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Stochastic quantisation

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outline

- euclidean quantum fields
- what is stochastic quantisation?
- varieties of stochastic quantisation
- properties of stochastically quantised measures
- open problems

Euclidean quantum fields (EQFs)

are particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$:

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + p(\varphi(x))\mathrm{d}x$$

for some non-linear function $p: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, e.g. a polynomial bounded below, exponentials, trig funcs.

Introduced in the '70-'80 as a tool to constructs models of (bosonic) quantum field theories in the sense of Wightman via the reconstruction theorem of Osterwalder–Schrader.

ill-defined representation:

- large scale problems: the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- small scale problems: sample paths are not expected to be function, but only distributions, the quantity $p(\varphi(x))$ does not make sense.

stochastic quantisation

▹ introduced in the '80 by Parisi/Wu (and similar methods by Nelson)

Idea: build the measure ν as invariant law to a stochastic partial differential equation

$$\partial_t \phi(t,x) = \frac{1}{2}[(\Delta_x - m^2)\phi(t,x) - p'(\phi(t,x))] + 2^{1/2}\xi(t,x)$$

$t \in \mathbb{R}$ fictious "simulation time", $x \in \mathbb{R}^d$, ξ space-time white noise,

$$\mathrm{Law}(\phi(t,\cdot)) = \nu$$

why is this a good idea?

stochastic quantisation is a **stochastic analysis** of EQFs

- Ito & Dœblin introduced a variety of analysis adapted to the sample paths of a stochastic process.
- consider a family of kernels $(P_t)_{t\geqslant 0}$ on $\mathbb{R}^d$ satisfying Chapman–Kolmogorov equation

$$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$$

which defines a probability $\mathbb{P}$ on $C(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$: the law of a continuous Markov process.

- sample paths have a "*tangent*" *process*. Ito identified it as a particular Lévy process: the Brownian motion $(W_t)_t$.
- stochastic calculus: from the local picture to the global structure via *stochastic differential equation* (SDE)

$$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$$

▹ these are the basic building blocks of **stochastic analysis**

▹ like in analysis, the fact that we can consider infinitesimal changes simplify the analysis and make appear universal underlying objects:

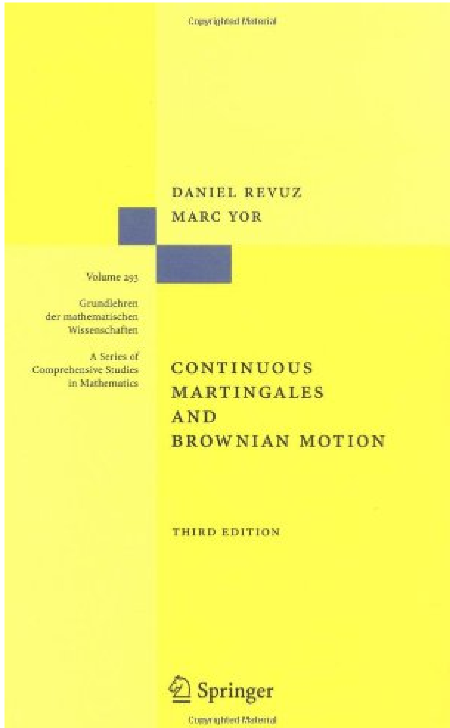
- polynomials → calculus, Taylor expansion
- Brownian motion and its functionals → Ito calculus, stochastic Taylor expansion

to have an analysis we need:

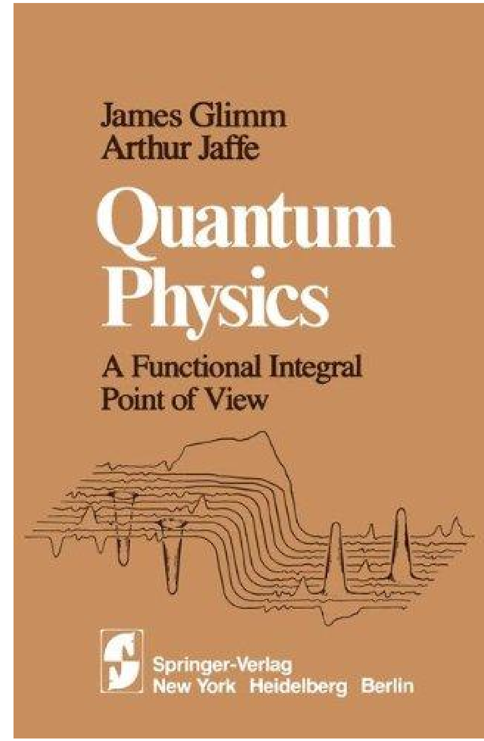
- a **change parameter** along which consider "change" (*time* for diffusions)
- a suitable **building block** for the infinitesimal changes (*Brownian motion* for diffusion)

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x, y) \in \mathcal{O} \subseteq \mathbb{R}^2$	global description	$P_t(x, \mathrm{d}y)$
$\alpha(x - x_0)^2 + \beta(y - y_0)^2 = \gamma$	·	$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$
t	change parameter	t
$x(t + \delta t) \approx x(t) + a\,\delta t + o(\delta t)$	local description	$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at + bt^2 + \cdots$	building block	$(W_t)_t$
$(\ddot{x}(t), \ddot{y}(t)) = F(x(t), y(t))$	local/global link	$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x, \mathrm{d}y)$	global description	$\nu \in \mathrm{Prob}(\mathscr{S}'(\mathbb{R}^d))$
$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$	·	$\frac{1}{Z} \int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-S(\varphi)} \mathrm{d}\varphi$
t	change parameter	t
$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t) a(x)^{-1} (y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x (\delta t)^{d/2}},$	local description	$\phi(t+\delta t) \approx \alpha \phi(t) + \beta \delta X(t) + \cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X = \frac{1}{2}[(\Delta_x - m^2) X] + 2^{1/2} \xi$
$\mathrm{d}X_t = a(X_t) \mathrm{d}W_t + b(X_t) \mathrm{d}t$	local/global link	$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2} \xi$



600 pages



535 pages

- **parabolic stochastic quantisation.** the parameter is an additional "fictious" coordinate $t \in \mathbb{R}$, playing the röle of a simulation time. The EQF is viewed as the invariant measure of a Markov process (SDE). Building block is a space-time white noise. [Parisi/Wu, Nelson, Jona-Lasinio/Mitter, Albeverio/Röckner, Da Prato/Debbusche, Hairer, Catellier/Chouk, Mourrat/Weber, G./Hofmanova, Albeverio/Kusuoka, Chandra/Moinat/Weber, Shen, Garban, many others...]

$$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **canonical stochastic quantisation.** same as for parabolic, but the evolution takes place in "phase space" and the SDE is second order in time, giving rise to a stochastic wave equation. [G./Koch/Oh, Tolomeo, Oh/Robert/Wang]

$$\partial_t^2 \phi + \partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **elliptic stochastic quantisation.** the parameter is a coordinate $z \in \mathbb{R}^2$. Building block is a white noise in $\mathbb{R}^{d+2}$. An elliptic stochastic partial differential equation describes the EQF as a function of the white noise. Link with supersymmetry. [Parisi/Sourlas, Klein/Landau/Perez, Albeverio/De Vecchi/G., Barashkov/De Vecchi]

$$-\Delta_z \phi(z,x) = \frac{1}{2}[(\Delta_x - m^2)\phi(z,x) - p'(\phi(z,x))] + 2^{1/2}\xi(z,x)$$

- **variational method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The EQF is described as the solution of a stochastic optimal control problem. [Barashkov/G.]
- **rg method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The effective action of the EQF satisfies an Hamilton–Jacobi–Bellmann equation. [Wilson, Wegner, Polchinski, Salmhofer, Brydges/Kennedy, Mitter, Gawedzki/Kupiainen, Brydges/Bauerschmidt/Slade, Bauerschmidt/Bodineau, Bauerschmidt/Hofstetter, also many others...]

features of stochastic quantisation

the interacting field ϕ is expressed as a function of the Gaussian free field X:

$$\phi(t) = F(X), \qquad \nu = \mathrm{Law}(\phi(t)) = F_*\mathrm{Law}(X) = F_*\mathrm{GFF}$$

