Grassmannian stochastic analysis

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Motivation: Stochastic quantisation of Fermionic Euclidean QFT

Outline

- What is stochasic quantisation (SQ).
- Fermionic EQFT → Grassmann algebras/integration (GA)
- Analysis/Probability on GA? Non-commutative probability setting.
- Grassmann Gaussian r.v./Brownian motion
- Stochastic differential equations and invariant „states“
- Random fields, Stochastic PDEs, large volume limit in Yukawa-type models.

☞ **Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions** | joint work with Sergio Albeverio, Luigi Borasi, Francesco C. De Vecchi. ArXiv:2004.09637

Stochastic quantisation

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SQ describes a **probability measure** (law of a RV) via an **equation** (for the RV). Basic example: Langevin dynamics (reversible Markovian dynamics wrt. to a prob. measure).

$$\mathrm{d}\psi = -\nabla V(\psi)\mathrm{d}t + \mathrm{d}B_t, \qquad t \geqslant 0.$$

$(\psi(t))_t$ is a stochastic process e.g. with values in $\mathbb{R}^N$. $V: \mathbb{R}^N \to \mathbb{R}$ a (potential) function. Invariant Gibbs measure μ:

$$\mathbb{E}[F(\psi(t))] \to \int_{\mathbb{R}^N} F(x) \frac{e^{-2V(x)}\mathrm{d}x}{Z_V}.$$

EQFT: Wick rotation of QFT. $t \to \tau = it$ $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ Euclidean space. Wightman functions → Schwinger functions.

Fermions: quantum particles satisfying Fermi–Dirac statistics (i.e. living in the antisymmetric tensor of one-particle states).

Euclidean Fermions

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▷ K. Osterwalder and R. Schrader. Euclidean Fermi fields and a Feynman-Kac formula for Boson-Fermions models. *Helvetica Physica Acta*, 46:277–302, 1973.

▷ $\Psi, \Psi^* \rightarrow \psi, \bar{\psi}$. Schwinger functions are given by

$$\langle O(\psi,\bar{\psi})\rangle = \frac{\int d\psi d\bar{\psi}\, O(\psi,\bar{\psi}) e^{-S_E(\psi,\bar{\psi})}}{\int d\psi d\bar{\psi} e^{-S_E(\psi,\bar{\psi})}} = \frac{\langle O(\psi,\bar{\psi}) e^{-V(\psi,\bar{\psi})}\rangle_C}{\langle e^{-V(\psi,\bar{\psi})}\rangle_C}$$

$$S_E(\psi,\bar{\psi}) = \frac{1}{2}(\psi, C\,\bar{\psi}) + V(\psi,\bar{\psi})$$

Berezin integral on a GA $\Lambda = \mathrm{GA}(\psi,\bar{\psi})$. the fields $\psi,\bar{\psi}$ are the generators of a Grassmann algebra: $\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$ ($\psi_\alpha^2 = 0$).

▷ Under $\langle\cdot\rangle_C$ the variables $\psi,\bar{\psi}$ are “Gaussian”.

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▷ A non-commutative probability space $(\mathscr{A}, \omega)$ is given by a C^*-algebra $\mathscr{A}$ and a *state* ω, a linear normalized positive functional on $\mathscr{A}$ (i.e. $\omega(aa^*) \geqslant 0$).

▷ For us: random variables with values in a Grassmann algebra Λ are algebra *homomorphisms*

$$\mathrm{Hom}(\Lambda, \mathscr{A})$$

(Not canonical!)

▷ Classical probability: $\mathscr{A} = L^\infty(\Omega; \mathbb{C})$, $\omega(a) = \int_\Omega a(\omega)\mathbb{P}(\mathrm{d}\omega)$, $\mathscr{M}$ manifold. $\mathscr{M}$ valued r.v. X:

$$X \in \mathrm{Hom}(L^\infty(\mathscr{M}), \mathscr{A}) \qquad f \in L^\infty(\mathscr{M}) \to X(f) \in \mathscr{A}, \qquad X(fg) = X(f)X(g), \quad X(f^*) = X(f)^*.$$

☞ L. Accardi, A. Frigerio, and J. T. Lewis. Quantum stochastic processes. *Kyoto University. Research Institute for Mathematical Sciences. Publications*, 18(1):97–133, 1982. 10.2977/prims/1195184017

Let V be a vector space and ΛV the GA generated by V:

$$v, \quad vw := v \otimes w - w \otimes v \, (v \neq w), \quad vv' \, v'', \ldots .$$

$\dim(\Lambda V) = 2^{\dim(V)}$.

