◇ ◇ ◇

A PDE construction of Φ^4_3

◇ ◇ ◇

M. Gubinelli, University of Bonn

(joint work with M. Hofmanová)

Euclidean QFT in $\mathbb{R}^3$

Theorem

There exists a family $(\nu^\lambda)_{\lambda>0}$ of probability measures on $\mathcal{S}'(\mathbb{R}^3)$ which are non-Gaussian, Euclidean invariant and reflection positive.

▷ Reflection (or Osterwalder–Schrader) positivity : $(\theta f)(x_0, x_1, x_2) = f(-x_0, x_1, x_2)$

$$\int_{\mathcal{S}'(\mathbb{R}^3)} \Big(\sum_i c_i e^{i\varphi(f_i)}\Big)\Big(\sum_i c_i^* e^{-i\varphi(\theta f_i)}\Big)\nu^\lambda(\mathrm{d}\varphi) \geqslant 0,$$

▷ Euclidean invariance and reflection positivity are key properties for the Euclidean approach to constructive quantum field theory, i.e. prove the existence of certain math-emaical objects describing the quantum physics of relativistic particles (here in $2+1$ dimensions).

▷ Schwinger functions:

$$S_n(f_1 \otimes \cdots \otimes f_n) := \int_{\mathcal{S}'(\mathbb{R}^3)} \varphi(f_1) \cdots \varphi(f_n) \nu^\lambda(\mathrm{d}\varphi).$$

Osterwalder–Schrader axioms (1975)

OS0. (Distribution property) Norm $\|\cdot\|_s$ on $\mathcal{S}'(\mathbb{R}^3_+)$ and $\beta > 0$

$$|S_n(f_1 \otimes \ldots \otimes f_n)| \leqslant (n!)^\beta \prod_{i=1}^{n} \|f_i\|_s. \qquad \forall n \geqslant 0,\ f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^3_+).$$

OS1. (Euclidean invariance) $(a, R).\, f_n(x) = f_n(a + R\,x)$, $(a, R) \in \mathbb{R}^3 \times \mathrm{O}(3)$

$$S_n((a, R).\, f_1 \otimes \ldots \otimes (a, R).\, f_n) = S_n(f_1 \otimes \ldots \otimes f_n),$$

OS2. (Reflection positivity) $(f_n \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^{3n}_<))_{n \in \mathbb{N}_0}$ (with finitely many nonzero elements)

$$\sum_{n, m \in \mathbb{N}_0} S_{n+m}(\overline{\theta f_n} \otimes f_m) \geqslant 0,$$

OS3. (Symmetry) $\forall \pi$ permutation of n elements

$$S_n(f_1 \otimes \cdots \otimes f_n) = S_n(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}).$$

The Φ^4_3 measure

▷ The Φ^4_3 measure ν^λ_Λ on $\Lambda \subseteq \mathbb{R}^3$ with $\lambda \geqslant 0$ is given by the formal prescription

$$\nu^\lambda_\Lambda(\mathrm{d}\phi) = \frac{e^{-\lambda V(\phi)}}{\mathcal{Z}}\mu(\mathrm{d}\phi), \qquad V(\phi) = \int_\Lambda \phi(x)^4 \mathrm{d}x,$$

where μ is the Gaussian measure on $\mathcal{S}'(\Lambda)$ with covariance $(\mu^2 - \Delta)^{-1}$.

▷ The measure μ is only supported on distributions of regularity $-1/2-\kappa$, therefore the potential V is not well defined $\Rightarrow$ need for renormalization.

▷ Regularization $\phi_T = \rho_T * \phi$ with $\rho_T \rightarrow \delta$ as $T \rightarrow \infty$ and introduction of *counterterms*

$$\nu^\lambda_{\Lambda,T}(\mathrm{d}\phi) = \frac{e^{-\lambda V_T(\phi_T)}}{\mathcal{Z}_T}\mu(\mathrm{d}\phi), \qquad V_T(\phi) = \int_\Lambda (\phi^4 - a_T\phi^2 - b_T)\mathrm{d}x \geqslant -C_T > -\infty.$$

Problem: Control the limit $T \rightarrow \infty$ and $\Lambda \rightarrow \mathbb{R}^3$ of the family $(\nu^\lambda_{\Lambda,T})_{\Lambda,T}$, describe the limiting object, prove the properties needed for applications to QFT (e.g. Osterwalder–Schrader axioms).

