A panorama of Singular SPDEs

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Outline

♣ Singular stochastic partial differential equations (SSPDEs) are a recent field of investigation, blossomed after 2013-2014 when M. Hairer solved the Kardar–Parisi–Zhang (KPZ) equation and lately found a theory of regularity structure which comprises essentially all the SSPDEs one could think of. (HAIRER 13,14)

♣ SSPDEs are PDEs with noise source terms which do not have a *formulation* in standard functional spaces. Meaning the analysis stops even before the problem of showing existence...

♣ In this talk I would like to first *motivate* SSPDEs, showing in which context (some of them) arise and hopefully conveying to you the idea that their structure is *rigid*. There is no much freedom in choosing them.

♣ The origin of this rigidity is *universality*: they describe large scale fluctuations of whole families of random fields, irrespective of microscopic details.

♣ Later on, time permitting, I will show how the intristic difficulties of SSPDEs can be handled using ideas from paradifferential calculus, in paricular *paraproducts* and *paracontrolled calculus* (G.–IMKELLER–PERKOWSKI 15).

♣ This approach is alternative to Hairer's regularity structure theory and currently cannot be applied to all coincievable SSPDEs. When it works delivers a theory which is more similar to standard PDE theories and therefore amenable to the full set of tools and tricks developed in PDE theory since long times.

♣ Both theories took inspiration from Lyons' *rough path theory*, which is concerned with one–dimensional signals (or random functions) and their non-linear transformations via solutions of driven differential equations. In particular rough path theory allows a deterministic treatment of stochastic differential equations.

♣ I will not have time to detail all the current research direction in SSPDEs. It is a very active field where some fundamental problems are still not well understood.

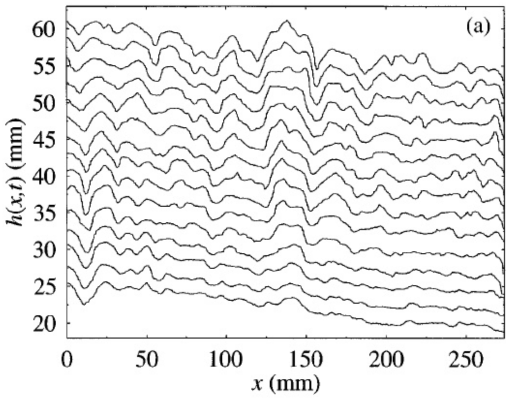
Growth of one dimensional interfaces

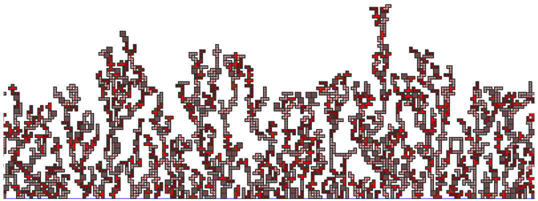
Three regimes

- "Real growth" e.g. ice and water at $10^{\circ}C$; non–reversible; fluctuations $O(t^{1/3})$; conjectured to rescale to **KPZ fixpoint**. Poorly understood. BORODIN, CORWIN, FERRARI, MATETSKI, QUASTEL, REMENIK, SASAMOTO, SPOHN and many others.

- "Coexistence" e.g. ice and water at $0^{\circ}C$; reversible; fluctuations $O(t^{1/4})$; rescales to Gaussian limit. Well understood. KIPNIS–OLLA–VARADHAN, ZHU, CHANG–YAU and many others.

