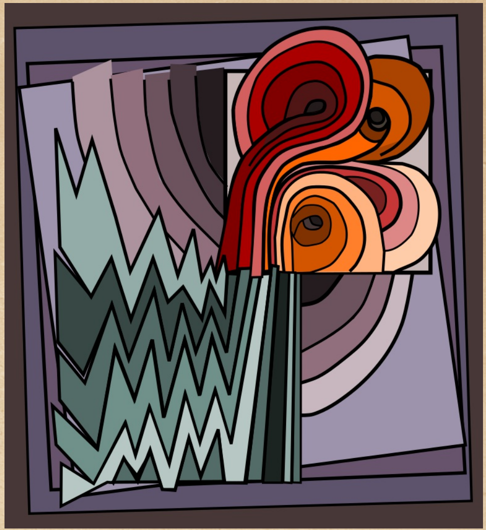
ETHZ – June 16th 2020

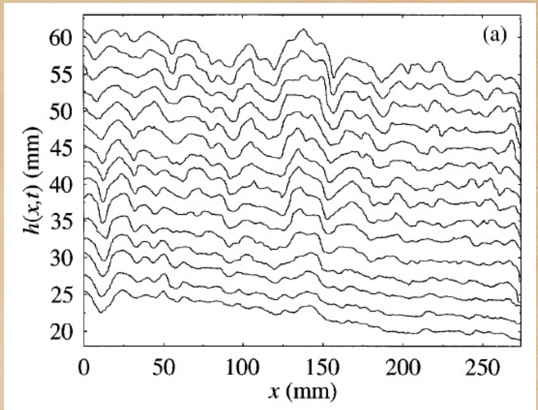
Weak universality, stochastic quantisation and singular SPDEs

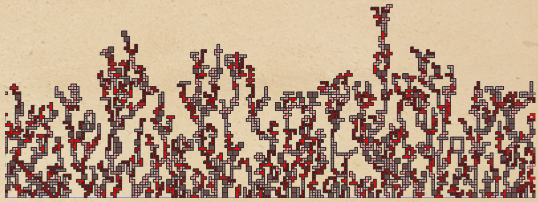


M. Gubinelli – IAM & HCM, University of Bonn

Growth of one dimensional interfaces

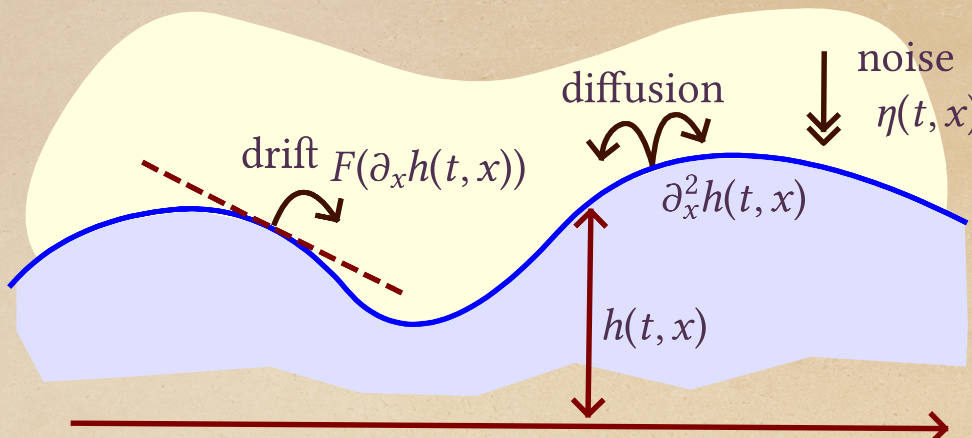
- "Finite growth" e.g. ice and water at $10^{\circ}C$; non–reversible; fluctuations $O(t^{1/3})$; conjectured to rescale to **KPZ fixpoint**. Poorly understood. BORODIN, CORWIN, FERRARI, MATETSKI, QUASTEL, REMENIK, SASAMOTO, SPOHN and many others.
- "Coexistence" e.g. ice and water at $0^{\circ}C$; reversible; fluctuations $O(t^{1/4})$; rescales to Gaussian limit. Well understood. KIPNIS–OLLA–VARADHAN, ZHU, CHANG–YAU and many others.
- "Slow growth" e.g. ice and water at $0.1^{\circ}C$; "nearly" reversible, fluctuations $O(t^{1/4})$, non-Gaussian; rescales to **KPZ equation**.