A long history

▷ *Constructive QFT.* ('70-'80) Glimm, Jaffe. Nelson. Segal. Guerra, Rosen, Simon...

▷ $(\Phi^4_3)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75)

▷ $(\Phi^4_3)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)

▷ *Other constructions.* Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)

▷ *Stochastic quantisation (d=2).* Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Röckner, R.Zhu, X.Zhu ('17)

▷ *Stochastic quantisation (d=3).* Hairer ('14) Kupiainen ('16) Catellier, Chouk ('17) Mourrat, Weber ('17) Hairer, Mattingly ('18) R.Zhu, X.Zhu ('18) G, Hofmanova ('18)

"Not only should one give a transparent proof of the dimension $d = 3$ construction, but as explained to me by Gelfand, one should make it sufficiently attractive that probabilists will take cognizance of the existence of a wonderful mathematical object."

(A. Jaffe, 2008)

Aim: construct the measure ν, take the $\Lambda \to \mathbb{R}^3$ limit and prove OS axioms via *dynamics*

Lattice approximation: $\Lambda_\varepsilon = \varepsilon\mathbb{Z}^d$, $\Lambda_{M,\varepsilon} = \varepsilon\mathbb{Z}^d \cap [-M/2, M/2)^d$.

▷ Langevin dynamics: $\varphi_\varepsilon = \varphi_\varepsilon(t,x)$, $t \geqslant 0$, $x \in \Lambda_{M,\varepsilon}$,

$$\dot{\varphi}_{M,\varepsilon} + (m^2 - \Delta)\varphi_{M,\varepsilon} + \lambda\varphi_{M,\varepsilon}^3 + (-3\lambda a_{M,\varepsilon} - 3\lambda^2 b_{M,\varepsilon})\varphi_{M,\varepsilon} = \xi_{M,\varepsilon},$$

$(\xi_{M,\varepsilon}(t,x))_{t \geqslant 0, x \in \Lambda_{M,\varepsilon}}$ collection of (time) white noises.

▷ Invariant measure (reflection positive, invariant under lattice translation)

$$\nu_{M,\varepsilon}^\lambda(\mathrm{d}\varphi) = \frac{e^{-\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} (|\nabla_\varepsilon \varphi|^2 + r_{M,\varepsilon}|\varphi|^2 + \frac{\lambda}{2}|\varphi|^4)}}{Z_{M,\varepsilon}} \prod_{x \in \Lambda_{M,e}} \mathrm{d}\varphi(x).$$

$$r_{M,\varepsilon} = m^2 - 3\lambda a_{M,\varepsilon} + 3\lambda^2 b_{M,\varepsilon}$$

▷ Prove results about $\nu_{M,\varepsilon}^\lambda$ when $M \to \infty, \varepsilon \to 0$ from *uniform* estimates on the PDE.

(Albeverio, Kusuoka ('18) in finite volume)

▷ From the PDE (ignoring renormalization)

$$\mathrm{d}\|\varphi(t)\|_{L^2}^2 + (m^2\|\varphi(t)\|_{L^2}^2 + \|\nabla\varphi(t)\|_{L^2}^2 + \lambda\|\varphi(t)\|_{L^4}^4)\mathrm{d}t = \langle\varphi(t), \xi(\mathrm{d}t)\rangle + C\mathrm{d}t.$$

▷ Stationarity gives estimates for the invariant measure:

$$\mathbb{E}(m^2\|\varphi(t)\|_{L^2}^2 + \|\nabla\varphi(t)\|_{L^2}^2 + \lambda\|\varphi(t)\|_{L^4}^4) = C.$$

▷ Too naive: C is not uniform in ε, M. $\varphi \notin L^2$ under ν^λ.

▷ Littlewood–Paley decomposition

$$f = \sum_{i \geqslant -1} \Delta_i f, \qquad g = \sum_{j \geqslant -1} \Delta_j g$$

with $\operatorname{supp}(\mathcal{F}\Delta_i f) \subseteq 2^i \mathcal{A}$, $i \geqslant 0$.

▷ Paraproducts (Bony, Meyer)

$$fg = \sum_{i,j: i < j-1} \Delta_i f \Delta_j g + \sum_{i,j: j < i-1} \Delta_i f \Delta_j g + \sum_{i,j: |i-j| \leqslant 1} \Delta_i f \Delta_j g$$

$$=: f \prec g + f \succ g + f \circ g$$

▷ "Better than products": $f \prec g$ is always well defined.