- estimates on ϕ obtained via two ingredients:
  - pathwise PDE estimates for the map F (in weighted Besov spaces)
  - probabilistic estimates for the GFF X
- coupling (φ, X)

  $$\phi = X + \psi$$

  where ψ is a random field which is more regular (i.e. smaller at small scale) than X (link with asymptotic freedom/perturbation theory)

  note that

  $$\nu = \mathrm{Law}(\varphi) \not\ll \mathrm{Law}(X(t)) = \mathrm{GFF}$$

estimates

- decomposition: $\phi = X + \psi$

$$\partial_t \psi = \frac{1}{2}[(\Delta_x - m^2)\psi - p'(X+\psi)]$$

- PDE estimates:

$$\|\psi(t)\| \leqslant H(\|X\|)$$

- tightness:

$$\int \|\varphi\|^p \nu(\mathrm{d}\varphi) = \mathbb{E}\|\psi(t)\|^p \leqslant \mathbb{E}[H(\|X\|)^p] < \infty$$

- tail-estimates:

$$\int e^{c\|\varphi\|^\alpha} \nu(\mathrm{d}\varphi) < \infty$$

[Moinat/Weber, Hofmanova/G., Hairer/Steele]

properties of the stochastically quantized EQF

Φ^4_3 measure. $p(\varphi)=\lambda\varphi^4-c\varphi^2$, $d=3$. [Detailed construction in Hofmanova/G. - CMP 2020]

▹ non-gaussianity:

$$\langle\varphi\varphi\varphi\varphi\rangle_c=\langle XXXX\rangle_c+4\langle XXX\psi\rangle_c+12\langle XX\psi\psi\rangle_c+4\langle X\psi\psi\psi\rangle_c+\langle\psi\psi\psi\psi\rangle_c$$

$$=4\langle XXX\psi\rangle_c+\cdots\neq 0$$

▹ renormalized cube:

$$[\![\varphi^3]\!]=\lim_{\varepsilon\to 0}[(\rho_\varepsilon*\varphi)^3-c_\varepsilon(\rho_\varepsilon*\varphi)]=[\![X^3]\!]+\left\{([\![X^2]\!]*_r\psi)+X\psi^2+\psi^3\right\}$$

result: $[\![\varphi^3]\!]$ is not a random variable but a distribution on $\mathrm{Cyl}(\mathscr{S}'(\mathbb{R}^3))$.

▹ Dyson–Schwinger equation (IBP formula for ν):

$$\int \mathrm{D}_\varphi F(\varphi)\nu(\varphi)=\int F(\varphi)\left\{(\Delta-m^2)\varphi-\lambda[\![\varphi^3]\!]\right\}\nu(\varphi)$$

outlook

Goal: develop a stochastic analysis of EQFs
(at least for superrenormalizable models)

- identify “building blocks” and describe EQFs (non-perturbatively) in terms of these simpler objects.
- small scales behaviour/renormalization: well understood in most models in some of the approaches (see e.g. recent results of Hairer et al. on Yang-Mills fields).
- coercivity (large fields problem) plays a key role for global control and infinite volume limit. So far, not understood at all for YM.
- uniqueness (high or low temp)? still open in most models, especially $\Phi^4_{2,3}$.
- some results on phase transition by Chandra/Gunaratnam/Weber.

open problems

- How to apply these ideas to gauge theories/geometric models? Higgs model, Yang-Mills? [Hairer/Zambotti/Chandra/Chevyrev/Shen/...] Coercivity not well understood.
- Grassmann fields? [partial progress in Albeverio/Borasi/De Vecchi/G., no renorm yet]
- Small coupling regime? (proof of Borel-summability?)
- Decay of correlations at high temperature? [some results Rana/Hofmanova/G.]
- Dyson-Schwinger eq. determine the measure?
- Use the approach for lattice unbounded spin systems?
- What about mass-less models on the lattice: $\nabla\varphi$ models?
- Weak-universality and models above the critical dimension?
- ...

thanks