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A variational method for Euclidean quantum fields

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[made with TeXmacs]

Euclidean quantum fields (EQFs)

a particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$:

EQF = regularity + Euclidean invariance + reflection positivity

Introduced in the '70-'80 as a tool to constructs models of (bosonic) quantum field theories in the sense of Wightman via the reconstruction theorem of Osterwalder–Schrader.

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + V(\varphi(x))\mathrm{d}x$$

for some non-linear function $V: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, e.g. a polynomial bounded below, exponentials, trig funcs. ill-defined representation:

- **large scale (IR) problems:** the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- **small scale (UV) problems:** sample paths are not expected to be function, but only distributions, the quantity $V(\varphi(x))$ does not make sense.

▷ Construct rigorously QM models which are compatible with special relativity, (finite speed of signals and Poincaré covariance of Minkowski space $\mathbb{R}^{n+1}$).

▷ Quantum field theory (QM with ∞ many degrees of freedom)

▷ Wightman axioms ('60-'70): Hilbert space, representation of the Poincaré group, fields operators (to construct local observables).

▷ Constructive QFT program: Hard to find models of such axioms. Examples in $\mathbb{R}^{1+1}$ were found in the $'60$.

▷ Euclidean rotation: $t \to it = x_0$ (imaginary time). $\mathbb{R}^{n+1} \to \mathbb{R}^d$ Minkowski → Euclidean

▷ Osterwalder–Schrader theorem : gives precise condition to perform the passage to/from Euclidean space (OS axioms for Euclidean correlation function).

▷ Surprise: in some cases the Euclidean theory is a probability measure on $\mathscr{S}'(\mathbb{R}^d)$.

▷ High point of CQFT: construction of Φ^4_3 (Euclidean version of a scalar field in $\mathbb{R}^{2+1}$ Minkowski space).

▷ simplest example of EQFT. We take a Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with covariance

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2 + |k|^2} \frac{\mathrm{d}k}{(2\pi)^d} = (m^2 - \Delta)^{-1}(x-y), \qquad x, y \in \mathbb{R}^d$$

and zero mean. Reflection positive, Eucl. covariant and regular. This is the GFF with mass $m > 0$.

▷ this measure can be used to construct a QFT in Minkowski space but unfortunately this theory is free, i.e. there is no interaction.

▷ note that $G(0) = +\infty$ if $d \geqslant 2$, this implies that the GFF is not a function.

▷ in particular GFF is a distribution of regulariy $(2-d)/2 - \kappa$ for any small $\kappa > 0$, e.g. locally in the sense of the scale of Besov–Holder spaces $(B^{\alpha}_{\infty,\infty})_{\alpha \in \mathbb{R}}$.

▷ heuristically we want

$$\nu(\mathrm{d}\varphi) = \frac{e^{\int_\Lambda V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi).$$

❶ go on a lattice: $\mathbb{R}^d \to \mathbb{Z}_\varepsilon^d = (\varepsilon\mathbb{Z})^d$ with spacing $\varepsilon > 0$ and make it periodic $\mathbb{Z}_\varepsilon^d \to \mathbb{Z}_{\varepsilon,L}^d = (\mathbb{Z}_\varepsilon / 2\pi L\mathbb{N})^d$.

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}_{\varepsilon,L}^d}} F(\varphi) e^{\overbrace{-\frac{1}{2}\sum_{x\in\mathbb{Z}_{\varepsilon,L}^d} |\nabla_\varepsilon\varphi(x)|^2 + m^2\varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}} \mathrm{d}\varphi$$

ε is an UV regularisation and L the IR one.

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below (otherwise integrab. problems). $d = 2, 3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ_d^4 measure.

❸ study the possible limit points (uniqueness? non-uniqueness? correlations? description?)

