IAMP One World mathematical physics seminar · December 20th, 2022

A stochastic analysis of EQFTs

the forward-backwards equation for Grassmann measures

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[made with TeXmacs]

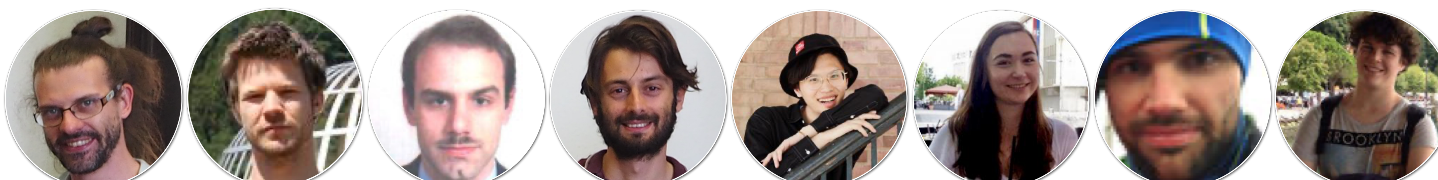
Part I · stochastic analysis & Euclidean QFTs

Part II · the FBSDE for Grassmann measures

in collaboration with:

S. Albeverio (Bonn), M. Hofmanová (Bielefeld), H. Koch (Bonn), T. Oh (Edinburgh)

Bonn/Oxford SPDE group



L. Borasi, N. Barashkov, F. de Vecchi, L. Fresta, R. Jin, S. Meyer, P. Rinaldi, M. Turra

Part I · stochastic analysis & Euclidean QFTs

Euclidean quantum fields

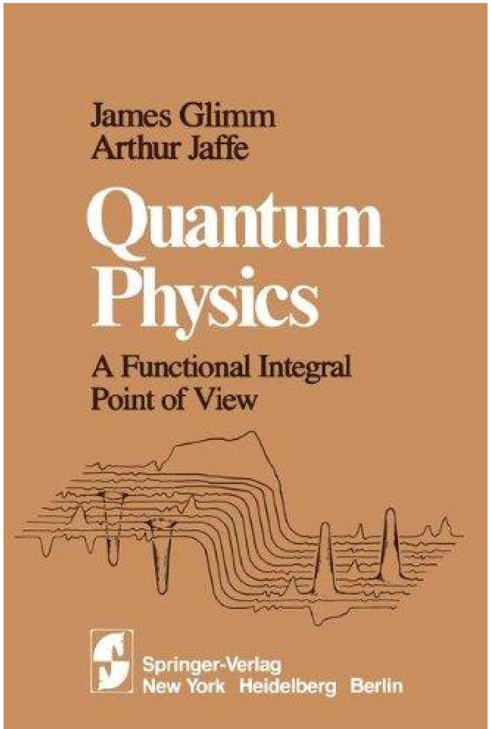
▷ Functional integral representation $\varphi: \mathbb{R}^d \to \mathbb{R}$, $d = 1, 2, 3, (4)$

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

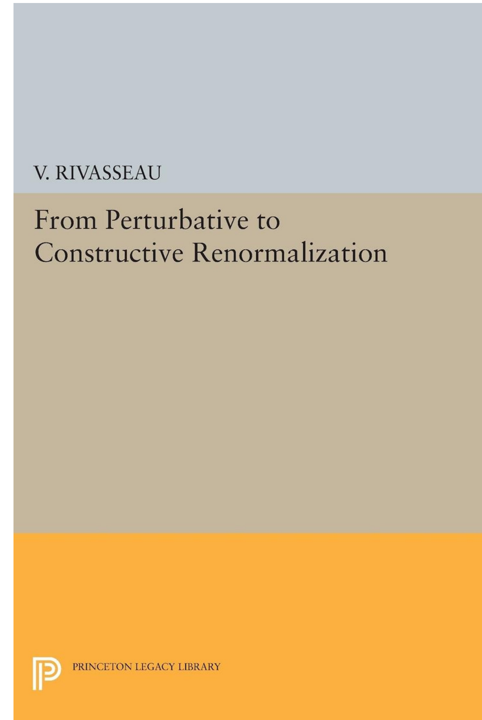
$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + V(\varphi(x))\mathrm{d}x$$

ill-defined:

- **large scale problems:** the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- **small scale problems:** sample paths are not expected to be functions, but only distributions, the quantity $V(\varphi(x))$ does not make sense.