A V-Grassmann random variable $\Psi \in \mathcal{G}(V) = \mathrm{Hom}(\Lambda V, \mathcal{A})$ is an algebra homomorphism from the Grassmann algebra ΛV into $\mathcal{A}$.

The *law* of $\Psi \in \mathcal{G}(V)$ is the family of its moments $\omega^{\Psi}(F) := \omega(\Psi(F))$ for all $F \in \Lambda V$, also represented by the linear functional $\omega^{\Psi} : \Lambda V \to \mathbb{R}$.

$$F = \sum_A F_A \, v_A \in \Lambda V, \qquad F(\Psi) := \Psi(F) = \sum_A F_A \, \Psi^A, \qquad \Psi^A := \Psi(v_A)$$

$$\Psi^{\alpha} \Psi^{\beta} = -\Psi^{\beta} \Psi^{\alpha}, \qquad \text{where } \Psi^{\alpha} = \Psi(v_{\alpha}).$$

Two GRV $X \in \mathscr{G}(V)$ and $Y \in \mathscr{G}(W)$

▷ we say that they are ***compatible*** if the linear map $Z: V \oplus W \to \mathscr{A}$ given by $Z(v) = X(v)$ if $v \in V$ and $Z(w) = Y(w)$ if $w \in W$, extends to an homomorphism $Z: \Lambda(V \oplus W) \to \mathscr{A}$.

(related to kinematic independence)

▷ *Independence.* If $(X_1, \ldots, X_n) \in \mathscr{G}(V_1 \oplus \cdots \oplus V_n)$ are compatible Grassmann variables with values in the probability space $(\mathscr{A}, \omega)$, then we say that $X_1, \ldots, X_n$ are (tensor) independent (with respect to the state ω) if, for all $F_j \in \Lambda V_j$, we have that

$$\omega\left(\prod_{j=1}^{k} X_j(F_j)\right) = \prod_{j=1}^{k} \omega(X_j(F_j)).$$

By GNS construction and Hilbert space products (via an involution), we can always arrange two given GRV to be independent. I.E. construct a product (non-comm) probability space.

Topology on $\mathscr{G}(V)$

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$\mathscr{G}(V)$ has a natural complete metric topology given by the distance

$$d_{G(V)}(X,Y) := \|X - Y\|_{G(V)} = \sup_{v\in V, |v|_V=1} \|X(v) - Y(v)\|_A, \tag{1}$$

where $\|\cdot\|_A$ is the natural norm in the $*$-algebra A.

Analysis. The embedding of ΛV into $\mathscr{A}$ allows to use the topology of $\mathscr{A}$ to do analysis on Grassmann algebras.

Analogy. Gaussian processes in Hilbert space. Abstract Wiener space. "a convenient place where to hang our (analytic) hat on".

Right and Left derivatives

$$\partial_R, \partial_L \colon \Lambda V \to \Lambda V \otimes V$$

$$\partial_R(f_1 \cdots f_n) = \sum_{k=1}^{n} (-1)^{n-k} (f_1 \cdots \not{f_k} \cdots f_n) \otimes f_k, \qquad f_1, \ldots, f_n \in V.$$

$$\partial_L(f_1 \cdots f_n) = \sum_{k=1}^{n} (-1)^{k-1} (f_1 \cdots \not{f_k} \cdots f_n) \otimes f_k, \qquad f_1, \ldots, f_n \in V.$$

Lemma. *Let $X, Y \in \mathcal{G}(V)$ be two compatible Grassmann random variables, then*

$$G(X+Y) = G(X) + \sum_{k=1}^{n} \frac{1}{k!} m[(X \otimes Y \otimes \cdots \otimes Y)(\partial_R^k G)] + O(\| Y \|_{\mathcal{G}(V)}^{n+1}), \qquad G \in \Lambda V \tag{2}$$

[$X + Y \in \mathcal{G}(V)$ is the unique hom. such that $(X+Y)(v) = X(v) + Y(v)$ for $v \in V \subseteq \Lambda V$]

Grassmann Gaussians

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Let V be a real pre-Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and with an antisymmetric bounded operator $C: V \to V$.

