Seminar on Stochastic Processes – Zürich, 2019/2/20

Two stochastic methods for Φ^4_3

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(joint work with M. Hofmanová and N. Barashkov)

Theorem

There exists a family $(\nu^\lambda)_{\lambda>0}$ of probability measures on $\mathcal{S}'(\mathbb{R}^3)$ which are non-Gaussian, Euclidean invariant and reflection positive.

▷ Reflection (or Osterwalder–Schrader) positivity : $(\theta f)(x_0,x_1,x_2) = f(-x_0,x_1,x_2)$

$$\int_{\mathcal{S}'(\mathbb{R}^3)} \left(\sum_i c_i e^{i\varphi(f_i)}\right)\left(\sum_i c_i^* e^{-i\varphi(\theta f_i)}\right)\nu^\lambda(\mathrm{d}\varphi) \geqslant 0,$$

▷ Euclidean invariance and reflection positivity are key properties for the Euclidean approach to constructive quantum field theory, i.e. prove the existence of certain mathemaical objects describing the quantum physics of relativistic particles (here in $2+1$ dimensions).

▷ Schwinger functions:

$$S_n(f_1 \otimes \cdots \otimes f_n) := \int_{\mathcal{S}'(\mathbb{R}^3)} \varphi(f_1)\cdots\varphi(f_n)\nu^\lambda(\mathrm{d}\varphi).$$

Osterwalder–Schrader axioms (1975)

OS0. (Distribution property) Norm $\|\cdot\|_s$ on $\mathcal{S}'(\mathbb{R}^3_+)$ and $\beta > 0$

$$|S_n(f_1 \otimes \ldots \otimes f_n)| \leqslant (n!)^\beta \prod_{i=1}^n \|f_i\|_s. \qquad \forall n \geqslant 0,\, f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^3_+).$$

OS1. (Euclidean invariance) $(a, R).\, f_n(x) = f_n(a + R x)$, $(a, R) \in \mathbb{R}^3 \times \mathrm{O}(3)$

$$S_n((a, R).\, f_1 \otimes \ldots \otimes (a, R).\, f_n) = S_n(f_1 \otimes \ldots \otimes f_n),$$

OS2. (Reflection positivity) $(f_n \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^{3n}_<))_{n \in \mathbb{N}_0}$ (with finitely many nonzero elements)

$$\sum_{n, m \in \mathbb{N}_0} S_{n+m}(\overline{\theta f_n} \otimes f_m) \geqslant 0,$$

OS3. (Symmetry) $\forall \pi$ permutation of n elements

$$S_n(f_1 \otimes \cdots \otimes f_n) = S_n(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}).$$

The Φ^4_3 measure

▷ The Φ^4_3 measure ν^λ_Λ on $\Lambda \subseteq \mathbb{R}^3$ with $\lambda \geqslant 0$ is given by the formal prescription

$$\nu^\lambda_\Lambda(\mathrm{d}\phi) = \frac{e^{-\lambda V(\phi)}}{\mathcal{Z}} \mu(\mathrm{d}\phi), \qquad V(\phi) = \int_\Lambda \phi(x)^4 \mathrm{d}x,$$

where μ is the Gaussian measure on $\mathcal{S}'(\Lambda)$ with covariance $(\mu^2 - \Delta)^{-1}$.

▷ The measure μ is only supported on distributions of regularity $-1/2 - \kappa$, therefore the potential V is not well defined $\Rightarrow$ need for renormalization.

▷ Regularization $\phi_T = \rho_T * \phi$ with $\rho_T \to \delta$ as $T \to \infty$ and introduction of *counterterms*

$$\nu^\lambda_{\Lambda,T}(\mathrm{d}\phi) = \frac{e^{-\lambda V_T(\phi_T)}}{\mathcal{Z}_T} \mu(\mathrm{d}\phi), \qquad V_T(\phi) = \int_\Lambda (\phi^4 - a_T \phi^2 - b_T) \mathrm{d}x \geqslant -C_T > -\infty.$$

Problem: Control the limit $T \to \infty$ and $\Lambda \to \mathbb{R}^3$ of the family $(\nu^\lambda_{\Lambda,T})_{\Lambda,T}$, describe the limiting object, prove the properties needed for applications to QFT (e.g. Osterwalder–Schrader axioms).

A long history

▷ *Constructive QFT.* ('70-'80) Glimm, Jaffe. Nelson. Segal. Guerra, Rosen, Simon...