535 pages



348 pages

other approaches

▷ (renormalized) Dyson–Schwinger equations / integration by parts formulas

$$\left\langle F(\varphi)\frac{\delta S(\varphi)}{\delta\varphi}+\frac{\delta F(\varphi)}{\delta\varphi}\right\rangle=0$$

[recent paper with M. Turra and F. de Vecchi · "A singular integration by parts formula for the exponential Euclidean QFT on the plane" · arXiv:2212.05584]

▷ Cohomological approach (Batalin–Vilkovisky) / factorisation algebras [e.g. Costello–Gwilliam]

stochastic quantisation

Parisi–Wu ('84) introduce a stationary stochastic evolution associated with the EQF

$$\partial_t \Phi(t,x) = -\frac{\delta S(\Phi(t,x))}{\delta \Phi} + \eta(t,x), \qquad t \geqslant 0, x \in \mathbb{R}^d,$$

with η space-time white noise

$$\langle \Phi(t,x_1) \cdots \Phi(t,x_n) \rangle = \frac{1}{Z} \int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(t,x_1) \cdots \varphi(t,x_n) e^{-S(\varphi)} \mathrm{d}\varphi, \qquad t \in \mathbb{R}$$

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transport interpretation: the map

$$\eta \mapsto \Phi(t,\cdot)$$

sends the Gaussian measure of the space-time white noise to the EQF measure.

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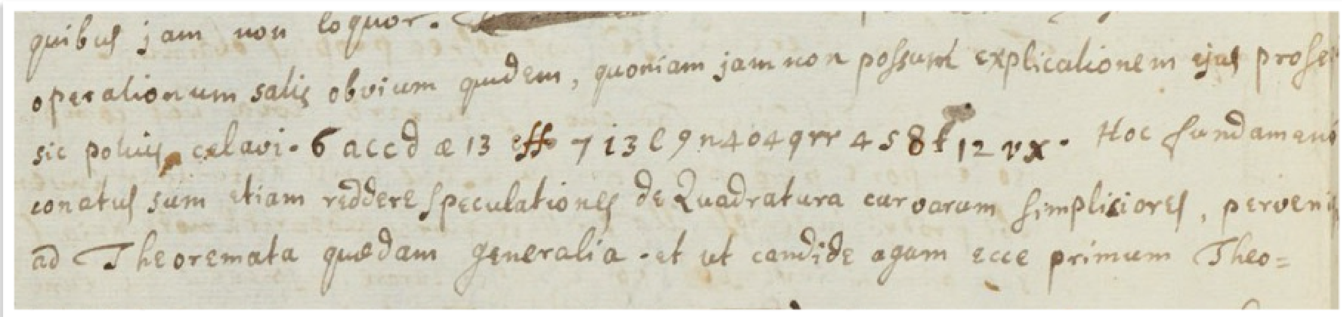
an (pre)history of stochastic quantisation (personal & partial)