▷ Resonant product $f \circ g$ well defined only if positive sum of regularities.

Paradifferential analysis of the problem

▷ φ be a *stationary* solution to

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a + 3b)\varphi + \varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

▷ *Ansatz* $\varphi = X + \eta$ where $(\partial_t - \Delta_\varepsilon + m^2)\underbrace{X}_{-1/2-\kappa} = \underbrace{\xi}_{-5/2-\kappa}$ (stationary) gives

$$(\partial_t - \Delta_\varepsilon + m^2)\eta + 3b\varphi + \underbrace{[\![X^3]\!]}_{-3/2-\kappa} + 3\eta\underbrace{[\![X^2]\!]}_{-1-\kappa} + 3\eta^2\underbrace{X}_{-1/2-\kappa} + \eta^3 = 0$$

- Instead of removing X^{Ψ} where $(\partial_t - \Delta_\varepsilon + m^2)X^{\Psi} = -[\![X^3]\!]$
- Let Y solve $(\partial_t - \Delta_\varepsilon + m^2)Y = -[\![X^3]\!] - 3(\Delta_{>L}[\![X^2]\!]) \succ Y$ (via fixed point)
- *Define* $\varphi = X + Y + \phi$ to have

$$(\partial_t - \Delta_\varepsilon + m^2)\phi + \phi^3 = -3[\![X^2]\!] \succ \phi - 3[\![X^2]\!] \circ \phi + \text{better (after renormalization)}$$

$$\frac{1}{2}\partial_t \|\phi\|_{L^{2,\varepsilon}}^2 + \|\phi\|_{L^{4,\varepsilon}}^4 + \langle \phi, (m^2 - \Delta_\varepsilon) \underbrace{\phi}_{1-\kappa} \rangle_\varepsilon$$

$$= \langle \phi, \underbrace{-3[\![X^2]\!] \succ \phi}_{-1-\kappa} \rangle_\varepsilon + \langle \phi, -3\underbrace{[\![X^2]\!]}_{-1-\kappa} \circ \phi \rangle_\varepsilon + \langle \phi, \text{better (after renormalization)} \rangle_\varepsilon$$

▷ *approximate duality*

$$\langle \phi, -3[\![X^2]\!] \circ \phi \rangle_\varepsilon - \langle -3[\![X^2]\!] \succ \phi, \phi \rangle_\varepsilon =: D(\phi, -3[\![X^2]\!], \phi)$$

bounded if the sum of the regularities of $\phi, -3[\![X^2]\!], \phi$ positive!

▷ combine with the Laplace term

$$\langle \phi, (m^2 - \Delta_\varepsilon)\phi + 2\cdot 3[\![X^2]\!] \succ \phi \rangle_\varepsilon$$

▷ complete the square using *elliptic paracontrolled Ansatz* (ψ is more regular than ϕ)

$$(m^2 - \Delta_\varepsilon)\psi := (m^2 - \Delta_\varepsilon)\phi + 3[\![X^2]\!] \succ \phi$$

- include a polynomial weight $\rho(x) = (1 + |x|^2)^{-\theta/2} \in L^4$ (= test by $\rho^4\phi$ instead of ϕ)
- denote $\mathbb{X}_{M,\varepsilon} = (X_{M,\varepsilon}, [\![X^2_{M,\varepsilon}]\!], X^{\Psi}_{M,\varepsilon}, \ldots)$
- uniformly in M, ε:

$$\frac{1}{2}\partial_t \|\rho^2\phi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho\phi_{M,\varepsilon}\|^4_{L^{4,\varepsilon}} + \|\rho^2\phi_{M,\varepsilon}\|^2_{H^{1-2\kappa,\varepsilon}} + \|\rho^2\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho^2\nabla_\varepsilon\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}}$$

$$\leqslant (|\log t| + 1) Q_\rho(\mathbb{X}_{M,\varepsilon}).$$

- the resonant product $[\![X^2]\!] \circ \phi$ not controlled; $[\![X^2]\!] \circ \psi$ also not
- analogy with PDE weak solutions (equation interpreted in a suitable duality sense)