▷ $(\Phi^4_3)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75)

▷ $(\Phi^4_3)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)

▷ *Other constructions.* Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)

▷ *Stochastic quantisation (d=2).* Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Röckner, R.Zhu, X.Zhu ('17)

▷ *Stochastic quantisation (d=3).* Hairer ('14) Kupiainen ('16) Catellier, Chouk ('17) Mourrat, Weber ('17) Hairer, Mattingly ('18) R.Zhu, X.Zhu ('18) G, Hofmanova ('18)

"Not only should one give a transparent proof of the dimension $d = 3$ construction, but as explained to me by Gelfand, one should make it sufficiently attractive that probabilists will take cognizance of the existence of a wonderful mathematical object."

(A. Jaffe, 2008)

One

Aim: construct the measure ν, take the $\Lambda \to \mathbb{R}^3$ limit and prove OS axioms via *dynamics*

Lattice approximation: $\Lambda_\varepsilon = \varepsilon \mathbb{Z}^d$, $\Lambda_{M,\varepsilon} = \varepsilon \mathbb{Z}^d \cap [-M/2, M/2)^d$.

▷ Langevin dynamics: $\varphi_\varepsilon = \varphi_\varepsilon(t,x)$, $t \geqslant 0$, $x \in \Lambda_{M,\varepsilon}$,

$$\dot{\varphi}_{M,\varepsilon} + (m^2 - \Delta)\varphi_{M,\varepsilon} + \lambda \varphi_{M,\varepsilon}^3 + (-3\lambda a_{M,\varepsilon} - 3\lambda^2 b_{M,\varepsilon})\varphi_{M,\varepsilon} = \xi_{M,\varepsilon},$$

$(\xi_{M,\varepsilon}(t,x))_{t \geqslant 0, x \in \Lambda_{M,\varepsilon}}$ collection of (time) white noises.

▷ Invariant measure (reflection positive, invariant under lattice translation)

$$\nu_{M,\varepsilon}^\lambda(\mathrm{d}\varphi) = \frac{e^{-\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} (|\nabla_\varepsilon \varphi|^2 + r_{M,\varepsilon} |\varphi|^2 + \frac{\lambda}{2} |\varphi|^4)}}{Z_{M,\varepsilon}} \prod_{x \in \Lambda_{M,e}} \mathrm{d}\varphi(x).$$

$$r_{M,\varepsilon} = m^2 - 3\lambda a_{M,\varepsilon} + 3\lambda^2 b_{M,\varepsilon}$$

▷ Prove results about $\nu_{M,\varepsilon}^\lambda$ when $M \to \infty, \varepsilon \to 0$ from *uniform* estimates on the PDE.

(Albeverio, Kusuoka ('18) in finite volume)

Energy method

- From the PDE (ignoring renormalization)

$$\mathrm{d}\|\varphi(t)\|_{L^2}^2 + (m^2\|\varphi(t)\|_{L^2}^2 + \|\nabla\varphi(t)\|_{L^2}^2 + \lambda\|\varphi(t)\|_{L^4}^4)\mathrm{d}t = \langle\varphi(t), \xi(\mathrm{d}t)\rangle + C\mathrm{d}t.$$

- Stationarity gives estimates for the invariant measure:

$$\mathbb{E}(m^2\|\varphi(t)\|_{L^2}^2 + \|\nabla\varphi(t)\|_{L^2}^2 + \lambda\|\varphi(t)\|_{L^4}^4) = C.$$

- Too naive: C is not uniform in ε, M. $\varphi \notin L^2$ under ν^λ.

▷ Littlewood–Paley decomposition

$$f = \sum_{i \geqslant -1} \Delta_i f, \qquad g = \sum_{j \geqslant -1} \Delta_j g$$

with $\operatorname{supp}(\mathcal{F}\Delta_i f) \subseteq 2^i \mathcal{A}$, $i \geqslant 0$.

▷ Paraproducts (Bony, Meyer)

$$fg = \sum_{i,j: i < j-1} \Delta_i f \Delta_j g + \sum_{i,j: j < i-1} \Delta_i f \Delta_j g + \sum_{i,j: |i-j| \leqslant 1} \Delta_i f \Delta_j g$$

$$=: f \prec g + f \succ g + f \circ g$$

▷ "Better than products": $f \prec g$ is always well defined.

▷ Resonant product $f \circ g$ well defined only if positive sum of regularities.