▷ for $d=2$ other choices are possible:

$$V_{\varepsilon}(\xi)=\lambda\xi^{2l}+\sum_{k=0}^{2l-1}a_{k,\varepsilon}\xi^{k},\quad V_{\varepsilon}(\xi)=a_{\varepsilon}\cos(\beta\xi)$$

$$V_{\varepsilon}(\xi)=a_{\varepsilon}\cosh(\beta\xi),\qquad V_{\varepsilon}(\xi)=a_{\varepsilon}\exp(\beta\xi)$$

▷ for $d=3$ "only" 4th order (6th order is critical).

▷ for $d=4$ all the possible limits are Gaussian (see recent work of Aizenmann-Duminil Copin, arXiv:1912.07973)

▷ Ito introduced stochastic analysis to study the law of diffusion processes via Brownian motion and the related stochastic calculus.

▷ various (equivalent?) stochastic analysis of EFQs are provided by stochastic quantisations

parabolic, elliptic, hyperbolic,...?

See e.g.

- M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.
- S. Albeverio, F. C. De Vecchi, and M. Gubinelli, `Elliptic Stochastic Quantization', *Annals of Probability* 48, no. 4 (July 2020): 1693–1741, https://doi.org/10.1214/19-AOP1404.
- S. Albeverio et al., `Grassmannian Stochastic Analysis and the Stochastic Quantization of Euclidean Fermions', *ArXiv:2004.09637 [Math-Ph]*, 25 May 2020, http://arxiv.org/abs/2004.09637.
- M. Gubinelli, H. Koch, and T. Oh, `Renormalization of the Two-Dimensional Stochastic Nonlinear Wave Equations', *Transactions of the American Mathematical Society*, 2018, 1, https://doi.org/10.1090/tran/7452.

another approach to the stochastic analysis of EQF

- N. Barashkov and M. Gubinelli, `A Variational Method for Φ^4_3 ', *Duke Mathematical Journal* 169, no. 17 (November 2020): 3339–3415, https://doi.org/10.1215/00127094-2020-0029.
- N. Barashkov and M. Gubinelli, `The Φ^4_3 Measure via Girsanov's Theorem', E.J.P 2021 (arXiv:2004.01513).
- N. Barashkov's PhD thesis, University of Bonn, 2021.
- N. Barashkov and M. Gubinelli, `On the variational description of Euclidean quantum fields in infinite volume' (in preparation)

based on a variational formula for Brownian functionals proved by Boué and Dupuis.

Theorem. *(Let* $(B_t)_{t \geqslant 0}$ *be a Brownian motion on* $\mathbb{R}^n$*, then for any bounded* $F: C(\mathbb{R}_+; \mathbb{R}^n) \to \mathbb{R}$ *we have*

$$\log \mathbb{E}[e^{F(B_\bullet)}] = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[F(B_\bullet + I(u)_\bullet) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

with $u: \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ *adapted to* B *and with*

$$I(u)_t := \int_0^t u_s \mathrm{d}s.$$

$$\frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s \approx H(\mathrm{Law}(B_\bullet + I(u)_\bullet) \,|\, \mathrm{Law}(B_\bullet)).$$

M. Boué and P. Dupuis, `A Variational Representation for Certain Functionals of Brownian Motion', *The Annals of Probability* 26, no. 4: 1641–59 https://doi.org/10.1214/aop/1022855876

$$\mathbb{E}[W_t(x)W_s(y)] = (t \wedge s)(m^2 - \Delta)^{-1}(x-y), \qquad t, s \in [0,1].$$

The BD formula gives

$$-\log \int e^{-F(\phi)} \mu(\mathrm{d}\phi) = -\log \mathbb{E}[e^{-F(W_1)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[F(W_1 + Z_1) + \frac{1}{2}\int_0^1 \| u_s \|_{L^2}^2 \mathrm{d}s\right],$$

where

$$Z_t = (m^2 - \Delta)^{-1/2} \int_0^t u_s \mathrm{d}s, \qquad u_t = (m^2 - \Delta)^{1/2} \dot{Z}_t$$

$$-\log \mathbb{E}[e^{-F(W_1)}] = \inf_{Z \in H^a} \mathbb{E}[F(W_1 + Z_1) + \mathscr{E}(Z_\bullet)],$$

with

$$\mathscr{E}(Z_\bullet) := \frac{1}{2} \int_0^1 \|(m^2 - \Delta)^{1/2} \dot{Z}_s\|_{L^2}^2 \mathrm{d}s = \frac{1}{2} \int_0^1 (\|\nabla \dot{Z}_s\|_{L^2}^2 + m^2 \|\dot{Z}_s\|_{L^2}^2) \mathrm{d}s$$