A simple asymmetric growth model

$$\partial_t h_\varepsilon(t,x) = \partial_x^2 h_\varepsilon(t,x) + \varepsilon^{1/2} F(\partial_x h_\varepsilon(t,x)) + \eta(t,x), \quad t \geq 0, \quad x \in \mathbb{R},$$



▹ η smooth Gaussian field with $O(1)$ stationary correlations. F even polynomial.

Rescaling

- Scaling transformation $\tilde{h}_\varepsilon(t,x) = \varepsilon^{1/2} h_\varepsilon(t/\varepsilon^2, x/\varepsilon)$.

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-1} F(\varepsilon^{1/2} \partial_x \tilde{h}_\varepsilon) + \xi_\varepsilon$$

- Noise $\xi_\varepsilon(t,x) = \varepsilon^{-3/2} \eta(t/\varepsilon^2, x/\varepsilon)$ converges to space–time white noise ξ

$$\mathbb{E}\left[\left(\iint \xi_\varepsilon(t,x)\varphi(t,x)\mathrm{d}t\mathrm{d}x\right)^2\right] \longrightarrow \iint (\varphi(t,x))^2 \mathrm{d}t\mathrm{d}x \qquad \text{as } \varepsilon \longrightarrow 0.$$

$$\mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta(x-x')$$

- Nonlinearity (heuristics):

$$\varepsilon^{-1} F(\varepsilon^{1/2} \partial_x \tilde{h}_\varepsilon) = \varepsilon^{-1} F(0) + \varepsilon^{-1/2} F'(0) \partial_x \tilde{h}_\varepsilon + F''(0)(\partial_x \tilde{h}_\varepsilon)^2 + O(\varepsilon^{1/2})$$

Hairer–Quastel weak universality

▹ Better heuristics: $\partial_t X_\varepsilon = \partial_x^2 X_\varepsilon + \xi_\varepsilon$ and $\tilde{h}_\varepsilon = X_\varepsilon + u_\varepsilon$ with $u_\varepsilon \in C^{1/2+}$

$$\varepsilon^{-1} F(\varepsilon^{1/2}\partial_x \tilde{h}_\varepsilon) = \varepsilon^{-1} F(\varepsilon^{1/2}\partial_x X_\varepsilon) + \varepsilon^{-1/2} F'(\varepsilon^{1/2}\partial_x X_\varepsilon)\partial_x u_\varepsilon + F''(\varepsilon^{1/2}\partial_x X_\varepsilon)(\partial_x u_\varepsilon)^2 + O(\varepsilon^{1/2})$$

Theorem. (Hairer–Quastel 15) *[Polynomial F, Gaussian η] $\exists(\lambda, c, \nu, \rho) = \Lambda(F, \eta)$ such that the random field*

$$H_\varepsilon(t,x) = \tilde{h}_\varepsilon(t, x - \rho t) - (\nu/\varepsilon + c)t,$$

converges in law in $C([0,T],\mathbb{T})$ to $H(t,x)$ solving

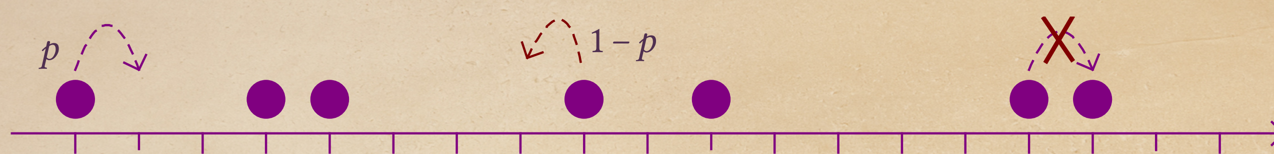
$$H(t,x) = \lambda^{-1}\log Z(t,x), \qquad \partial_t Z = \partial_x^2 Z(t,x) + \lambda Z(t,x)\xi(t,x)$$

*(**Hopf–Cole solution**, the product $Z\xi$ is understood according to Ito calculus).*

Other interface growth models

▹ **WASEP** (Weakly asymmetric simple exclusion) Particles on $\mathbb{Z}$ moves independently, only one particle per size; jump left with rate p, right with rate $1-p$.