▷ φ be a *stationary* solution to

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a+3b)\varphi + \varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

▷ *Ansatz* $\varphi = X + \eta$ where $(\partial_t - \Delta_\varepsilon + m^2)\underbrace{X}_{-1/2-\kappa} = \underbrace{\xi}_{-5/2-\kappa}$ (stationary) gives

$$(\partial_t - \Delta_\varepsilon + m^2)\eta + 3b\varphi + \underbrace{[\![X^3]\!]}_{-3/2-\kappa} + 3\eta\underbrace{[\![X^2]\!]}_{-1-\kappa} + 3\eta^2\underbrace{X}_{-1/2-\kappa} + \eta^3 = 0$$

- Instead of removing X^{Ψ} where $(\partial_t - \Delta_\varepsilon + m^2)X^{\Psi} = -[\![X^3]\!]$
- Let Y solve $(\partial_t - \Delta_\varepsilon + m^2)Y = -[\![X^3]\!] - 3(\Delta_{>L}[\![X^2]\!]) \succ Y$ (via fixed point)
- *Define* $\varphi = X + Y + \phi$ to have

$$(\partial_t - \Delta_\varepsilon + m^2)\phi + \phi^3 = -3[\![X^2]\!] \succ \phi - 3[\![X^2]\!] \circ \phi + \text{better (after renormalization)}$$

$$\frac{1}{2}\partial_t \|\phi\|_{L^{2,\varepsilon}}^2 + \|\phi\|_{L^{4,\varepsilon}}^4 + \langle \phi, (m^2 - \Delta_\varepsilon) \underbrace{\phi}_{1-\kappa} \rangle_\varepsilon$$

$$= \langle \phi, \underbrace{-3[\![X^2]\!] \succ \phi}_{-1-\kappa} \rangle_\varepsilon + \langle \phi, -3\underbrace{[\![X^2]\!]}_{-1-\kappa} \circ \phi \rangle_\varepsilon + \langle \phi, \text{better (after renormalization)} \rangle_\varepsilon$$

▷ *approximate duality*

$$\langle \phi, -3[\![X^2]\!] \circ \phi \rangle_\varepsilon - \langle -3[\![X^2]\!] \succ \phi, \phi \rangle_\varepsilon =: D(\phi, -3[\![X^2]\!], \phi)$$

bounded if the sum of the regularities of $\phi, -3[\![X^2]\!], \phi$ positive!

▷ combine with the Laplace term

$$\langle \phi, (m^2 - \Delta_\varepsilon)\phi + 2 \cdot 3[\![X^2]\!] \succ \phi \rangle_\varepsilon$$

▷ complete the square using *elliptic paracontrolled Ansatz* (ψ is more regular than ϕ)

$$(m^2 - \Delta_\varepsilon)\psi := (m^2 - \Delta_\varepsilon)\phi + 3[\![X^2]\!] \succ \phi$$

Weighted energy estimates

- include a polynomial weight $\rho(x) = (1 + |x|^2)^{-\theta/2} \in L^4$ ($=$ test by $\rho^4\phi$ instead of ϕ)
- denote $\mathbb{X}_{M,\varepsilon} = (X_{M,\varepsilon}, [\![X^2_{M,\varepsilon}]\!], X^{\Psi}_{M,\varepsilon}, \ldots)$
- uniformly in M, ε:

$$\frac{1}{2}\partial_t \|\rho^2 \phi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho\phi_{M,\varepsilon}\|^4_{L^{4,\varepsilon}} + \|\rho^2\phi_{M,\varepsilon}\|^2_{H^{1-2\kappa,\varepsilon}} + \|\rho^2\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho^2\nabla_\varepsilon\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}}$$

$$\leqslant (|\log t| + 1) Q_\rho(\mathbb{X}_{M,\varepsilon}).$$

- the resonant product $[\![X^2]\!] \circ \phi$ not controlled; $[\![X^2]\!] \circ \psi$ also not
- analogy with PDE weak solutions (equation interpreted in a suitable duality sense)

Tightness

▷ Recall

- $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + Y_{M,\varepsilon} + \phi_{M,\varepsilon}$ is stationary with law $\nu_{M,\varepsilon}$
- $X_{M,\varepsilon}$ stationary, $Y_{M,\varepsilon}$ not stationary $\Rightarrow$ $\phi_{M,\varepsilon}$ not stationary

▷ Alternative *stationary* decomposition

$$\varphi_{M,\varepsilon} = X_{M,\varepsilon} + X^{\Psi}_{M,\varepsilon} + \zeta_{M,\varepsilon}$$

Theorem

- *The family of joint laws of* $(\varphi_{M,\varepsilon}, \mathbb{X}_{M,\varepsilon})$ *evaluated at some* $t \geqslant 0$ *is tight.*
- *Any limit measure* μ *satisfies for all* $p \in [1, \infty)$

$$\mathbb{E}_\mu \|\varphi\|^{2p}_{H^{-1/2-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{2p}_{L^2(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{2}_{H^{1-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{4}_{B^0_{4,\infty}(\rho)} < \infty.$$

- $\mathrm{Law}_\mu(\varphi_t)$ *is Non-Gaussian, OS positive, translation invariant (missing rotations).*

Integration by parts formula

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi) \propto \exp\left\{-2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2}|\varphi|^2 + \frac{1}{4}|\varphi|^4\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\varphi(x)$$

