate any potential satisfying hypothesis **QC** and such that $\partial^2 V+m^2>\epsilon$ with potentials V_T of the previous form the theorem is proved.

Elliptic stochastic quantisation in $d>0$

we use the previous results to prove a version of dimensional reduction for elliptic SPDEs in dimension $d+2$ for $d>0$.

In particular we consider the equation

$$(-\Delta_x - \Delta_z + m^2)\phi + \mathfrak{a}_\epsilon^{*2} * (f(x')\, g(z')\, \partial V(\phi)) = \mathfrak{a}_\epsilon * (\xi(x,z))$$

where $(x,z) \in \mathbb{R}^2 \times \mathbb{R}^d$, $\mathfrak{a}_\epsilon \colon \mathbb{R}^d \to \mathbb{R}$ is a mollifier, g is a smooth function with compact support on $\mathbb{R}^d$ and ξ is a $d+2$ dimensional white noise.

It is possible to prove that there exists at last a weak solution ν_0 to the previous equation such that

$$\exp\left[\int_{\mathbb{R}^{d+2}} f'(x)\, g(z)\, \phi(x,z)\, \mathrm{d}x\mathrm{d}z\right] \mathrm{d}\nu_0\Big|_{\phi(0,\cdot)} \approx \text{"}\exp\left[-4\pi\left(K_\epsilon(\varphi)+\int_{\mathbb{R}^d} g\, V(\varphi)\, \mathrm{d}z\right)\right]\mathscr{D}\varphi\text{"}$$

where $\mathscr{D}\varphi$ is a formal Lebesgue measure on the space of functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ and

$$K_\epsilon(\varphi) = \int_{\mathbb{R}^d} \left(|\mathscr{A}_\epsilon^{-1} \Delta_z \varphi|^2 + m^2 \varphi^2\right) \mathrm{d}z,$$

where $\mathscr{A}_\epsilon(h) = \mathfrak{a}_\epsilon * h$.

For $d=1$ it is possible to take the limit $\mathfrak{a}_\epsilon \to \delta_0$ since $(-\Delta+m^2)^{-1}\xi$ is a continuous function.

For $d \geqslant 2$ the equation becomes singular when $\epsilon \to 0$, and we need renormalization for defining the composition of ϕ with the function V.

In the case $d = 2$ and $V = \exp(\alpha x)$ (for $|\alpha| < 4\pi$) the renormalization leaves V convex.

Then we show that the solution to the singular elliptic SPDE

$$(-\Delta_x - \Delta_z + m^2)\phi + \alpha :e^{\alpha\phi} := \xi,$$

where $:\exp(\alpha\phi) := "\exp\left(\alpha\phi - \frac{\mathbb{E}[\alpha^2\phi^2]}{2}\right)"$, has the law

$$\phi(x,z) \sim "\exp\left[-4\pi\left(K_0(\varphi) + \int_{\mathbb{R}^d} :\exp(\alpha\varphi(z)): \mathrm{d}z\right)\right] \mathscr{D}\varphi"$$

open problems

- IR cut-off removal in the non-convex case;
- Invariance with respect to translations and uniqueness in non-convex case;
- Possible extensions of dimensional reduction to elliptic SPDEs with multiplicative noise;
- Probabilistic proof of dimensional reduction, explaining the role of supersymmetry.

thanks