For $p=1/2$ reversible dynamics, large scale gaussian fluctuations. For $p=1/2+\varepsilon$ rescales to Hopf–Cole solution of KPZ (BERTINI–GIACOMIN, CMP 97)



▹ **Ginzburg–Landau $\nabla\varphi$ interface model.** Interacting Brownian motions on $\mathbb{Z}$

$$\mathrm{d}x^i = (pV'(r^{i+1}) - (1-p)V'(r^i))\mathrm{d}t + \mathrm{d}B^i, \qquad i \in \mathbb{Z}, \quad r^i = x^i - x^{i-1}.$$

For $p=1/2$ reversible dynamics. large scale gaussian fluctuations.

For $p=1/2+\varepsilon$, rescales to the Hopf–Cole solution of the KPZ equation (DIEHL–G.–PERKOWSKI CMP16)

KPZ equation

Formally, H solves the Kardar–Parisi–Zhang equation:

$$\partial_t H(t,x) = \partial_x^2 H(t,x) - \lambda[(\partial_x H(t,x))^2 - \infty] + \xi(t,x), \qquad t \geq 0, x \in \mathbb{T}.$$

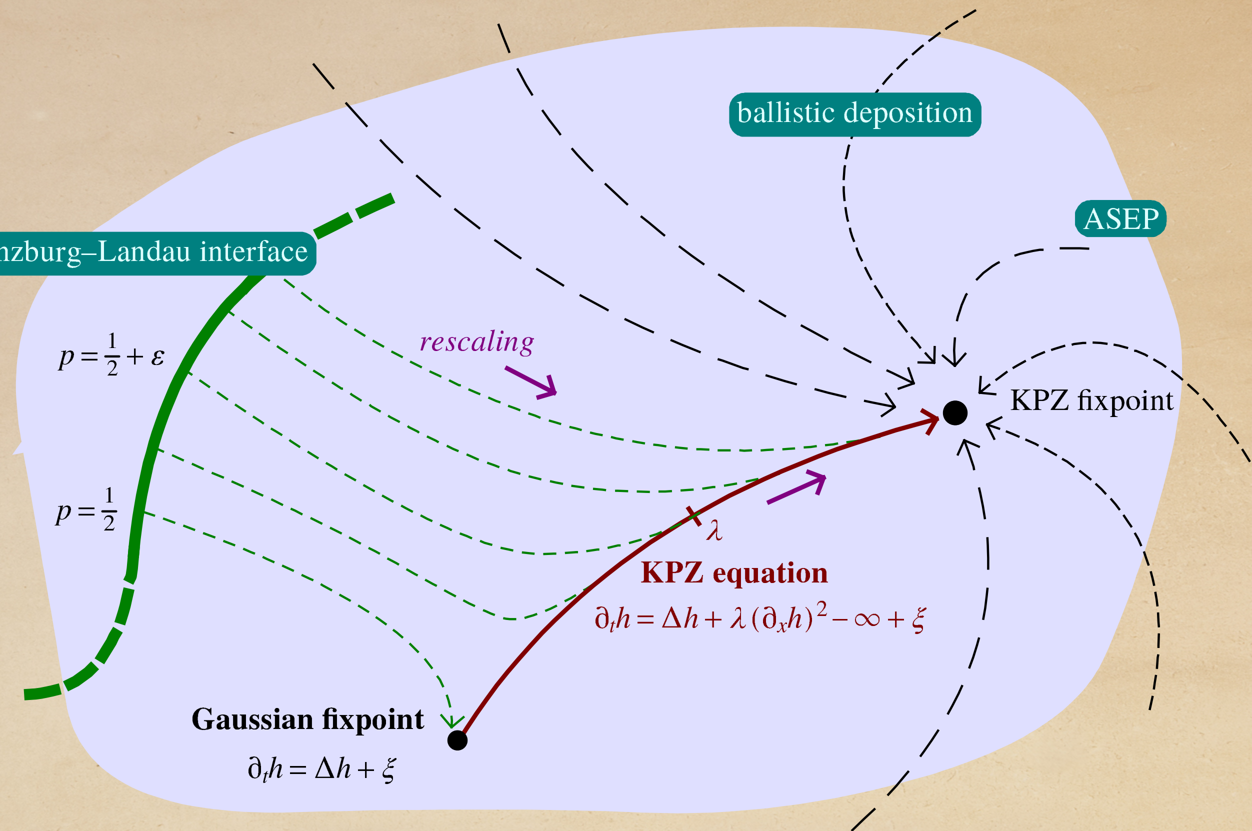
Problem: Not well posed. $H \in C([0,T]; C^{1/2-\kappa}(\mathbb{T}))$. ($\infty$ coming from Ito correction)

▹ HAIRER (Ann.Math. 13). Solution theory for the KPZ based on rough paths (LYONS)

▹ GONÇALVES–JARA (10, ARMA 13). Solution theory for KPZ based on martingale problem. Refined martingale problem (G.–JARA, SPDE/AC 13). Uniqueness (G.–PERKOWSKI, JAMS 18)

▹ HAIRER (Inv.Math. 14), G.–PERKOWSKI (CMP 17) solutions theories based on regularity structures and paracontrolled distributions.