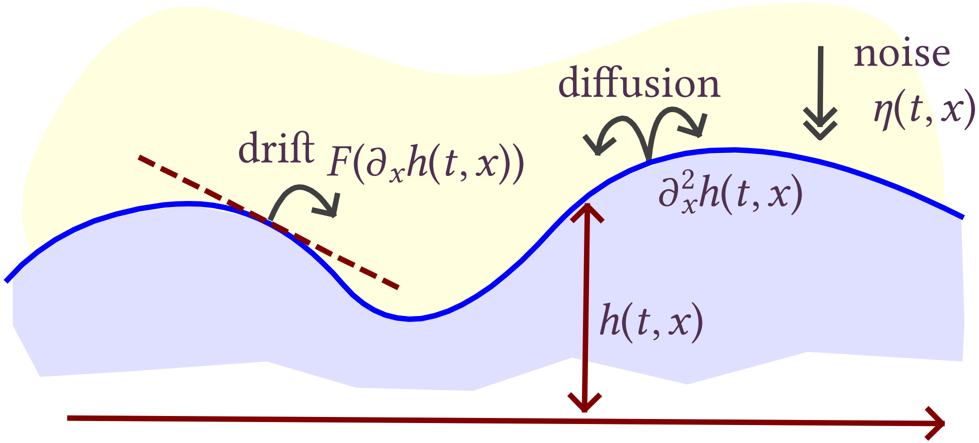
- "Slow growth" e.g. ice and water at $0.1^{\circ}C$; "nearly" reversible, fluctuations $O(t^{1/4})$, non-Gaussian; rescales to **KPZ equation**.





A simple growth model

$$\partial_t h_\varepsilon(t,x) = \partial_x^2 h_\varepsilon(t,x) + \varepsilon^{1/2} F(\partial_x h_\varepsilon(t,x)) + \eta(t,x), \quad t \geq 0, \quad x \in \mathbb{R},$$



▹ η smooth Gaussian field with $O(1)$ stationary correlations. F even polynomial.

Rescaling

- Scaling transformation $\tilde{h}_\varepsilon(t,x) = \varepsilon^{1/2} h_\varepsilon(t/\varepsilon^2, x/\varepsilon)$.

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-1} F\big(\varepsilon^{1/2}\partial_x \tilde{h}_\varepsilon\big) + \xi_\varepsilon$$

- Noise $\xi_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(t/\varepsilon^2, x/\varepsilon)$ converges to space–time white noise

$$\mathbb{E}\Big[\Big(\int\!\!\int \xi_\varepsilon(t,x)\varphi(t,x)\mathrm{d}t\mathrm{d}x\Big)^2\Big] \longrightarrow \int\!\!\int (\varphi(t,x))^2 \mathrm{d}t\mathrm{d}x \qquad \text{as } \varepsilon \longrightarrow 0.$$

Hairer–Quastel weak universality

Theorem. (Hairer–Quastel 15) $\exists C_0, c$ *s.t.*

$$\lambda = \int_{\mathbb{R}} F''(C_0^{1/2}x)\gamma(\mathrm{d}x), \qquad \nu = \int_{\mathbb{R}} F(C_0^{1/2}x)\gamma(\mathrm{d}x). \qquad \gamma = \textit{Normal law.}$$

Then the random field

$$H_\varepsilon(t,x) = \tilde{h}_\varepsilon(t,x) - (\nu/\varepsilon + c)t,$$

converges in law in $C([0,T],\mathbb{T})$ *to* $H(t,x)$ *solving*

$$H(t,x) = \lambda^{-1}\log Z(t,x), \qquad \partial_t Z = \partial_x^2 Z(t,x) + \lambda Z(t,x)\xi(t,x)$$

*(**Hopf–Cole solution**, the product* $Z\xi$ *is understood according to Ito calculus).*

Other interface growth models

▹ **WASEP** (Weakly asymmetric simple exclusion) Particles on $\mathbb{Z}$ moves independently, only one particle per size; jump left with rate p, right with rate $1-p$.

For $p = 1/2$ reversible dynamics, large scale gaussian fluctuations. For $p = 1/2 + \varepsilon$ rescales to Hopf–Cole solution of KPZ (Bertini–Giacomin, CMP 97)

▹ **Ginzburg–Landau $\nabla\varphi$ interface model.** Interacting Brownian motions on $\mathbb{Z}$

$$\mathrm{d}x^i = (pV'(r^{i+1}) - (1-p)V'(r^i))\mathrm{d}t + \mathrm{d}B^i, \qquad i \in \mathbb{Z}, \quad r^i = x^i - x^{i-1}.$$

For $p = 1/2$ reversible dynamics. large scale gaussian fluctuations.