Φ_2^4 in a bounded domain Λ

Fix a compact region $\Lambda \Subset \mathbb{R}^2$ and consider the Φ_2^4 measure θ_Λ on $\mathscr{S}'(\mathbb{R}^2)$ with interaction in Λ and given by

$$\theta_\Lambda(\mathrm{d}\phi) := \frac{e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}{\int e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)} \qquad \phi \in \mathscr{S}'(\mathbb{R}^2) \tag{1}$$

with interaction potential $V_\Lambda(\phi) := \int_\Lambda \phi^4 - c\int_\Lambda \phi^2$. For any $f: \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let

$$e^{-\mathscr{W}_\Lambda(f)} := \int e^{-f(\phi)}\theta_\Lambda(\mathrm{d}\phi).$$

We have the variational representation, $Z = Z_1$, $Z_\bullet = (Z_t)_{t\in[0,1]}$:

$$\mathscr{W}_\Lambda(f) = \inf_{Z\in H^a} F^{f,\Lambda}(Z_\bullet) - \inf_{Z\in H^a} F^{0,\Lambda}(Z_\bullet)$$

where

$$F^{f,\Lambda}(Z_\bullet) := \mathbb{E}[f(W+Z) + \lambda V_\Lambda(W+Z) + \mathscr{E}(Z_\bullet)].$$

renormalized potential

$$V_\Lambda(W+Z) = \int_\Lambda \Big\{ \underbrace{W^4 - cW^2}_{\mathbb{W}^4} + 4\underbrace{\Big[W^3 - \frac{c}{4}W\Big]}_{\mathbb{W}^3} Z + 6\underbrace{\Big[W^2 - \frac{c}{6}\Big]}_{\mathbb{W}^2} Z^2 + 4WZ^3 + Z^4 \Big\}$$

take $c = 12\mathbb{E}[W^2(x)] = +\infty$

$$V_\Lambda(W+Z) = \int_\Lambda \{4\mathbb{W}^3 Z + 6\mathbb{W}^2 Z^2 + 4WZ^3 + Z^4\} + \cdots$$

$$\mathbb{W}^n \in \mathscr{C}^{-n\kappa}(\Lambda) = B^{-n\kappa}_{\infty,\infty}(\Lambda)$$

Here $B^{-\kappa}_{\infty,\infty}(\Lambda)$ is an Hölder–Besov space. A distribution $f \in \mathscr{S}'(\mathbb{T}^d)$ belongs to $B^{\alpha}_{\infty,\infty}(\Lambda)$ iff for any $n \geqslant 0$

$$\|\Delta_n f\|_{L^\infty} \leqslant (2^n)^{-\alpha} \|f\|_{B^{\alpha}_{\infty,\infty}(\Lambda)}$$

where $\Delta_n f = \mathscr{F}^{-1}(\varphi_n(\cdot)\mathscr{F} f)$ and φ_n is a function supported on an annulus of size $\approx 2^n$. We have $f = \sum_{n \geqslant 0} \Delta_n f$. If $\alpha > 0$ $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d)$ is a space of functions otherwise they are only distributions.

Euler–Lagrange equation for minimizers

Lemma. *There exists a minimizer* $Z = Z^{f,\Lambda}$ *of* $F^{f,\Lambda}$. *Any minimizer satisfies the Euler–Lagrange equations*

$$\begin{aligned}&\mathbb{E}\left(4\lambda\int_\Lambda Z^3 K + \int_0^1\int_\Lambda (\dot{Z}_s(m^2-\Delta)\dot{K}_s)\mathrm{d}s\right)\\ = \;& \mathbb{E}\left(\int_\Lambda f'(W+Z)K + \lambda\int_\Lambda (\mathbb{W}^3+\mathbb{W}^2 Z + 12 W Z^2) K\right)\end{aligned}$$

for any K *adapted to the Brownian filtration and such that* $K \in L^2(\mu, H)$.