- F a cylinder functional on $S'(\Lambda_{M,\varepsilon})$: $F(\varphi) = \Phi(\varphi(f_1), \ldots, \varphi(f_n))$
- (finite dimensional) integration by parts gives

$$\int \mathrm{D}F(\varphi)\nu_{M,\varepsilon}(\mathrm{d}\varphi) = 2\int F(\varphi)[\varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + (m^2 - \Delta_\varepsilon)\varphi]\nu_{M,\varepsilon}(\mathrm{d}\varphi).$$

To pass to the limit:

- use the stationary decomposition $\varphi = X + X^{\Upsilon} + \zeta$
- φ^3 is problematic
  - $[\![X^2]\!] \circ \zeta$ – not well-defined based on the energy estimates so far
  - If ρ is the Gaussian free field then $[\![\rho^3]\!]$ exists only as an *Hida distribution*
  - $[\![X^3]\!]$ is a space-time distribution

- Let $h: \mathbb{R} \to \mathbb{R}$ smooth with $\operatorname{supp} h \subset \mathbb{R}_+$ and $\int_{\mathbb{R}} h \, \mathrm{d}t = 1$
- Let $[\![\varphi^3]\!] := \varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi$ we get

$$\int F(\varphi)[\![\varphi^3]\!]\nu_{M,\varepsilon}(\mathrm{d}\varphi) = \mathbb{E}[F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!]] = \mathbb{E}\left[\int_{\mathbb{R}} h(t)F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!]\,\mathrm{d}t\right]$$

Theorem

$$\int \mathrm{D}F(\varphi)\nu(\mathrm{d}\varphi) = 2\int F(\varphi)[(m^2 - \Delta)\varphi]\nu(\mathrm{d}\varphi) + 2J(F),$$

$$J(F) := \mathbb{E}\left[\int_{\mathbb{R}} h(t)F(\varphi(t))[\![\varphi^3]\!](t)\,\mathrm{d}t\right] "=" \int F(\varphi)[\![\varphi^3]\!]\nu(\mathrm{d}\varphi)$$

$$[\![\varphi^3]\!] = [\![X^3]\!] + 3[\![X^2]\!] \succ (-X^{\Psi} + \zeta) + 3[\![X^2]\!] \prec (-X^{\Psi} + \zeta) + \cdots.$$

▷ operator product expansion, Schwinger–Dyson equations

Two

▷ Regularization $\phi_T = \rho_T * \phi$ with $\rho_T \to \delta$ as $T \to \infty$:

$$\nu^{\lambda}_{\Lambda,T}(\mathrm{d}\phi) = \frac{e^{-\lambda V_T(\phi_T)}}{\mathcal{Z}_T}\mu(\mathrm{d}\phi), \qquad V_T(\phi) = \int_{\Lambda}(\phi^4 - a_T\phi^2 - b_T)\mathrm{d}x \geqslant -C_T > -\infty.$$

▷ As $T \to \infty$ fluctuations at different scales adds up independently into $(\phi_T)_T$.

▷ Wilson ('83) Polchinski ('84) Brydges, Kennedy ('87) Brydges, Dimock, Hurd ('95) Brydges, Slade, P.K.Mitter ('14)

▷ **Aim.** Present a new proof of existence of the limit $\nu_T \to \nu$ at fixed Λ and a *description* of the limit measure ν as a *variational problem* via a stochastic approach

$\mathbb{P}, \mathbb{E}$ Wiener measure, X canonical process.

Theorem

(Boué–Dupuis) *We have the variational representation*

$$-\log \mathbb{E}[e^{-F(X)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[F\left(X + \int_0^{\cdot} u_s \mathrm{d}s \right) + \frac{1}{2} \int_0^{\infty} |u_s|^2 \mathrm{d}s \right].$$

▷ Control problem (non–Markovian in general). Useful to get estimates and large deviations.

▷ The controlled process $X + \int_0^{\cdot} u_s \mathrm{d}s$ features explicitly the "free" part X and more regular drift part, similar to solutions to SDEs.

▷ Boué–Dupuis ('98), X. Zhang ('09), Lehec ('13), Üstünel ('14).