Renormalization group picture



Non-gaussian fluctuations in three dimensions

▹ Scalar fields in $d = 3$ dimensions can be used to describe (mesoscopic) magnetization in ferromagnetic system or (Euclidean) scalar quantum fields in $2 + 1$ dimensions.

▹ We look for "universal" non-Gaussian models for scalar fluctuations in three-dimensions by perturbing a Gaussian model (as we did for the KPZ equation)

▹ A natural family $\Gamma(\mu)$ of centered Gaussian models has covariance

$$\mathbb{E}[X(x)X(y)] = (\mu - \Delta)^{-1}(x, y), \qquad x, y \in \mathbb{R}^3.$$

▹ Under rescaling R_ε which fixes $\Gamma(0)$ the parameter μ grows: $R_\varepsilon\Gamma(\mu) = \Gamma(\varepsilon^{-2}\mu)$, leading to the *high temperature* fixpoint $\mu \longrightarrow \infty$, where correlations are absent in the macroscopic scale.

Dynamical model

▹ Promote $X(x)$ to a *time dependent* random field satisfying the Langevin equation

$$\partial_t X(t,x) = -(\mu - \Delta)X(t,x) + \xi(t,x).$$

New key ingredient: the space-time white noise ξ, a universal source of randomness. The original field $X(x)$ is the invariant measure of the dynamics.

▹ *Nonlinear perturbation*: introduce the family of dynamic Ginzburg–Landau models $\mathrm{DGL}(F,\eta)$ of the form

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) - F(\varphi(t,x)) + \eta(t,x)$$

where η is a smooth Gaussian noise with finite range correlations. A model for noisy reaction-diffusion system.

- ▹ Scaling transformation R_ε (we want to keep diffusion and noise nontrivial):

$$\varphi_\varepsilon(t,x) = \varepsilon^{-1/2}\varphi(t/\varepsilon^2, x/\varepsilon), \qquad \eta_\varepsilon(t,x) = \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon),$$

- ▹ Equation for $R_\varepsilon \mathrm{DGL}(F,\eta) = \mathrm{DGL}(\varepsilon^{-2}F(\varepsilon^{1/2}\cdot), \eta_\varepsilon)$

$$\partial_t\varphi_\varepsilon = \Delta\varphi_\varepsilon - \varepsilon^{-5/2}F(\varepsilon^{1/2}\varphi_\varepsilon) + \eta_\varepsilon$$

- ▹ If $F(\varphi) = a_1\varphi + a_3\varphi^3 + \cdots$ then

$$\varepsilon^{-5/2}F(\varepsilon^{1/2}\varphi_\varepsilon) = \varepsilon^{-2}a_1\varphi + \varepsilon^{-1}a_3\varphi^3 + \varepsilon^0 a_5\varphi^5 + \varepsilon^1 a_7\varphi^7 + \cdots$$

- ▹ **Two relevant directions**: associated to φ and φ^3:
  - φ points towards the high temperature (Gaussian) limit
  - φ^3 points in a new (non-Gaussian) direction

Weak-universality for reaction-diffusion equations

Consider

$$\partial_t \varphi_\varepsilon(t,x) - \Delta\varphi_\varepsilon(t,x) = -F_\varepsilon(\varepsilon^{1/2}\varphi_\varepsilon(t,x)) + \eta_\varepsilon(t,x), \qquad t\in[0,T], x\in\mathbb{T}^3.$$