Tightness

▷ Recall

- $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + Y_{M,\varepsilon} + \phi_{M,\varepsilon}$ is stationary with law $\nu_{M,\varepsilon}$
- $X_{M,\varepsilon}$ stationary, $Y_{M,\varepsilon}$ not stationary $\Rightarrow$ $\phi_{M,\varepsilon}$ not stationary

▷ Alternative *stationary* decomposition

$$\varphi_{M,\varepsilon} = X_{M,\varepsilon} + X^{\Psi}_{M,\varepsilon} + \zeta_{M,\varepsilon}$$

Theorem

- *The family of joint laws of* $(\varphi_{M,\varepsilon}, \mathbb{X}_{M,\varepsilon})$ *evaluated at some* $t \geqslant 0$ *is tight.*
- *Any limit measure* μ *satisfies for all* $p \in [1, \infty)$

$$\mathbb{E}_\mu \|\varphi\|^{2p}_{H^{-1/2-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{2p}_{L^2(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{2}_{H^{1-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{4}_{B^0_{4,\infty}(\rho)} < \infty.$$

- $\mathrm{Law}_\mu(\varphi_t)$ *is Non-Gaussian, OS positive, translation invariant (missing rotations).*

Nontriviality

$$L(\varphi, \varphi, \varphi, \varphi) := \langle (\Delta_i \varphi)^4(0) \rangle - 3\langle (\Delta_i \varphi)^2(0) \rangle^2$$

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$ to get

$$L(\varphi, \varphi, \varphi, \varphi) = \underbrace{L(X, X, X, X)}_{=0} + 4L(X, X, X, X^{\Psi}) + R$$

- show that

$$|R| \lesssim 2^{i(1/2+5\kappa)}$$

- whereas explicit computation yields

$$L(X, X, X, X^{\Psi}) \approx -2^i$$

Integration by parts formula

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi) \propto \exp\left\{-2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2}|\varphi|^2 + \frac{1}{4}|\varphi|^4\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\varphi(x)$$

- F a cylinder functional on $S'(\Lambda_{M,\varepsilon})$: $F(\varphi) = \Phi(\varphi(f_1), \ldots, \varphi(f_n))$
- (finite dimensional) integration by parts gives

$$\int \mathrm{D}F(\varphi)\nu_{M,\varepsilon}(\mathrm{d}\varphi) = 2\int F(\varphi)[\varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + (m^2 - \Delta_\varepsilon)\varphi]\nu_{M,\varepsilon}(\mathrm{d}\varphi).$$

To pass to the limit:

- use the stationary decomposition $\varphi = X + X^{\Psi} + \zeta$
- φ^3 is problematic
  - $[\![X^2]\!] \circ \zeta$ – not well-defined based on the energy estimates so far
  - If ρ is the Gaussian free field then $[\![\rho^3]\!]$ exists only as an *Hida distribution*
  - $[\![X^3]\!]$ is a space-time distribution

φ^3 as an "Hida" distribution

- Let $h\colon \mathbb{R} \to \mathbb{R}$ smooth with $\operatorname{supp} h \subset \mathbb{R}_+$ and $\int_{\mathbb{R}} h \, \mathrm{d}t = 1$
- Let $[\![\varphi^3]\!] := \varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi$ we get

$$\int F(\varphi)[\![\varphi^3]\!]\nu_{M,\varepsilon}(\mathrm{d}\varphi) = \mathbb{E}[F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!]] = \mathbb{E}\left[\int_{\mathbb{R}} h(t) F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!] \, \mathrm{d}t\right]$$

Theorem

$$\int \mathrm{D}F(\varphi)\nu(\mathrm{d}\varphi) = 2\int F(\varphi)[(m^2 - \Delta)\varphi]\nu(\mathrm{d}\varphi) + 2J(F),$$

$$J(F) := \mathbb{E}\left[\int_{\mathbb{R}} h(t) F(\varphi(t))[\![\varphi^3]\!](t) \, \mathrm{d}t\right] \text{"="} \int F(\varphi)[\![\varphi^3]\!]\nu(\mathrm{d}\varphi)$$

$$[\![\varphi^3]\!] = [\![X^3]\!] + 3[\![X^2]\!] \succ (-X^{\Psi} + \zeta) + 3[\![X^2]\!] \prec (-X^{\Psi} + \zeta) + \cdots.$$

▷ operator product expansion, Schwinger–Dyson equations

Thank you.