An example

Let $F(X) \geqslant 0$ be Lipshitz, i.e.

$$|F(X+I(u))-F(X)| \leqslant L\|I(u)\|_{L^\infty([0,1])} \leqslant L\int_0^1 |u_s|\mathrm{d}s$$

Then

$$\log \mathbb{E}[e^{\lambda F(X)}] = \sup_u \mathbb{E}_{\mathbb{P}}\left[\lambda F(X+I(u)) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

$$\leqslant \mathbb{E}_{\mathbb{P}}\left[\lambda F(X) + \lambda L\, \|I(u)\|_{L^\infty} - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

$$\leqslant \mathbb{E}_{\mathbb{P}}[\lambda F(X) + \underbrace{\frac{1}{2}\int_0^1 (2\lambda L|u_s| - |u_s|^2)\mathrm{d}s}_{\leqslant -\frac{1}{2}\lambda^2 L^2}] \leqslant \mathbb{E}_{\mathbb{P}}[\lambda F(X)] - \frac{1}{2}\lambda^2 L^2.$$

We conclude that F has Gaussian tails. The only additional information needed is $\mathbb{E}_{\mathbb{P}}[|F(X)|] < +\infty$. L can be random, i.e. $L = L(X)$.

Enters "time"

▷ Fix $\Lambda = \mathbb{T}^3$. Let X be a cylindrical Brownian motion on $L^2(\Lambda)$ and

$$Y_t = \int_0^t \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} \mathrm{d}X_s, \qquad \int_0^t \sigma_s(\mathrm{D})^2 \mathrm{d}s = \rho_t(\mathrm{D})^2,$$

with $\mathrm{D} = |-\Delta|^{1/2}$ $\langle \mathrm{D} \rangle = (1 + \mathrm{D}^2)^{1/2}$, $\rho_t(\mathrm{D}) = \rho(\mathrm{D}/t)$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ smooth, compactly supported and with $\rho(0) = 1$. Then

$$\mathbb{E}_{\mathbb{P}}[Y_T(f) Y_S(g)] = \int_0^{T \wedge S} \left\langle \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} f, \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} g \right\rangle \mathrm{d}s = \left\langle f, \frac{\rho_{T \wedge S}(\mathrm{D})^2}{\langle \mathrm{D} \rangle^2} g \right\rangle,$$

- Y_∞ is a Gaussian free field (massive)
- $Y_T \sim \rho_T * Y_\infty \sim \rho_T * \phi$
- $(Y_t)_t$ is a martingale

Variational problem for the pressure

Boué–Dupuis formula:

$$-\log \mathcal{Z}_T = -\log \mathbb{E}[e^{-\lambda V_T(Y_T(X))}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\left[\lambda V_T(Y_T + Z_T) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s\right]$$

with

$$Y_T = \int_0^T \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} \mathrm{d}X_s, \qquad Z_t = I_t(u) := \int_0^t \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} u_s \mathrm{d}s.$$

▷ *Regularity estimate*

$$\sup_{0 \leqslant t \leqslant T} \|I_t(v)\|_{H^1}^2 \lesssim \int_0^T \|v_s\|_{L^2}^2 \mathrm{d}s.$$

▷ When $d = 2$ we can choose the renomalization constants such that

$$\Theta_T(u) := \lambda V_T(Y_T + Z_T) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s = \Psi_T(u) + \Phi_T(u)$$

$$\Psi_T(u) := \lambda \int_\Lambda [\![Y_T]\!]^4 + 4\lambda \int_\Lambda [\![Y_T^3]\!] Z_T + 6\lambda \int_\Lambda [\![Y_T^2]\!] Z_T^2 + 4\lambda \int_\Lambda [\![Y_T]\!] Z_T^3$$

$$\Phi_T(u) := \underbrace{\lambda \int_\Lambda Z_T^4 + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s}_{\text{good terms}}$$

where $[\![Y_T^k]\!]$ are Wick polynomials of the (smooth) Gaussian field $(Y_T)_T$. In particular $T \mapsto [\![Y_T^k]\!]$ is a martingale.

▷ Standard estimates show that $[\![Y_T^k]\!] \in C([0, \infty], \mathcal{C}^{-\kappa}(\Lambda))$ almost surely with $L^p(\mathbb{P})$ norms for all $p \geqslant 1$ and $\kappa < 0$. Here $\mathcal{C}^\alpha(\Lambda) = B_{\infty,\infty}^\alpha(\Lambda)$ are Hölder–Besov spaces of regularity $\alpha \in \mathbb{R}$.

Now the game is to control the terms without sign with the good terms. Let $W_T = Y_T$.

$$\left| 4\lambda \int_\Lambda [\![W_T^3]\!] Z_T \right| \leqslant 4\lambda \|[\![W_T^3]\!]\|_{H^{-1}} \|Z_T\|_{H^1} \leqslant C(\delta, d)\lambda^2 \|[\![W_T^3]\!]\|_{H^{-1}}^2 + \delta \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s$$

$$\left| 6\lambda \int_\Lambda [\![W_T^2]\!] Z_T^2 \right| \leqslant \frac{C^2 \lambda^3}{2\delta} \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}^4 + \delta \left(\|Z_T\|_{W^{1,2}}^2 + \lambda \|Z_T\|_{L^4}^4 \right)$$

$$\left| 4\lambda \int_\Lambda W_T Z_T^3 \right| \leqslant C E(\lambda) \|W_T\|_{W^{-1/2-\varepsilon,p}}^K + \delta \left(\|Z_T\|_{W^{1,2}}^2 + \lambda \|Z_T\|_{L^4}^4 \right)$$