For $p = 1/2 + \varepsilon$, rescales to the Hopf–Cole solution of the KPZ equation (Diehl–G.–Perkowski CMP16)

KPZ equation

Formally H solves the Kardar–Parisi–Zhang equation:

$$\partial_t H = \partial_x^2 H - \lambda[(\partial_x H)^2 - \infty] + \xi.$$

Problem: Not well posed. $H \in C([0,T]; C^{1/2-\kappa})$. ($\infty$ coming from Ito correction)

▹ Hairer (Ann.Math. 13). Solution theory for the KPZ based on rough paths (Lyons)

▹ Gonçalves–Jara (10, ARMA 13). Solution theory for KPZ based on martingale problem. Refined martingale problem (G.–Jara, SPDE/AC 13). Uniqueness (G.–Perkowski, JAMS 18)

▹ Hairer (Inv.Math. 14), G.–Perkowski (CMP 17) solutions theories based on regularity structures and paracontrolled distributions.

Mourrat–Weber convergence result

Theorem. (Mourrat–Weber, CPAM17) *Take $\gamma = \varepsilon^{1/2}$, $\varepsilon = N^{-1}$, and let*

$$\varphi_\varepsilon(t,x) = \varepsilon^{-1/2} h_\gamma(t/\varepsilon, x/\varepsilon)$$

and $\beta - 1 = \varepsilon(C_\gamma + A)$ where

$$C_\gamma = \sum_{\omega \in \mathbb{Z}^2, 0 < |\omega| < \gamma^{-1}} \frac{1}{4\pi^2 |\omega|^2} + O(1) \simeq \log \gamma^{-1}$$

Then $\varphi_\varepsilon \longrightarrow \varphi$ in law in $\mathcal{D}(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}^2))$.

Problem: Equation solved by φ?

Guessing the equation

$$h_\gamma(t,x) = h_\gamma(0,x) + \int_0^t \mathcal{L} h_\gamma(s,x) \mathrm{d}s + M_t$$

$$\mathcal{L} h_\gamma(s,x) = -h_\gamma(s,x) + (\kappa_\gamma * \tanh(\beta h_\gamma(s)))(x)$$

$$= \kappa_\gamma * h_\gamma(s,x) - h_\gamma(s,x) + (\beta - 1)(\kappa_\gamma * h_\gamma)(s,x) - \frac{\beta^3}{3}(\kappa_\gamma * h_\gamma^3)(s,x) + \cdots$$

- Rescaling

$$\varphi_\varepsilon(t) = \varphi_\varepsilon(0) + \int_0^t \underbrace{\varepsilon^{-1}(\kappa_\gamma *_\varepsilon - 1)}_{\Delta_\gamma} \varphi_\varepsilon(s) + \underbrace{\varepsilon^{-1}(\beta - 1)}_{C_\gamma + A}(\kappa_\gamma *_\varepsilon \varphi_\varepsilon(s)) - \frac{\beta^3}{3}(\kappa_\gamma *_\varepsilon \varphi_\varepsilon^3(s)) \mathrm{d}s + \cdots + m_t^\varepsilon$$

The dynamic Φ^4_2 model

Guess: φ solves the stochastic quantisation equation (SQE) or dynamical Φ^4_2 model:

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) + (\infty + A)\,\varphi(t,x) - \frac{1}{3}\varphi(t,x)^3 + \xi(t,x), \qquad t \geqslant 0, x \in \mathbb{T}^2.$$

where ξ is space–time white noise.

Problem: Equation is not well posed (maybe already clear from ∞ present there...)

- ▹ Linear equation: $\partial_t X = \Delta X + AX + \xi$
- ▹ Regularity $X \in C([0,T]; \mathcal{C}^{-\kappa})$ almost surely with $\kappa > 0$ arbitrarily small.
- ▹ $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ Besov–Hölder spaces. $f \in \mathcal{C}^\alpha \Longleftrightarrow \|\Delta_i f\|_{L^\infty} \lesssim 2^{-i\alpha}$ for all $i \geqslant -1$.
- ▹ $(\Delta_i)_{i \geqslant -1}$ Littlewood–Paley decomposition. $\operatorname{supp}(\widehat{\Delta_i f}) \subseteq 2^i \mathcal{A}$. $f = \sum_{i \geqslant -1} \Delta_i f$ for all $f \in \mathcal{S}'$.