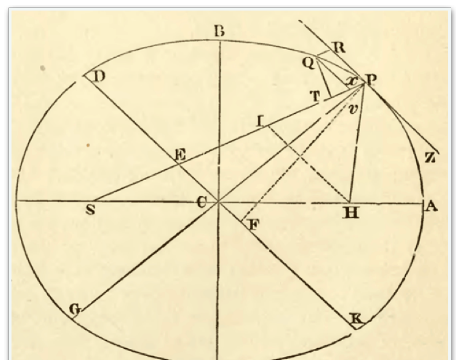
- 1984 – Parisi/Wu – SQ (for gauge theories)
- 1985 – Jona-Lasinio/Mitter – “On the stochastic quantization of field theory” (rigorous SQ for Φ^4_2 on bounded domain)
- 1988 – Damgaard/Hüffel – review book on SQ (theoretical physics)
- 1990 – Funaki – Control of correlations via SQ (smooth reversible dynamics)
- 1990–1994 – Kirillov – “Infinite-dimensional analysis and quantum theory as semimartingale calculus”, “On the reconstruction of measures from their logarithmic derivatives”, “Two mathematical problems of canonical quantization.”
- 1993 – Ignatyuk/Malyshev/Sidoravichius – “Convergence of the Stochastic Quantization Method I,II” [Grassmann variables + cluster expansion]
- 2000 – Albeverio/Kondratiev/Röckner/Tsikalenko – “A Priori Estimates for Symmetrizing Measures…” [Gibbs measures via IbP formulas]
- 2003 – Da Prato/Debussche – “Strong solutions to the stochastic quantization equations”
- 2014 – Hairer – Regularity structures, local dynamics of Φ^4_3
- 2017 – Mourrat/Weber – coming down from infinity for Φ^4_3
- 2018 – Albeverio/Kusuoka – “The invariant measure and the flow associated to Φ^4_3…”
- 2021 – Hofmanova/G. – Global space-time solutions for Φ^4_3 and verification of axioms
- 2020-2021 – Chandra/Chevyrev/Hairer/Shen – SQ for Yang–Mills 2d/3d (local theory)

what is stochastic quantisation?



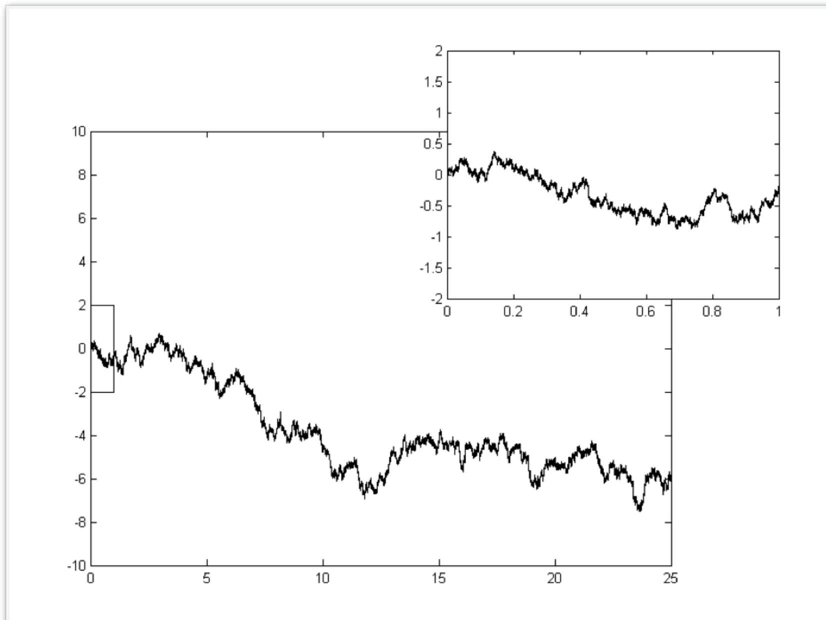
quibus jam non loquor.
operationum satis obvium quidem, quoniam jam non possum explicationem ejus prose
sic potius celavi. 6accdæ13eff7i3l9n4o4qrr4s8t12vx. Hoc fundament
conatus sum etiam reddere speculationes de Quadratura curvarum simpliciores, perveni
ad Theoremata quaedam generalia. et ut candide agam ecce primum Theo-

Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa (Newton)