Therefore

$$-K_T + (1-\delta)\Phi_T(u) \leqslant \mathbb{E}[\Psi_T(u) + \Phi_T(u)] \leqslant K_T + (1+\delta)\Phi_T(u),$$

which implies

$$\sup_T |\log \mathcal{Z}_T| = \sup_T \left| \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}[\Psi_T(u) + \Phi_T(u)] \right| \lesssim O(\lambda^2).$$

▷ In three dimensions W_∞ is more irregular and as a consequence we get uniform estimates for the Wick powers only in the following spaces

$$[\![W_T]\!] \in \mathcal{C}^{-1/2-\kappa}, [\![W_T^2]\!] \in \mathcal{C}^{-1-\kappa},$$

and $[\![W_T^3]\!]$ does not even converge as a distribution.

▷ As a consequence we cannot hope to control the term $\int_\Lambda [\![W_T^3]\!] Z_T$, and $\int_\Lambda [\![W_T^2]\!] Z_T^2$ as we did in two dimensions. We only have control of Z_T in H^1 and L^4.

▷ By perturbative considerations one expects further divergences (beyond Wick ordering) therefore the functional to minimize is now

$$\mathbb{E}\left[\lambda \int_\Lambda \mathbb{W}_T^3 Z_T + \frac{\lambda}{2} \int_\Lambda \mathbb{W}_T^2 Z_T^2 + 4\lambda \int_\Lambda W_T Z_T^3\right]$$

$$-\mathbb{E}\left[2\gamma_T \int_\Lambda W_T Z_T + \gamma_T \int_\Lambda Z_T^2\right] + \mathbb{E}\left[\lambda \int_\Lambda Z_T^4 + \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s\right].$$

where we introduced the convenient notations: $\mathbb{W}_t^3 := 4[\![W_t^3]\!]$, $\mathbb{W}_t^2 := 12[\![W_t^2]\!]$.

▷ We aim to "complete the square" in order to eliminate the terms which we cannot control. So we control the system which a drift of the form

$$u_s = -\lambda J_s(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) + w_s$$

$$\dot{Z}_s = J_s u_s = -\lambda J_s^2(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) + \dot{K}_s$$

where w is a free control and $J_s = \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D})$.

▷ *Paraproducts.* $fg = f \prec g + f \circ g + f \succ g$. (Bony, Meyer ('80))

▷ The cost of such a drift is

$$\frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s = \frac{\lambda^2}{2}\int_0^T \int_\Lambda (J_s(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s))^2 \mathrm{d}s$$

$$-\lambda \int_0^T \int_\Lambda (\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s)\dot{Z}_s \mathrm{d}s + \frac{1}{2}\int_0^T \|w_s\|^2 \mathrm{d}s$$

▷ Integration by parts in the time variable allows to transform the mixed terms in this cost to

$$-\lambda \int_0^T \int_\Lambda (\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) \dot{Z}_s \mathrm{d}s = -\lambda \int_\Lambda (\mathbb{W}_T^3 + \mathbb{W}_T^2 \succ Z_T) Z_T$$

$$+\lambda \int_0^T \int_\Lambda \left(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ \dot{Z}_s\right) Z_s \mathrm{d}s + \text{martingale}$$

which after some analysis will cancel the terms

$$\lambda \int_\Lambda (\mathbb{W}_T^3 Z_T + \mathbb{W}_T^2 Z_T^2)$$

modulo some nice remainder.

▷ The quadratic term generated by the new cost looks like (again after some integration by parts)

$$\frac{\lambda^2}{2}\int_0^T\int_\Lambda (J_s(\mathbb{W}_s^3+\mathbb{W}_s^2\succ Z_s))^2 \mathrm{d}s = \frac{\lambda^2}{2}\int_0^T\int_\Lambda (J_s(\mathbb{W}_s^3))^2 \mathrm{d}s$$

$$+\frac{\lambda^2}{2}\int_0^T\int_\Lambda [(J_s(\mathbb{W}_s^2\succ Z_s))^2 - 2\dot{\gamma}_s Z_s^2]\mathrm{d}s$$

$$+\lambda^2\int_0^T\int_\Lambda [(J_s(\mathbb{W}_s^3))(J_s(\mathbb{W}_s^2\succ Z_s)) - 2\dot{\gamma}_s W_s Z_s]\mathrm{d}s + \lambda^2\int_0^T\int_\Lambda \dot{\gamma}_s[(Z_s)^2+2W_sZ_s]\mathrm{d}s$$