Theorem 1. (FURLAN, G. PTRF 2018) *There exists a map* $\Lambda:(F,\eta)\mapsto\lambda=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^4$ *such that if* $(F_\varepsilon)_\varepsilon\subseteq C^9_{\exp}$, *and* $\Lambda(F_\varepsilon,\eta_\varepsilon)\longrightarrow\lambda\in\mathbb{R}^4$ *then* $\varphi_\varepsilon\longrightarrow\varphi$ *in* $C([0,T];\mathcal{S}'(\mathbb{T}^3))$ *in probability. Here* φ *is the solution of the* Φ^4_3 *dynamical model:*

$$\partial_t\varphi(t,x) - \Delta\varphi(t,x) = -\lambda_3(\varphi^3-\infty) - \lambda_2(\varphi^2-\infty) - \lambda_1\varphi - \lambda_0 + \xi(t,x).$$

In particular, the law of φ *depends only on* λ *and not on other details of* η *or* F *and is not Gaussian. (If* F_ε *odd, then* $\lambda_2=\lambda_0=0$*).*

[Other results by HAIRER, XU (2018/2019), XU, SHEN (2017)]

Euclidean Quantum Field theories

Link between probability measures on distributions and relativistic quantum mechanical systems

$x \in \mathbb{R}^d$, $\theta x = (x_1, \ldots, x_{d-1}, -x_d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d \geqslant 0\}$. G Euclidean group of $\mathbb{R}^d$ together with reflection θ. $f^g(x) = f(g^{-1}x)$ for $g \in G$.

▹ μ probability measure on $\mathcal{S}'(\mathbb{R}^d)$ and $S(f) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\varphi(f)} \mu(\mathrm{d}\varphi)$ satisfying

1. *Euclidean invariance*: $S(f^g) = S(f)$ for all $g \in G$.
2. *Reflection positivity*: $\forall (f_\alpha \in \mathcal{S}(\mathbb{R}^d_+))_\alpha$, the matrix $(S(f_\alpha - f_\beta^\theta))_{\alpha,\beta}$ is positive definite.
3. *Exponential bounds*: for some k and some norm $\|\cdot\|$: $|S(f)| \leqslant e^{\|f\|^k}$ for all $f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$.

Osterwalder–Schrader reconstruction: Then $\exists$ a *relativistic quantum theory* on an Hilbert space $\mathcal{H}$ equipped with a unitary representation of the Poincaré group. Hamiltonian is positive and has a Poincaré invariant vacuuum vector. [see GLIMM, JAFFE “Quantum Physics”]

Euclidean Φ^4_3 model

Measures that satisfy all these properties are rare.

When $d=3$ we know only the Gaussian free field μ, namely the Gaussian measure with covariance

$$\int_{\mathcal{S}'(\mathbb{R}^3)} \varphi(f)\varphi(g)\mu(\mathrm{d}\varphi) = \langle f, (1-\Delta)^{-1} g\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^3),$$

and the Φ^4_3 measure, formally given by

$$\nu(\mathrm{d}\varphi) = \frac{\exp(-\lambda \int_{\mathbb{R}^3} (\varphi^4/4 - \infty\varphi^2/2)\mathrm{d}x)}{Z_\lambda} \mu(\mathrm{d}\varphi).$$

(Brydges, Federbush, Frölich, Glimm, Guerra, Jaffe, Gallavotti, Mitter, Nelson, Rivasseau, Rosen, Simon, Spencer, and many others, '70-'80)

▹ Rigorously this measure can be constructed on a bounded domain $\Lambda \subseteq \mathbb{R}^3$ and with an ultra-violet cutoff ε and a mass counterterm a_ε

$$\nu_\varepsilon(\mathrm{d}\varphi) = \frac{\exp(-\lambda \int_\Lambda (\varphi_\varepsilon^4/4 - a_\varepsilon \varphi_\varepsilon^2/2)\mathrm{d}x)}{Z_{\lambda,\varepsilon}} \mu(\mathrm{d}\varphi)$$

where $\varphi_\varepsilon = \rho_\varepsilon * \varphi$ and $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$ with smooth regularizer ρ.

Main problem: control the limit as $\varepsilon \longrightarrow 0$ of ν_ε. We expect $\nu \not\ll \mu$.