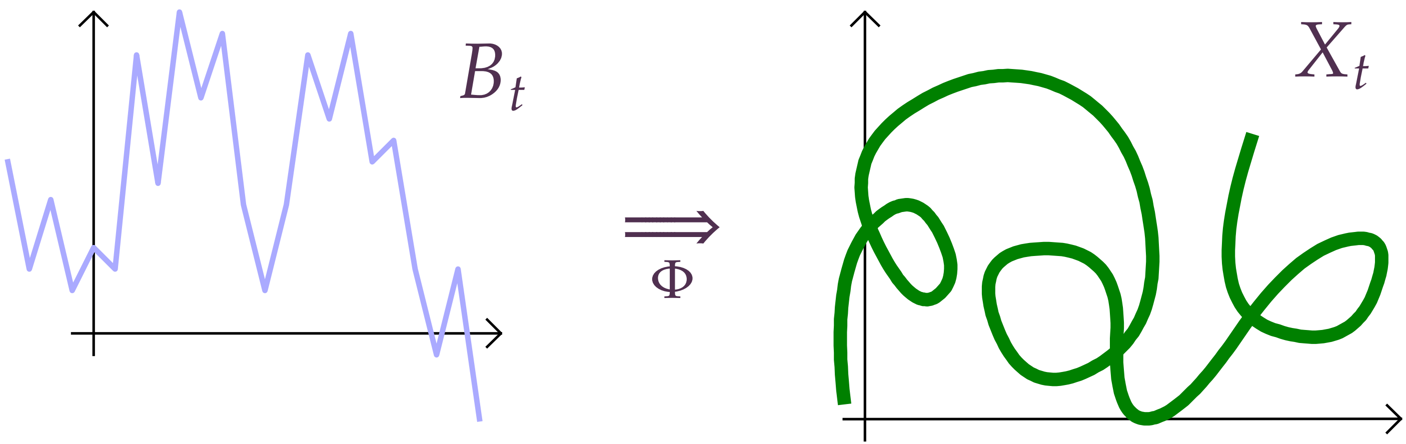
[Given an equation involving any number of fluent quantities to find the fluxions, and vice versa]

The word "*random*" comes from a French hunting term: "*randon*" designates the erratic course of the deer which zigzags trying to escape the dogs. The word also gave "*randonnée*" (hiking) in French.



Ito's idea

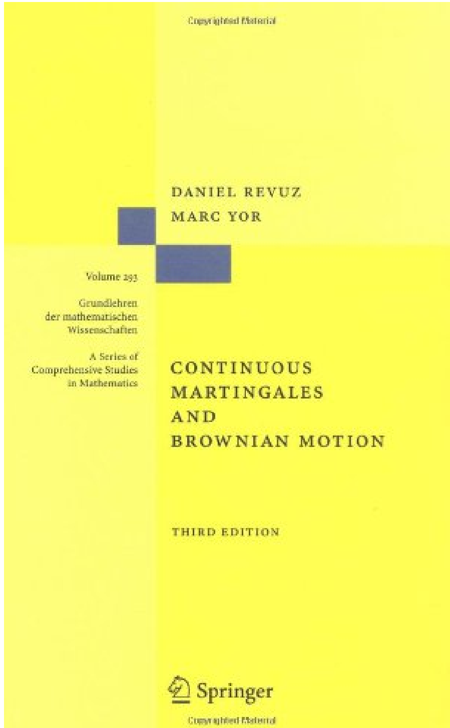
Ito arrived to his calculus while trying to understand Feller's theory of diffusions an evolution in the space of probability measures and he introduced stochastic differential equations to define a map (**the Ito map**) which send Wiener measure to the law of a diffusion.



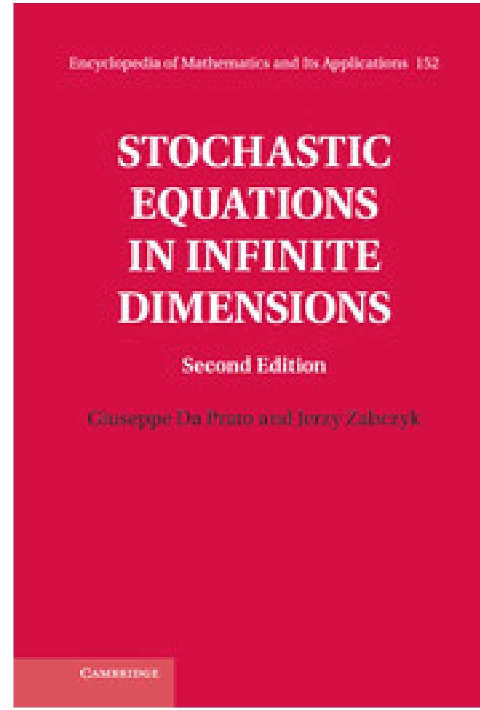
stochastic analysis

> *[...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.*
>
> (D. Stroock, “Elements of stochastic calculus and analysis ”, Springer, 2018)

Nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…



600 pages



492 pages

stochastic analysis

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y)\in\mathcal{O}\subseteq\mathbb{R}^2$	global description	$P_t(x,\mathrm{d}y)$
$\alpha(x-x_0)^2+\beta(y-y_0)^2=\gamma$	·	$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t)\approx x(t)+a\delta t+o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at+bt^2+\cdots$	building block	$(W_t)_t$
$(\ddot{x}(t),\ddot{y}(t))=F(x(t),y(t))$	local/global link	$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$

▷ other examples: rough paths, regularity structures, SLE,...

stochastic quantisation as a stochastic analysis

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x,\mathrm{d}y)$	global description	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi$
$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$		$\left\langle F(\varphi)\frac{\delta S(\varphi)}{\delta\varphi}+\frac{\delta F(\varphi)}{\delta\varphi}\right\rangle=0$
t	change parameter	t
$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$	local description	$\phi(t+\delta t)\approx\alpha\phi(t)+\beta\delta X(t)+\cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X=\frac{1}{2}[(\Delta_x-m^2)X]+\xi$
$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$	local/global link	$\partial_t\phi=\frac{1}{2}[(\Delta_x-m^2)\phi-V'(\phi)]+\xi$

stochastic analysis of EQFs

- **parabolic stochastic quantisation**

$$\partial_t \phi(t) = \frac{1}{2}[(\Delta_x - m^2)\phi(t) - V'(\phi(t))] + \xi(t)$$

[**MG**, M. Hofmanová · Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space · Comm. Math. Phys. 2019 | **MG**, M. Hofmanová · A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory · Comm. Math. Phys. 2021]

- **canonical stochastic quantisation** · singular stochastic wave equations

$$\partial_t^2 \phi(t) + \partial_t \phi(t) = \frac{1}{2}[(\Delta_x - m^2)\phi(t) - V'(\phi(t))] + \xi(t)$$

[**MG**, H. Koch, T. Oh · Renormalization of the two-dimensional stochastic non- linear wave equations · Trans. Am. Math. Soc. 2018 | **MG**, H. Koch, and T. Oh · Paracontrolled Approach to the Three-Dimensional Stochastic Nonlinear Wave Equation with Quadratic Nonlinearity · Jour. Europ. Math. Soc. 2022]

- **elliptic stochastic quantisation** · supersymmetric proof

$$-\Delta_z \phi(z) = \frac{1}{2}[(\Delta_x - m^2)\phi(z) - V'(\phi(z))] + \xi(z), \quad z \in \mathbb{R}^2$$

[S. Albeverio, F. De Vecchi, **MG** · Elliptic Stochastic Quantization · Ann. Prob. 2020]

- **variational method/FBSDE** · stochastic control problem · Γ-convergence

$$\log \int e^{f(\varphi) - S(\varphi)} \mathrm{d}\varphi = \inf_u \mathbb{E}\left[f(\Phi^u_\infty) + V(\Phi^u_\infty) + \frac{1}{2}\int_0^\infty |u_s| \mathrm{d}s \right]$$

scale parameter $t \in [0, \infty]$ · $\Phi^u_t = X_t + \int_0^t J_s u_s \mathrm{d}s$

[N. Barashkov, **MG** · A Variational Method for Φ^4_3 · Duke Math. Jour. 2020]

Part II · the FBSDE for Grassmann measures

Euclidean Fermions

Fermions: quantum particles satisfying Fermi–Dirac statistics

EQFT: Wick rotation of QFT. $t \to \tau = it$, $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ Euclidean space. Wightman functions → Schwinger functions.

$$\Psi, \Psi^* \to \psi, \bar{\psi}.$$

☞ K. Osterwalder and R. Schrader. Euclidean Fermi fields and a Feynman-Kac formula for Boson-Fermions models. *Helvetica Physica Acta*, 46:277–302, 1973.