Da Prato–Debussche trick

(see also Bourgain for dispersive equations)

Da prato–Debussche (Ann.Prob.03). Write $\varphi = X + \psi$ where

$$\partial_t X = \Delta X + AX + \xi, \qquad \partial_t \psi = \Delta \psi + A\psi + \infty(X + \psi) - \frac{1}{3}(X + \psi)^3$$

$$\infty(X + \psi) - \frac{1}{3}(X + \psi)^3 = -\frac{1}{3}\underbrace{(X^3 - 3\infty X)}_{[\![X^3]\!]} - \underbrace{(X^2 - \infty)}_{[\![X^2]\!]}\psi - X\psi^2 - \frac{1}{3}\psi^3.$$

▹ *Wick powers* $[\![X^2]\!], [\![X^3]\!] \in C([0, T]; \mathcal{C}^{-\kappa})$. **Well posed equation** for ψ

$$\partial_t \psi = \Delta \psi + A\psi - \frac{1}{3}[\![X^3]\!] - [\![X^2]\!]\psi - X\psi^2 - \frac{1}{3}\psi^3$$

since $\psi \in C([0, T]; \mathcal{C}^{2-\kappa})$ by parabolic regularity and product continuous in $\mathcal{C}^{2-\kappa} \times \mathcal{C}^{-\kappa}$.

Euclidean Quantum Field theories

▹ $x \in \mathbb{R}^d$, $\theta x = (x_1, \ldots, x_{d-1}, -x_d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d \geq 0\}$. G Euclidean group of $\mathbb{R}^d$ together with reflection θ. $f^g(x) = f(g^{-1}x)$ for $g \in G$.

▹ μ measure on $\mathcal{S}'(\mathbb{R}^d)$ and $S(f) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{\varphi(f)} \mu(\mathrm{d}\varphi)$ satisfying

1. *Euclidean invariance*: $S(f^g) = S(f)$ for all $g \in G$.
2. *Reflection positivity*: $\forall (f_i \in \mathcal{S}(\mathbb{R}^d_+))_i$, the matrix $(S(\bar{f}_i - f_j^\theta))_{i,j}$ is positive definite.
3. *Exponential bounds*: for some k and some norm: $|S(f)| \leq e^{\|f\|^k}$.

▹ Then ∃ a *relativistic quantum theory* on an Hilbert space $\mathcal{H}$ equipped with a unitary representation of the Poincaré group. Hamiltonian is positive and has a Poincaré invariant vacuuum vector.

[see GLIMM, JAFFE “Quantum Physics”]

Euclidean Φ^4_3 model

Measures that satisfy all these properties are rare. When $d=3$ we know only the Gaussian free field μ, namely the Gaussian measure with covariance

$$\int_{\mathcal{S}'(\mathbb{R}^3)} \varphi(f)\varphi(g)\mu(\mathrm{d}\varphi) = \langle f, (1-\Delta)^{-1} g\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^3),$$

and the Φ^4_3 measure, formally given by

$$\nu(\mathrm{d}\varphi) = \frac{\exp(-\lambda \int_{\mathbb{R}^3} (\varphi^4/4 - \infty \varphi^2/2)\mathrm{d}x)}{Z_\lambda} \mu(\mathrm{d}\varphi).$$

(Brydges, Federbush, Frölich, Glimm, Guerra, Jaffe, Gallavotti, Mitter, Nelson, Rivasseau, Rosen, Simon, Spencer, and many others, '70-'80)

▹ Rigorously this measure can be constructed on a bounded domain $\Lambda \subseteq \mathbb{R}^3$ and with an ultra-violet cutoff ε and a mass counterterm a_ε

$$\nu_\varepsilon(\mathrm{d}\varphi) = \frac{\exp(-\lambda \int_\Lambda (\varphi_\varepsilon^4/4 - a_\varepsilon \varphi_\varepsilon^2/2)\mathrm{d}x)}{Z_{\lambda,\varepsilon}} \mu(\mathrm{d}\varphi)$$

where $\varphi_\varepsilon = \rho_\varepsilon * \varphi$ and $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$ with smooth regularizer ρ.