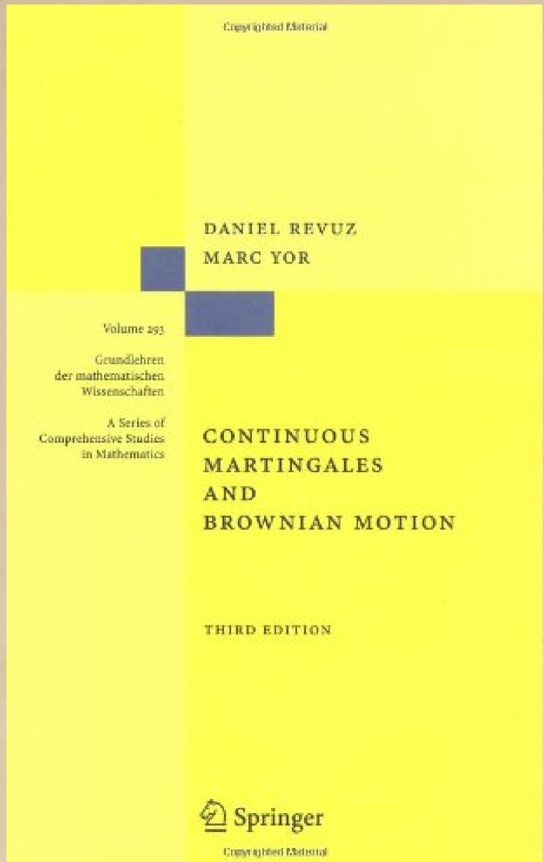
▹ Under μ we have $\varphi \in C^{-1/2-\kappa}$ almost surely.

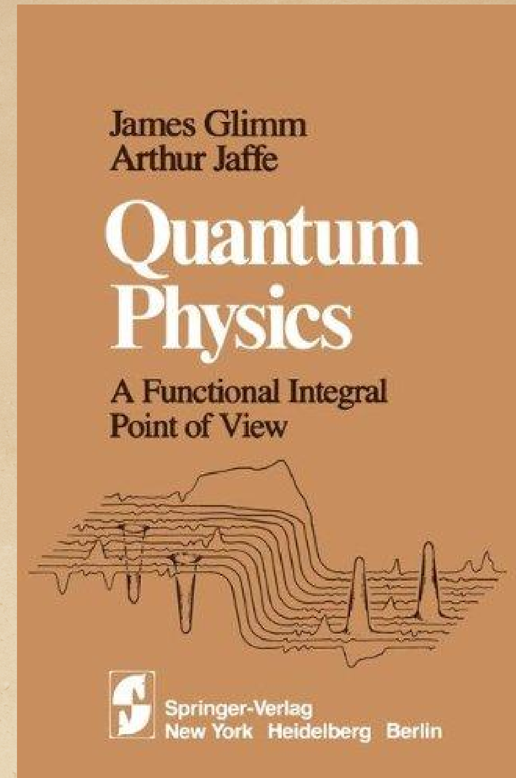
Stochastic analysis

Ito and Dœblin wanted to study diffusion processes via their *sample paths*

Measures	*Samples*
$(\mu_t)_t \subseteq \Pi(S)$	$X\colon \Omega \longrightarrow C(\mathbb{R}_+, S)$
$\mu_t(\mathrm{d}y) = \int P_{t-s}(x, \mathrm{d}y)\mu_s(\mathrm{d}x)$	$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t$

- lower dimensional problem
- more tools (e.g. fixpoint theorems)
- more intuition
- ***canonical*** *reference object* $(B_t)_t$


Copyrighted Material
DANIEL REVUZ
MARC YOR
Volume 293
Grundlehren der mathematischen Wissenschaften
A Series of Comprehensive Studies in Mathematics
CONTINUOUS MARTINGALES AND BROWNIAN MOTION
THIRD EDITION
Springer
Copyrighted Material


James Glimm
Arthur Jaffe
Quantum Physics
A Functional Integral Point of View
Springer-Verlag
New York Heidelberg Berlin

Stochastic quantisation

Relation between a stochastic differential equation and a probability measure

(broadly speaking)

▹ Nelson and Parisi–Wu ('84) advocated the *constructive* use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for the use of Euclidean quantum field theory (in particular gauge theories)

▹ Theoretical version of MCMC methods

(Parabolic) stochastic quantisation

$\Lambda = \text{finite set}, \mathbb{T}^d, \mathbb{R}^d$

equation	$\partial_t \phi(t) = -\dfrac{\delta V(\phi(t))}{\delta \phi} + \sqrt{2}\xi(t),$	$\phi: \mathbb{R}_+ \times \Lambda \longrightarrow \mathbb{R}$
measure	$\phi(t) \sim \nu(\mathrm{d}\varphi) = \dfrac{e^{-V(\varphi)}}{Z} \mathrm{d}\varphi \in \mathrm{Prob}(\Lambda \longrightarrow \mathbb{R})$	