▷ A (V-)Grassmann (centered) Gaussian variable with correlation C is a random variable $X \in \mathscr{G}(V)$ such that

$$\omega(X(G)X(f)) = \omega(X(\langle \partial_R G, Cf \rangle)), \qquad G \in \Lambda V, f \in V. \tag{3}$$

We also require that $\|X\|_{\mathscr{G}(V)} < \infty$, i.e that the map $X: V \to \mathscr{A}$ must be continuous with respect the topology induced on V by the pre-Hilbert product structure and the (norm) topology of $\mathscr{A}$.

▷ Wick's rule: $\omega(X(f_1) \cdots X(f_n)) = 0$ if n is odd and if $n = 2k$ is even

$$\omega(X(f_1) \cdots X(f_{2k})) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{k} \langle f_{\sigma(2i-1)}, Cf_{\sigma(2i)} \rangle_V. \tag{4}$$

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▷ V-valued white noise: $\Xi \in \mathcal{G}(L^2(\mathbb{R}_+) \otimes V)$ with correlation

$$\langle \mathbb{1}_{[0,t]} \otimes v, \mathbb{1}_{[0,s]} \otimes Cw \rangle.$$

▷ V-valued Brownian motion $B_t(v) = \Xi(\operatorname{sgn}(t)\, \mathbb{1}_{[0,t]} \otimes v)$ for $v \in V$ and extended: $B_t \in \mathcal{G}(V)$.

$$\| B_t(v) - B_s(v) \|_A \leqslant |t - s|^{1/2} \| v \|_V, \qquad t, s \geqslant 0, v \in V.$$

$B = (B_t)_{t \in \mathbb{R}_+}$ is a Gaussian process with continuous trajectories. We have that $B_0(v) = 0$,

$$\omega(B_t(v)) = 0, \qquad \omega(B_t(v) B_s(w)) = \langle v, Cw \rangle\, (t \wedge s), \qquad t, s \geqslant 0, \quad v, w \in V.$$

Note that

$$\sup_{t \in [0,T]} \| B_t \|_{G(V)} \lesssim T.$$

(Additive) SDEs

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We say that a function $\Psi_\cdot \in C^0(\mathbb{R}, \mathcal{G}(V))$ is a solution to the additive noise SDE with coefficient G and starting at $-T \in \mathbb{R}$ if, for any $t \geqslant -T$ and any $v \in V$, we have

$$\Psi_t(v) - \Psi_{-T}(v) = \int_{-T}^{t} G(\Psi_s, v)\mathrm{d}s + B_t(v) - B_{-T}(v),$$

where $\Psi_{-T}(v)$ is an element of $\mathcal{G}(V)$, $G: V \to \Lambda_{\mathrm{odd}} V$ is a linear map and the integral is the Bochner integral with respect to the norm of $\mathcal{A}$.

Local existence: by Picard iteration.

Uniqueness: easy.

Global existence? All SDE have polynomial coefficients.

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Let $V = W \oplus W$ and $C \in \mathscr{L}_a(V)$ and $G\colon V \to \Lambda V$ such that

$$C = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad v^i_\alpha \to G(v_\alpha) := (-1)^i v^i_\alpha \sum_\beta \gamma_{\alpha,\beta}(v^1_\beta\, v^2_\beta), \qquad v^1_\gamma, v^2_\gamma \in W$$

If we denote by $\Psi^{i,\alpha}_t := \Psi_t(v^i_\alpha)$ and $B^{i,\alpha}_t := B_t(v^i_\alpha)$ we obtain

$$\Psi^{i,\alpha}_t = \Psi^{i,\alpha}_{-T} + \int_{-T}^t \Psi^{i,\alpha}_s \cdot \left((-1)^i \sum_\beta \gamma_{\alpha,\beta} (\Psi^{1,\beta}_s \Psi^{2,\beta}_s) \right) \mathrm{d}s + B^{i,\alpha}_t - B^{i,\alpha}_{-T}.$$