Euclidean fermion fields $\psi, \bar{\psi}$ form a Grassmann algebra

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \quad (\psi_\alpha^2 = 0).$$

Schwinger functions

- Schwinger functions are given by a Berezin integral on $\Lambda = \mathrm{GA}(\psi, \bar{\psi})$

$$\langle O(\psi,\bar{\psi})\rangle = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-S_E(\psi,\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi} e^{-S_E(\psi,\bar{\psi})}} = \frac{\langle O(\psi,\bar{\psi}) e^{-V(\psi,\bar{\psi})}\rangle_C}{\langle e^{-V(\psi,\bar{\psi})}\rangle_C}$$

$$S_E(\psi,\bar{\psi}) = \frac{1}{2}(\psi, C\,\bar{\psi}) + V(\psi,\bar{\psi}) \qquad \langle O(\psi,\bar{\psi})\rangle_C = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-\frac{1}{2}(\psi, C\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{-\frac{1}{2}(\psi, C\bar{\psi})}}$$

- Under $\langle\cdot\rangle_C$ the variables $\psi,\bar{\psi}$ are “Gaussian” (Wicks' rule):

$$\langle \psi(x_1)\cdots\psi(x_{2n})\rangle_C = \sum_{\sigma} (-1)^{\sigma} \langle \psi(x_{\sigma(1)})\psi(x_{\sigma(2)})\rangle_C \cdots \langle \psi(x_{\sigma(2n-1)})\psi(x_{\sigma(2n-1)})\rangle_C$$

algebraic probability

▷ a non-commutative probability space $(\mathscr{A},\omega)$ is given by a C^*-algebra $\mathscr{A}$ and a *state* ω, a linear normalized positive functional on $\mathscr{A}$ (i.e. $\omega(aa^*)\geqslant 0$).

▷ a random variable is an algebra homomorphism into $\mathscr{A}$

☞ L. Accardi, A. Frigerio, and J. T. Lewis. Quantum stochastic processes. *Kyoto University. Research Institute for Mathematical Sciences. Publications*, 18(1):97–133, 1982. 10.2977/prims/1195184017

example. (classical) random variable X with values on a manifold $\mathscr{M}$?

$$\Omega \xrightarrow{X} \mathscr{M} \xrightarrow{f} \mathbb{R}$$

$$f\in L^\infty(\mathscr{M};\mathbb{C})\to X(f)\in\mathscr{A}=L^\infty(\Omega;\mathbb{C}),\qquad X(fg)=X(f)X(g),\quad X(f^*)=X(f)^*.$$

algebraic data: $\mathscr{A}=L^\infty(\Omega;\mathbb{C})$, $\omega(a)=\int_\Omega a(\omega)\mathbb{P}(\mathrm{d}\omega)$, $X\in\mathrm{Hom}_*(L^\infty(\mathscr{M}),\mathscr{A})$.

Grassmann probability

▷ random variables with values in a Grassmann algebra Λ are algebra homomorphisms

$$\mathscr{G}(V) = \mathrm{Hom}(\Lambda V, \mathscr{A})$$

The embedding of ΛV into $\mathscr{A}$ allows to use the topology of $\mathscr{A}$ to do analysis on Grassmann algebras.

$$d_{\mathscr{G}(V)}(X,Y) := \|X - Y\|_{\mathscr{G}(V)} = \sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathscr{A}},$$

analogy. Gaussian processes in Hilbert space. Abstract Wiener space. "a convenient place where to hang our (analytic) hat on".