where we have introduced an abitrary function $(\gamma_s)_s$. In this expression now the first term is divergent but independend of the control, the two middle terms can be shown to be finite provided the counterterm γ is chosen appropriately and finally, the last term is compensated by

$$2\gamma_T\int_\Lambda W_TZ_T + \gamma_T\int_\Lambda Z_T^2.$$

Let us see how does it work for

$$A = \frac{\lambda^2}{2}\int_0^T\int_\Lambda [(J_s(\mathbb{W}_s^2 \succ Z_s))^2 - 2\dot{\gamma}_s Z_s^2]\mathrm{d}s.$$

▷ *Commutator lemma.* $J_s\mathbb{W}_s^2 \in \mathcal{C}^{-\kappa}$ and $Z_s \in H^{1/2-\kappa}$

$$\int_\Lambda (J_s(\mathbb{W}_s^2 \succ Z_s))^2 = \int_\Lambda (J_s(\mathbb{W}_s^2 \succ Z_s)) \circ (J_s(\mathbb{W}_s^2 \succ Z_s))$$

$$\simeq \int_\Lambda (J_s\mathbb{W}_s^2) \circ (J_s\mathbb{W}_s^2) Z_s^2 + \int_\Lambda \underbrace{C(J_s\mathbb{W}_s^2, J_s\mathbb{W}_s^2, Z_s)}_{\in B_{1,1}^{0+}}$$

Therefore

$$A = \frac{\lambda^2}{2}\int_0^T\int_\Lambda \underbrace{[(J_s\mathbb{W}_s^2) \circ (J_s\mathbb{W}_s^2) - 2\dot{\gamma}_s]}_{\mathbb{W}^{2\diamond 2} \in \mathcal{C}^{-\kappa}} Z_s^2 \mathrm{d}s$$

Similarly

$$\mathbb{W}_s^{2\diamond 3} := (J_s\mathbb{W}_s^3) \circ (J_s\mathbb{W}_s^2) - 2\dot{\gamma}_s W_s \in \mathcal{C}^{-1/2-\kappa}$$

Renormalized control problem

$$\mathbb{W}_T := (W_T, \mathbb{W}_T^2, \mathbb{W}_T^3, \mathbb{W}^{2\diamond 2}, \mathbb{W}_s^{2\diamond 3}) \in \mathfrak{W} = \mathcal{C}^{-1/2-\kappa} \times \mathcal{C}^{-1-\kappa} \times \mathcal{C}^{-3/2-\kappa} \times \mathcal{C}^{-\kappa} \times \mathcal{C}^{-1/2-\kappa}$$

▷ We have shown that

$$\begin{aligned} -\log \mathcal{Z}_T(\lambda) &= \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda V_T(Y_T + I_T(u)) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s\right] \\ &= \inf_{l \in \mathbb{H}_a} \mathbb{E}\left[E_T(Z(l), K(l)) + \lambda \|Z_T(l)\|_{L^4}^4 + \frac{1}{2}\int_0^\infty \|l_s\|_{L^2}^2 \mathrm{d}s\right] \\ &=: \inf_{l \in \mathbb{H}_a} \tilde{F}_T(l) \end{aligned}$$

where $Z = Z(l) \in H^{1/2-\varepsilon}$ and $K = K(l) \in H^{1-\varepsilon}$ solve the integral equations

$$Z_t(l) = -\lambda \int_0^t J_s^2 \mathbb{W}_s^3 \mathrm{d}s + K_t(l), \qquad K_t(l) = -\lambda \int_0^t J_s^2 (\mathbb{W}_s^2 \succ Z_s(l)) \mathrm{d}s + \int_0^t J_s l_s \mathrm{d}s.$$

▷ Estimates of the form

$$|E_T(Z(l), K(l))| \leqslant C \|\mathcal{W}_T\|_S^K + \delta \|Z_T(l)\|_{L^4}^4 + \delta \|K(l)\|_{H^{1-\varepsilon}}^2.$$

Variational setting. (X, l) canonical variables on $C([0,\infty],\mathfrak{W}) \times L^2_w([0,\infty)\times\Lambda)$

$$\mathcal{X} := \{\mu \in P(C([0,\infty],S) \times L^2_w([0,\infty)\times\Lambda)) \mid \mu = \mathrm{Law}_{\mathbb{P}}(W, u) \text{ for some } u \in \mathbb{H}_a\}.$$