- The measure ν is described via *white noise*
- Markov process, invariant measures, ergodicity

Dynamic Φ_d^4

$$V(\varphi) = \int \frac{1}{2}|\nabla\varphi|^2 + \frac{m^2 - \infty}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4.$$

$$\partial_t\varphi = \Delta\varphi - \lambda(\varphi^3 - \infty\varphi) - m^2\varphi + \sqrt{2}\xi \qquad \mathbb{R}^3 \times \mathbb{R}_+$$

(d=2) Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Tsatsoulis, Weber ('16) Röckner, R.Zhu, X.Zhu ('17)

▹ $d = 3$ is more singular: regularity structures (Hairer), paracontrolled distributions (G. Imkeller, Perkowski)

(HAIRER Inv.Math 14) Local solution theory based on regularity structures. (CATELLIER–CHOUK 15, AOP18) Local solution theory based on paracontrolled distributions (G.–IMKELLER–PERKOWSKI F.Math.Π 15). Renormalization group approach (Kupiainen, AIHP15)

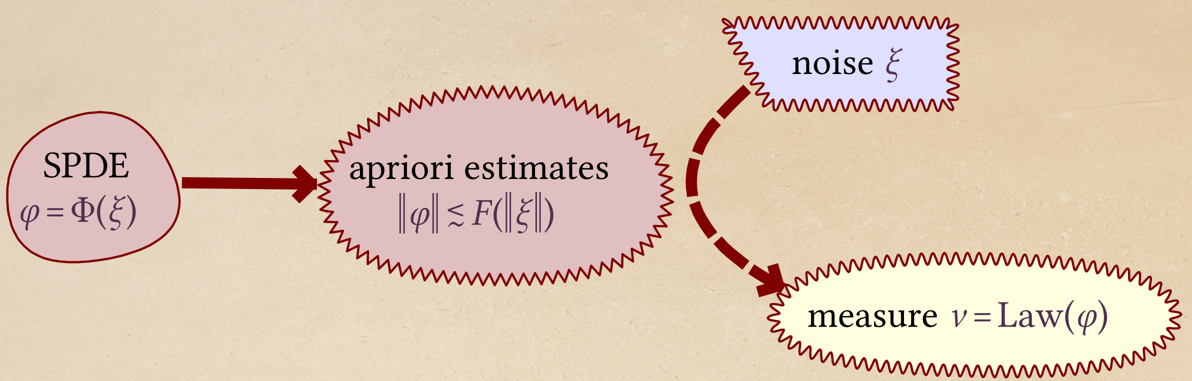
Recent developments

- ▹ Global space–time solutions in $\mathbb{R}^2$ (Mourrat–Weber CMP17)
- ▹ Ergodicity for dynamical Φ^4_2 (Röckner–Zhu–Zhu CMP17)
- ▹ Convergence of lattice discretizations ($\mathbb{T}^3$) (Hairer–Matetski). Complete proof of invariance of Φ^4_3 wrt. the dynamics.
- ▹ Global solution in time on $\mathbb{T}^3$ (Mourrat–Weber CMP17). Coming down from infinity.
- ▹ Tightness for the Φ^4_3 measure via dynamics (Albeverio–Kusuoka 18)
- ▹ Global space–time solutions in $\mathbb{R}^3$ for parabolic equations and global solutions to elliptic equations in $\mathbb{R}^4, \mathbb{R}^5$ related to the Φ^4_2, Φ^4_3 measures via (conjectured) dimensional reduction. (G.–Hofmanová 18).

A PDE construction of Φ^4_3

Reflection positivity + Euclidean invariance $\Longrightarrow$ singularities, infinite volume limit

G., Hofmanová ('18) – construction of Φ^4_3 on $\mathbb{R}^3$ via stochastic quantisation and verification of (most of) the axioms.



▹ Much like Ito's approach to diffusions / Markovianity does not play any role

▹ Mix of: analysis of (low regularity) PDEs in weighted spaces, paradifferential calculus, stochastic analysis of multilinear Gaussian functionals, convergence of finite element methods.