back to QFT: IR & UV problems

QFT requires to consider the formula (Fermionic path integral)

$$\langle O(\psi,\bar\psi)\rangle_{C,V} = \frac{\langle O(\psi,\bar\psi)e^{-V(\psi,\bar\psi)}\rangle_C}{\langle e^{-V(\psi,\bar\psi)}\rangle_C}$$

with local interaction

$$V(\psi,\bar\psi) = \int_{\mathbb{R}^d} P(\psi(x),\bar\psi(x))\mathrm{d}x$$

and singular covariance kernel (due to reflection positivity)

$$\langle\bar\psi(x)\psi(y)\rangle \propto |x-y|^{-\alpha}$$

this gives an ill-defined representation

- **large scale (IR) problems**
- **small scale (UV) problems**

well understood in the constructive QFT literature (Gawedzki, Kupiainen, Lesniewski, Rivasseau, Seneor, Magnen, Feldman, Salmhofer, Mastropietro, Giuliani,...)

what about stochastic quantisation for Grassmann measures?

☞ Ignatyuk/Malyshev/Sidoravichius | "Convergence of the Stochastic Quantization Method I,II", 1993. [Grassmann variables + cluster expansion]

weak topology + solution of equations in law + infinite volume limit but no removal of the UV cutoff

*

☞ "Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions" | joint work with Sergio Albeverio, Luigi Borasi, Francesco C. De Vecchi. arXiv:2004.09637 (PTRF)

algebraic probability viewpoint + strong solutions via Picard interation + infinite volume limit but no removal of the UV cutoff

☞ "A stochastic analysis of subcritical Euclidean fermionic field theories" | joint work with Francesco C. De Vecchi and Luca Fresta. arXiv:2210.15047

alg. prob. + forward-backward SDE + infinite volume limit & removal of IR cutoff in the whole subcritical regime

Grassmann stochastic analysis

- ▷ filtration $(\mathscr{A}_t)_{t\geqslant 0}$, conditional expectation $\omega_t\colon \mathscr{A}\to\mathscr{A}_t$,

$$\omega_t(ABC) = A\omega_t(B)C, \qquad A, C \in \mathscr{A}_t.$$

- ▷ Brownian motion $(B_t)_{t\geqslant 0}$ with $B_t \in \mathscr{G}(V)$

$$\omega(B_t(v)B_s(w)) = \langle v, Cw\rangle (t\wedge s), \qquad t, s \geqslant 0, v, w \in V.$$

$$\|B_t - B_s\| \lesssim |t-s|^{1/2}.$$

- ▷ Ito formula

$$\Psi_t = \Psi_0 + \int_0^t B_u(\Psi_u)\mathrm{d}u + X_t, \qquad \omega(X_t \otimes X_s) = C_{t\wedge s}$$

$$\omega_s(F_t(\Psi_t)) = \omega_s(F_s(\Psi_s)) + \int_s^t \omega_s[\partial_u F_u(\Psi_u) + \mathscr{L}F_u(\Psi_u)]\mathrm{d}u,$$

$$\mathscr{L}_u F_u = \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle B_u, \mathrm{DF}_u\rangle$$

the forward-backward SDE

[joint work with Francesco C. De Vecchi and Luca Fresta]

let Ψ be a solution of

$$\mathrm{d}\Psi_s = \dot{C}_s\,\omega_s(\mathrm{D}V(\Psi_T))\mathrm{d}s + \mathrm{d}X_s, \qquad s \in [0,T], \quad \Psi_0 = 0.$$

where $(X_t)_t$ is Gaussian martingale with covariance $\omega(X_t \otimes X_s) = C_{t\wedge s}$. Then

$$\omega(\mathrm{e}^{V(X_T)})\omega(\mathrm{e}^{-V(\Psi_T)}) = 1$$

and

$$\omega(O(\Psi_T)) = \frac{\omega(O(X_T)\mathrm{e}^{V(X_T)})}{\omega(\mathrm{e}^{V(X_T)})} = \frac{\langle O(\psi)e^{V(\psi)}\rangle_{C_T}}{\langle e^{V(\psi)}\rangle_{C_T}}$$

for any O.