Note that $G\colon V \to \Lambda V$

$$G(v_\alpha) = \langle C \partial_R U, v_\alpha \rangle, \qquad U = \sum_\beta \gamma_{\alpha,\beta}(v^1_\beta\, v^2_\beta)(v^1_\alpha\, v^2_\alpha) \in \Lambda V.$$

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Theorem. *There is a unique (global in time) solution to the SDE.*

Proof. Write $\Theta = \Psi - B$, then $\Theta^{\mathfrak{a}} = \Theta_t^{a_1} \cdots \Theta_t^{a_n}$ with $\mathfrak{a} = a_1 \cdots a_n$ satisfy a finite-dimensional system of **linear** equations:

$$\mathrm{d}\Theta_t^{\mathfrak{a}} = \sum_{\mathfrak{b},\mathfrak{c}} c_{\mathfrak{b},\mathfrak{c}}^{\mathfrak{a}} B_t^{\mathfrak{b}} \Theta_t^{\mathfrak{c}}.$$

Conclude by applying Grownwall inequality to

$$\sum_{\mathfrak{a}} \| \Theta_t^{\mathfrak{a}} \|_{\mathcal{A}} \leqslant \sum_{\mathfrak{a}} \| \Theta_{-T}^{\mathfrak{a}} \|_{\mathcal{A}} + |c| \int_{-T}^{t} \sum_{\mathfrak{c}} \| \Theta_s^{\mathfrak{c}} \|_{\mathcal{A}} \sum_{\mathfrak{b}} \| B_s^{\mathfrak{b}} \|_{\mathcal{A}} \mathrm{d}s.$$

□

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$C: V \to V$ is an antisymmetric linear map and

$$Q_C \in \mathcal{L}(\Lambda V \otimes V \otimes V, \Lambda V) \qquad (f \otimes v \otimes w) \mapsto Q_C(f \otimes v \otimes w) = \langle v, Cw \rangle_V \cdot f$$

$G: V \to \Lambda V$, $K = \sum_{i=1}^{N} K_i \otimes v_i \in \Lambda V \otimes V$ we denote by $K \cdot G \in \Lambda V$

$$K \cdot G = \sum_{i=1,\ldots,N} K_i G(v_i).$$

Theorem. *Let Ψ_t be the solution to the SDE then for any $H \in \Lambda V$ and $-T \leqslant s \leqslant t$ we have*

$$\omega(H(\Psi_t)) = \omega(H(\Psi_s)) + \int_s^t \omega\left(\Psi_s\left(\partial_R H \cdot G + \frac{1}{2} Q_C(\partial_R^2 H)\right)\right) \mathrm{d}s.$$

Invariant state

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If ω is a positive state, $X \in \mathcal{G}(V)$ and $U \in \Lambda V$ we define the (generally nonpositive) state

$$\omega^{X,U}(\cdot) = \omega(\cdot e^{-2U(X)})$$

Definition. *We say that* e^{-2U} *is an invariant measure of the SDE with initial condition* Ψ_{-T} *if for any* $H \in \Lambda V$ *and any* $t, s \geqslant -T$ *we have*

$$\omega^{\Psi_{-T},U}(H(\Psi_s)) = \omega^{\Psi_{-T},U}(H(\Psi_t)) = \omega(H(\Psi_{-T})e^{-2U(\Psi_{-T})}).$$

Theorem. *For any even $U \in \Lambda_{\mathrm{even}} V$, the SDE*

$$\Psi_t(v) = X(v) + \int_{-T}^{t} (\Psi_s(Av) + \Psi_s(\langle C\partial_R U, v\rangle))\mathrm{d}s + B_t(v) - B_{-T}$$

i.e. the SDE with drift

$$G(\cdot) = A\cdot + \langle C\partial_R U, \cdot\rangle,$$

where B is a Brownian motion with correlation C and X is an independent Gaussian initial condition with correlation

$$C_A := \int_0^\infty e^{A^{\mathrm{T}}s} C e^{As} \mathrm{d}s$$

has $\omega^{X,U}$ as invariant measure provided

$$A^{\mathrm{T}} C_A - C_A A = 0.$$

Stationary solution

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Take the function G with the special form

$$G(v) = Av + \lambda\tilde{G}(v)$$

where $A \in \mathcal{L}(V, V)$ is a linear map such that $A < 0$, $\tilde{G}\colon V \to \Lambda_{\mathrm{odd}} V$, and $\lambda \in \mathbb{R}$ small.