Main problem: control the limit as $\varepsilon \longrightarrow 0$ of ν_ε. We expect $\nu \not\ll \mu$.

▹ Under μ we have $\varphi \in \mathcal{C}^{-1/2-\kappa}$ almost surely.

Stochastic quantisation

Idea: (Parisi–Wu, 81) Find a (fictious) dynamics which has ν_ε as invariant measure and use the dynamics to construct ν.

A possible choice (Langevin dynamics)

$$\partial_t \varphi = \Delta \varphi - \lambda \rho_\varepsilon * (\varphi_\varepsilon^3 - a_\varepsilon \varphi_\varepsilon) + \xi$$

where ξ is space–time white noise. **Problem:** How to take the limit $\varepsilon \longrightarrow 0$?

▹ It is expected that $a_\varepsilon = a^0/\varepsilon + \lambda a^1 \log(\varepsilon) + a_\varepsilon^2$ where $a_\varepsilon^2 \longrightarrow a^2$ as $\varepsilon \longrightarrow 0$.

Wick ordering does not suffice. Da Prato–Debussche trick does not suffice.

▹ (Hairer Inv.Math 14) Local solution theory based on regularity structures.

▹ (Catellier–Chouk 15, AOP18) Local solution theory based on paracontrolled distributions (G.–Imkeller–Perkowski F.Math.Π 15).

Recent developments

▹ Global space–time solutions in $\mathbb{R}^2$ (Mourrat–Weber CMP17)

▹ Ergodicity for dynamical Φ_2^4 (Röckner–Zhu–Zhu CMP17)

▹ Convergence of lattice discretizations ($\mathbb{T}^3$) (Hairer–Matetski). Complete proof of invariance of Φ_3^4 wrt. the dynamics.

▹ Global solution in time on $\mathbb{T}^3$ (Mourrat–Weber CMP17). Coming down from infinity.

▹ Tightness for the Φ_3^4 measure via dynamics (Albeverio–Kusuoka 18)

▹ Global space–time solutions in $\mathbb{R}^3$ for parabolic equations and global solutions to elliptic equations in $\mathbb{R}^4$, $\mathbb{R}^5$ related to the Φ_2^4, Φ_3^4 measures via (conjectured) dimensional reduction. (G.–Hofmanová 18).

▹ Local theory for *hyperbolic* Φ_2^4 model (G.–Koch–Oh. TAMS18)

$$\partial_t^2 \varphi - \Delta \varphi = -(\varphi^3 - \infty \varphi) + \xi.$$

Invariance principle for a population model

▹ $v^\varepsilon(t,x)$ population size at (t,x). $F \in C^2$, F'' bounded, $F(0)=0$.

$$\partial_t v^\varepsilon(t,x) = \Delta_{\mathbb{Z}^2} v^\varepsilon(t,x) + F(v^\varepsilon(t,x))\eta^\varepsilon(x), \qquad x \in \mathbb{Z}^2, t \geqslant 0.$$

▹ $(\eta^\varepsilon(x))_{x\in\mathbb{Z}^2}$ i.i.d. family with $\mathrm{Var}[\eta^\varepsilon(x)] = \varepsilon^2$ and $\mathbb{E}[\eta^\varepsilon(x)] = -F'(0)\varepsilon^2 c_\varepsilon$ for suitable $c_\varepsilon \simeq |\log \varepsilon|$.

- $F(u) = u$: discrete parabolic Anderson model in a small potential.
- $F(u) = u(C-u)$: restricted resources $u \leqslant C$.

Theorem 1. (Martin–Perkowski 17) *Fix $v^\varepsilon(0,x) = \mathbb{I}_{x=0}$. Let $u^\varepsilon(t,x) = v^\varepsilon(t/\varepsilon^2, x/\varepsilon)$. Then $u^\varepsilon \longrightarrow u$ (in law) where u solves*

$$\partial_t u = \Delta u + F'(0)u\xi - F'(0)u\infty, \qquad u(0) = \delta_0.$$

(linear continuous 2d Anderson model). Here ξ is a space white noise in $d=2$.