▷ Then

$$-\log \mathcal{Z}_T(\lambda) = \inf_{\mu\in\mathcal{X}} F_T(\mu) = \inf_{\mu\in\bar{\mathcal{X}}} F_T(\mu)$$

where, for $T \in [0,\infty]$,

$$F_T(\mu) := \mathbb{E}_\mu\left[E_T(Z(l), K(l)) + \lambda\|Z_T(l)\|^4_{L^4(\Lambda)} + \frac{1}{2}\int_0^\infty \|l_s\|^2_{L^2}\mathrm{d}s\right].$$

▷ The choice of $\mathcal{X}$ is dictated by the fact that the family $(F_T)_T$ is now equicoercive, namely that there exists a compact $\mathcal{K} \subseteq \mathcal{X}$ such that

$$\inf_{x\in\mathcal{K}} F_T(x) = \inf_{x\in\mathcal{X}} F_T(x), \qquad \text{for all } T.$$

▷ Finally using the continuity of the map E and the lower semicontinuity of the L^4 and entropy terms we establish

$$\Gamma\text{-}\lim_{T\to\infty} F_T = F_\infty.$$

Namely that

- For every sequence $\mu^T \to \mu$ in $\bar{\mathcal{X}}$:

$$F_\infty(\mu) \leqslant \liminf_T F_T(\mu^T),$$

- For every $\mu \in \bar{\mathcal{X}}$ there exists a sequence $\mu^T \to \mu$ in $\bar{\mathcal{X}}$ such that

$$F_\infty(\mu) \geqslant \limsup_T F_T(\mu^T).$$

▷ A consequence of Γ–convergence is the convergence of minima:

$$\lim_{T\to\infty} (-\log \mathcal{Z}_T) = \lim_{T\to\infty} \inf_{\bar{\mathcal{X}}} F_T = \min_{\bar{\mathcal{X}}} F_\infty.$$

We obtain *explicit* variational formula for the limiting functional

$$-\log \mathcal{Z}_\infty(f) = \inf_{l \in \mathbb{H}_a} \mathbb{E}\left[-\int_\Lambda f Z_\infty(l) + E_\infty(Z(l), K(l)) + \lambda \|Z_\infty(l)\|^4_{L^4(\Lambda)} + \frac{1}{2}\int_0^\infty \|l_s\|^2_{L^2} \mathrm{d}s \right]$$

defined for all $f \in \mathcal{S}(\Lambda)$ with

$$\mathcal{Z}_\infty(f) = \lim_T \mathcal{Z}_T(f), \qquad \mathcal{Z}_T(f) = \mathcal{Z}_T \mathbb{E}_\nu[e^{\int_\Lambda f\phi_T}] = \int e^{\int_\Lambda f\phi_T - \lambda V_T(\phi_T)} \mu(\mathrm{d}\phi).$$

▷ The interest of this formula lies in the fact that the Φ^4_3 measure is not absolutely continuous wrt. the Gaussian free field, so an explicit description was lacking.

▷ The variational formula seems a promising way to extract informations from this measure. E.g. large deviations, weak universality, pathwise properties, etc...

$$E_\infty(Z(l), K(l)) = E_\infty(Z, K) = \sum_{i=1}^{6} \Upsilon_\infty^{(i)}$$

with

$$\Upsilon_\infty^{(1)} := \frac{\lambda}{2}\kappa^{(2)}(\mathbb{W}_\infty^2, K_\infty, K_\infty) + \frac{\lambda}{2}⨍(\mathbb{W}_\infty^2 \prec K_\infty)K_\infty - \lambda^2⨍(\mathbb{W}_\infty^2 \prec \mathbb{W}_\infty^{[3]})K_\infty$$

$$\Upsilon_\infty^{(2)} = 0$$

$$\Upsilon_\infty^{(3)} := \lambda\int_0^\infty ⨍(\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t \mathrm{d}t$$

$$\Upsilon_\infty^{(4)} := 4\lambda⨍\mathbb{W}_\infty K_\infty^3 + 12\lambda^2⨍(\mathbb{W}_\infty \mathbb{W}_\infty^{[3]})K_\infty^2 + 12\lambda^3⨍\mathbb{W}_\infty(\mathbb{W}_\infty^{[3]})^2 K_\infty$$

$$\Upsilon_\infty^{(5)} := -2\lambda^2\int_0^\infty ⨍\gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t$$

$$\Upsilon_\infty^{(6)} := -\lambda^2⨍\mathbb{W}_\infty^{2\diamond[3]}K_\infty - \lambda^2\int_0^T ⨍\mathbb{W}_t^{2\diamond 2}(Z_t^\flat)^2\mathrm{d}t + \frac{\lambda^2}{2}\int_0^\infty \kappa_t^{(1)}(\mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)$$

and

$$|\gamma_t| + \langle t\rangle|\dot{\gamma}_t| \lesssim \lambda^2 \log\langle t\rangle.$$

Thank you.