Theorem. *(stationary solution) Under the previous hypotheses, for $|\lambda|$ small enough, the following equation*

$$\Psi_t^s(v) = \lambda \int_{-\infty}^{t} \tilde{G}(\Psi_\tau^s, e^{A(t-\tau)} v)\mathrm{d}\tau + B_t^A(v)$$

where $B_t^A(v) = \Xi(\mathbb{I}_{(-\infty,t]}(\cdot) \otimes e^{A(t-\cdot)} v)$, has a unique solution.

▷ Key fact for proof : $\sup_{t\in\mathbb{R}} \|B_t^A\|_{\mathcal{G}(V)} < +\infty$.

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Theorem. *If*

$$G(\cdot) = A\cdot + \lambda\langle C\partial_R U, \cdot\rangle,$$

for $|\lambda|$ *small enough, we have that, for any* $t \in \mathbb{R}$ *and for any* $H \in \Lambda V$*,*

$$\frac{\omega^{X,U}(H(X))}{\omega^{X,U}(1)} = \frac{\omega(H(X)e^{-2U(X)})}{\omega(e^{-2U(X)})} = \omega(H(\Psi_t^s))$$

where X *is a Gaussian* V*-random variable with covariance* C_A.

Series expansion of solutions

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$$\Psi_t(v) = \Phi_t(v) + \int_0^t \Psi_s(e^{-(t-s)}F(v))\mathrm{d}s, \qquad t \geqslant 0, v \in V,$$

$$\Psi_t = \sum_\tau J_\tau(\Phi)(t) = J_\bullet(\Phi)(t) + J_{[\bullet\bullet\bullet]}(\Phi)(t) + J_{[[\bullet\bullet\bullet]\bullet\bullet]}(\Phi)(t) + \cdots + J_{[[\bullet\bullet\bullet][\bullet[\bullet\bullet[\bullet\bullet\bullet]]\bullet]\bullet]}(\Phi)(t) + \cdots$$

The series is indexed by (planar) trees τ which have branches of order 3 and where J is a multilinear integral operator such that

$$J_\bullet(\Phi)(t)^\alpha = \Phi_t(v_\alpha)$$

$$J_{[\tau_1\tau_2\tau_3]}(\Phi)(t)^\alpha = \sum_{\alpha_1,\alpha_2,\alpha_3} \int_0^t e^{-(t-s)}\lambda^\alpha_{\alpha_1,\alpha_2,\alpha_3} J_{\tau_1}(\Phi)(s)^{\alpha_1} J_{\tau_2}(\Phi)(s)^{\alpha_2} J_{\tau_3}(\Phi)(s)^{\alpha_3}\mathrm{d}s$$

where $\bullet$ denotes the simple tree and $[\tau_1,\ldots,\tau_3]$ the tree with branches $\tau_1,\ldots,\tau_3$.

Global solutions (II)

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Theorem. *There exists an increasing function* $E(t)$ *depending on* $N, |\lambda|, m$ *such that for all* $m > 0$,

$$\sup_{s \leqslant t} \| \Psi_s \| \leqslant E(t),$$

where recall that $(\Psi_t)_{t \geqslant 0}$ *is the unique solution of the equation.*

Key Lemma. *Assume* $m \geqslant 1$. *For any* $n \geqslant 1$ *and any* $t_1, \ldots, t_n \in [0, T]$, $\alpha_1, \ldots, \alpha_n \in \{1, \ldots, N\}$ *we have*

$$\| \Phi_{t_1}^{\alpha_1} \cdots \Phi_{t_n}^{\alpha_n} \| \leqslant \frac{C^{n+1} T^{n/2}}{(n!)^{1/2}}$$

where C *is a universal constant depending only on* N.

Outlook

- Convenient language for Euclidean fermions. Replace Berezin integral.
- In the paper: construction of solutions of SPDEs in the whole space driven by white noise, but with regularized interaction.

Open problems

- Ito integral? Stochastic calculus? Equations with multiplicative noise?

$$\int B_t \mathrm{d} B_t$$

  is an unbounded operator…
- Renormalization? Wick products can be unbounded operators…
- How to handle unbounded operators?

Thanks, and happy new year!