Anderson Hamiltonian

▹ **"Toy" problem.** Give a well defined meaning to the operator

$$H = \Delta + \xi$$

in $L^2(\mathbb{T}^2)$ where ξ is a space–white noise. Domain?

Observation: the domain does not contain smooth functions... $\xi \in C^{-1-\kappa}$.

▹ **Paraproducts:** $fg = f \prec g + f \circ g + f \succ g$ where $f \prec g = \sum_{i<j-1} \Delta_i f \Delta_j g$ and $f \circ g = \sum_{|i-j|=1} \Delta_i f \Delta_j g$.

$$Hf = \Delta f + \xi \succ f + \xi \circ f + \xi \prec f$$

Assume $f \in H^\alpha$. Then $\xi \succ f \in H^{-1-2\kappa}$ for any $\alpha \geqslant 0$, $\xi \circ f \in H^{\alpha-1-2\kappa}$ for $\alpha > 1 - 2\kappa$, $\xi \prec f \in H^{\alpha-1-2\kappa}$ for any $\alpha \in \mathbb{R}$.

Paracontrolled Ansatz

Let $X = (1-\Delta)^{-1}\xi \in \mathcal{C}^{1-\kappa}$ and **assume**

$$f - f \prec X = f^{\sharp} \in H^2.$$

Then $f \in H^{1-\kappa}$ and

$$\Delta f = \Delta(f \prec X + f^{\sharp}) = (\Delta f) \prec X + f \prec \Delta X + \nabla f \prec \nabla X + \Delta f^{\sharp}$$

$$Hf = \Delta f + \xi \succ f + \xi \circ f + \xi \prec f = \xi \circ f + \underbrace{(\Delta f) \prec X + f \prec X + \nabla f \prec \nabla X + \Delta f^{\sharp} + \xi \prec f}_{\in H^{-2\kappa}}$$

Commutator lemma:

$$\xi \circ f = \xi \circ (f \prec X) + \xi \circ f^{\sharp} = f\, \xi \circ X + \underbrace{\mathrm{Comm}(f, \xi, X)}_{H^{1-3\kappa}} + \xi \circ f^{\sharp}.$$

Renormalization

$$(H-\infty)f = f\big(\xi\circ X-\infty\big) + \xi\circ f^\sharp + \underbrace{\mathrm{Comm}(f,\xi,X)}_{H^{1-3\kappa}} + \underbrace{(\Delta f)\prec X + f\prec X + \nabla f\prec\nabla X + \Delta f^\sharp + \xi\prec f}_{\in H^{-2\kappa}}$$

Define

$$\xi\diamond X = \xi\circ X-\infty = \lim_{\varepsilon\longrightarrow 0}(\xi_\varepsilon\circ X_\varepsilon - c_\varepsilon)\in\mathcal{C}^{-\kappa}.$$

Finally for all $f\in L^2$ such that $f-f\prec X\in H^2$ we have a well defined expression

$$H_{\mathrm{ren}}f = f(\xi\diamond X)+\xi\circ f^\sharp+\cdots$$

▹ **Rigorously**: $H_\varepsilon=\Delta+\xi_\varepsilon-c_\varepsilon\longrightarrow H_{\mathrm{ren}}$ in operator resolvent sense.

More involved proof in $\mathbb{T}^3$. (Allez-Chouk, G.–Ugurcan–Zachhuber, Labbé)

Conclusions

We have seen several results of convergence of *microscopic* models to scaling limits given by non–Gaussian random fields.

These random fields are conjectured to be *universal*, independent of specific details of the microscopic model.

They solve SSPDEs, equations that, on first sight are not well defined due to the presence of singular non-linear terms and renormalizations via subtraction of formal infinite quantities.

I tried to make clear that such infinities are not misterious. They are just manifestations of phenmomena taking place on a different (larger) scale. And the price to pay for universality.

Nowadays we dispose various tools: regularity structures, paracontrolled distributions, the approach of Otto–Weber, RG approach by Kupiainen, and we try to tackle to more and more difficult questions...

Thank you.