▷ this FBSDE provides a stochastic quantisation of the Grassmann Gibbs measure along the interpolation $(X_t)_t$ of its Gaussian component

the backwards step

let F_t be such that $F_T = \mathrm{D}V$. By Ito formula

$$B_s := \omega_s(\mathrm{D}V(\Psi_T)) = \omega_s(F_T(\Psi_T))$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

letting $R_t = B_t - F_s(\Psi_s)$ we have now the forwards-backwards system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

with

$$Q_u := \partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{DF}_u\rangle$$

solution theory

▷ standard interpolation for $C_\infty = (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}$, $\gamma \leqslant d/2$. $\chi \in C^\infty(\mathbb{R}_+)$, compactly supported around 0:

$$C_t := (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}\chi(2^{-2t}(-\Delta_{\mathbb{R}^d})), \qquad \|\dot{C}\|_{\mathscr{L}(L^\infty,L^\infty)} \lesssim 2^{2\gamma-d}, \|\dot{C}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim 2^{2\gamma}$$

▷ the system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

can be solved by standard fixpoint methods for small interaction, uniformly in the volume since X stays bounded as long as $T < \infty$:

$$\|X_t\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{\gamma t}.$$

▷ decay of correlations can be proved by coupling different solutions (Funaki '96).

▷ limit $T \to \infty$ requires renormalization when $\gamma \in [0, d/2]$.

relation with the continuous RG

if we take F such that $Q=0$ we have $R=0$ and then

$$\Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s))\mathrm{d}s + X_t,$$

with

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{D} \mathrm{F}_u \rangle = 0, \qquad F_T = \mathrm{D}V.$$

define the effective potential V_t by the solution of the HJB equation

$$\partial_u V_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} V_u + \langle \mathrm{D}V_u, \dot{C}_u \mathrm{D}\mathrm{V}_u \rangle = 0, \qquad V_T = V.$$

then $F_t = \mathrm{D}V_t$ and the FBSDE computes the solution of the RG flow equation along the interacting field.

▷ so far a full control of the Fermionic HJB equation has not been achieved (work by Brydges, Disertori, Rivasseau, Salmhofer,...). Fermionic RG methods rely on a discrete version of the RG iteration.

approximate flow equation

thanks for the FBSDE we are not bound to solve exactly the flow equation and we can proceed to approximate it.

▷ **linear approximation.** take

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u = 0, \qquad F_T = \mathrm{D}V.$$

this corresponds to Wick renormalization of the potential V:

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \, (F_s(\Psi_s) + R_s) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{D}F_u(\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

the key difficulty is to show uniform estimates for

$$\int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u$$

as $T \to \infty$. we cannot expect better than $\|\Psi_t\| \approx \|X_t\| \approx 2^{\gamma t}$.

polynomial truncation

a better approximation is to truncate the equation to a (large) finite polynomial degree

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u + \Pi_{\leqslant K} \langle F_u, \dot{C}_u \mathrm{D} F_u \rangle = 0$$

where $\Pi_{\leqslant K}$ denotes projection on Grassmann polynomials of degree $\leqslant K$ and take

$$F_t(\psi) = \sum_{k \leqslant K} F_t^{(k)} \psi^{\otimes k}.$$

With this approximation one can solve the flow equation and get estimates

$$\|F_t^{(k)}\| \leqslant \frac{2^{(\alpha - \beta k) t}}{(k+1)^2}, \qquad t \geqslant 0,$$

with $\alpha = 3\beta$, $\beta = d/2 - \gamma$, provided the initial condition $F_T = \mathrm{D} V$ is appropriately renormalized.

FBSDE in the full subcritical regime

with the truncation Π_K we have

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \, (F_s(\Psi_s) + R_s) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{DF}_u \rangle (\Psi_u)] \mathrm{d}u + \int_t^T \omega_t [\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

but now observe that

$$\|\Psi_t\| \approx \|X_t\| \lesssim 2^{\gamma t} \qquad \|F_t^{(k)} \Psi_t^{\otimes k}\| \lesssim 2^{(\gamma k - \beta(k-3))t}$$

which is exponentially small for k large as long as $\gamma \leqslant d/4$ (full subcrititcal regime).

now the term

$$\int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{DF}_u \rangle (\Psi_u)] \mathrm{d}u$$

can be controlled uniformly as $T \to \infty$ and also the full FBSDE system. (!)

thanks