Varieties of stochastic quantisation: canonical stochastic quantisation

equation

$$\begin{cases} \partial_t \phi(t) = \underbrace{-\frac{\delta H(\phi(t), \dot{\phi}(t))}{\delta \dot{\phi}}}_{\text{Hamiltonian dynamics}} \\ \partial_t \dot{\phi}(t) = -\frac{\delta H(\phi(t), \dot{\phi}(t))}{\delta \phi} \underbrace{- \gamma \dot{\phi}(t) + \sqrt{2}\,\xi(t),}_{\text{linear Langevin dynamics}} \end{cases} \quad \phi, \dot{\phi} \colon \mathbb{R} \times \Lambda \longrightarrow \mathbb{R}$$

$$H(\varphi, \dot{\varphi}) := V(\varphi) + \frac{\gamma}{2} \dot{\varphi}^2$$

measure

$$(\phi(t), \dot{\phi}(t)) \sim \nu(\mathrm{d}\varphi \mathrm{d}\dot{\varphi}) = \frac{e^{-H(\varphi, \dot{\varphi})}}{Z} \mathrm{d}\varphi \mathrm{d}\dot{\varphi} \in \mathrm{Prob}(\Lambda \longrightarrow \mathbb{R}^2)$$

▹ Introduced by Ryang, Saito and Shigemoto ('85).

Singular stochastic wave equations

For Φ_d^4, $d = 1, 2, 3$

$$V(\varphi) = \int \frac{1}{2}|\nabla \varphi|^2 + \frac{m^2 - \infty}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4,$$

$$\partial_t^2 \phi = \Delta\phi + (m^2 - \infty)\phi + \lambda\phi^3 - \gamma\partial_t\phi + \sqrt{2}\xi,$$

Problem: no Schauder estimates, scaling arguments less clear.

Conjecture: *same* renormalization constants of the static measure!

- $d = 1$. Tolomeo ('18) unique ergodicity.
- $d = 2$. G, Koch, Oh ('18) local well-posedness (any polynomial), G, Koch, Oh, Tolomeo (in preparation) global well-posedness.
- $d = 3$. G, Koch, Oh ('18) only quadratic nonlinearity.

Elliptic stochastic quantisation

equation	$\Delta_z \phi(z) = -\frac{\delta V(\phi(z))}{\delta \phi} + \xi(z), \qquad \phi \colon \mathbb{R}^2 \times \Lambda \longrightarrow \mathbb{R}$
measure	$\phi(z) \sim \nu(\mathrm{d}\varphi) = \frac{e^{-4\pi V(\varphi)}}{Z} \mathrm{d}\varphi \in \mathrm{Prob}(\Lambda \longrightarrow \mathbb{R})$

Discovered perturbatively by Imry, Ma ('75), Young ('77). Non–perturbative "proof" by Parisi and Sourlas ('79-'82) using *supersymmetry*

$$(\mathrm{SPDE})_{d+2} \xrightarrow[\text{“Girsanov”}]{} (\mathrm{SUSY\ EQFT})_{d+2} \xrightarrow[\text{dimensional reduction}]{} (\mathrm{measure})_d$$

Gaussian case

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 \qquad \Delta_z \varphi(z) = -m^2 \varphi(z) + \xi(z), \qquad z \in \mathbb{R}^2$$

$$\varphi(z) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot z}}{|k|^2 + m^2} \frac{\eta(\mathrm{d}k)}{2\pi}$$

$$\mathbb{E}[\varphi(0)^2] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}k}{(|k|^2 + m^2)^2} = \frac{1}{(2\pi)^2 m^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}k}{(|k|^2 + 1)^2} = \frac{1}{4\pi m^2} \int_0^\infty \frac{\mathrm{d}\rho^2}{(\rho^2 + 1)^2} = \frac{1}{4\pi m^2}$$

$$\varphi(0) \sim e^{-4\pi \frac{m^2}{2} \phi^2} \mathrm{d}\phi \sim e^{-4\pi V(\phi)} \mathrm{d}\phi$$

Rigorous results

▹ Rigorous proof of dimensional reduction by KLEIN, LANDAU AND PEREZ ('84)

▹ Recently complete proof by ALBEVERIO, G. AND DE VECCHI (AOP '18). First for Λ finite dimensional $+$ technical conditions. Then extended to (some) renormalized EQFT.

Stochastic quantisation of Liouville action up to the critical value of $\sigma^2 < 8\pi$ in $\Lambda = \mathbb{T}^2$

$$V(\varphi) = \int_{\mathbb{T}^2} \frac{1}{2} |\nabla \varphi|^2 + \alpha e^{\sigma\varphi - \sigma^2 \infty}$$

Thanks.