▷ technically one really needs a relaxation to discuss minimizers, we ignore this all along this talk. the actualy object of study is the law of the pair $(\mathbb{W}, Z)$ and not the process Z. (similar as what happens in the Φ_3^4 paper) we use polynomial weights $\rho(x) = (1 + \ell |x|)^{-n}$ for large $n > 0$ and small $\ell > 0$.

Theorem. *There exists a constant C independent of $|\Lambda|$ such that, for any minimizer Z of $F^{f,\Lambda}(\mu)$ and any spatial weight $\rho : \Lambda \to [0,1]$ with $|\nabla \rho| \leqslant \varepsilon \rho$ for some $\varepsilon > 0$ small enough, we have*

$$\mathbb{E}\left(4\lambda \int_\Lambda \rho Z_1^4 + \int_0^1 \int_{\mathbb{R}^2} ((m^2 - \Delta)^{1/2} \rho^{1/2} \dot{Z}_s)^2 \mathrm{d}s\right) \leqslant C.$$

Proof. test the Euler–Lagrange equations with $K = \rho Z$ and then estimate the bad terms with the good terms and objects only depending on $\mathbb{W}$, e.g.

$$\left|\int_\Lambda \rho \mathbb{W}^3 Z\right| \leqslant C_\delta \| \mathbb{W}^3 \|_{H^{-1}(\rho^{1/2})}^2 + \delta \| Z \|_{H^1(\rho^{1/2})}^2,$$

$$\left|\int_\Lambda \rho \mathbb{W}^2 Z^2\right| \leqslant C_\delta \| \rho^{1/8} \mathbb{W}^2 \|_{C^{-\varepsilon}}^4 + \delta(\| \rho^{1/4} \bar{Z} \|_{L^4}^4 + \| \rho^{1/2} \bar{Z} \|_{H^{2\varepsilon}}^2), \cdots$$

$$\mathscr{W}_\Lambda(f) = \inf_Z F^{f,\Lambda}(Z) - \inf_Z F^{0,\Lambda}(\mu) = F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

Therefore

$$F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{f,\Lambda}) \leqslant \mathscr{W}_\Lambda(f) \leqslant F^{f,\Lambda}(Z^{0,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

and since, for any g,

$$F^{f,\Lambda}(Z^{g,\Lambda}) - F^{0,\Lambda}(Z^{g,\Lambda}) = \mathbb{E}[f(W + Z^{g,\Lambda}) + \lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})]$$

$$- \mathbb{E}[\lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})] = \mathbb{E}[f(W + Z^{g,\Lambda})]$$

$$\boxed{\mathbb{E}[f(W + Z^{f,\Lambda})] \leqslant \mathscr{W}_\Lambda(f) \leqslant \mathbb{E}[f(W + Z^{0,\Lambda})]}$$

Consequence: tightness of $(\theta_\Lambda)_\Lambda$ in $\mathscr{S}'(\mathbb{R}^2)$ and optimal exponential bounds (cfr. Hairer/Steele)

$$\sup_\Lambda \int \exp(\delta \|\phi\|^4_{W^{-\kappa,4}(\rho)}) \theta_\Lambda(d\phi) < \infty.$$

Euler–Lagrange equation in infinite volume

The family $(Z^{f,\Lambda})_\Lambda$ is also converging (provided we look at the relaxed problem) and any limit point $Z = Z^f$ satisfies a EL equation:

$$\mathbb{E}\left\{\int_{\mathbb{R}^2} f'(W+Z)K + 4\lambda\int_{\mathbb{R}^2} [\![(W+Z)^3]\!]K + \int_0^1\int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\dot{K}_s \mathrm{d}s\right\} = 0$$

for any test process K (adapted to $\mathbb{W}$ and to Z).

<u>**a new kind of stochastic "elliptic" problem**</u>

Open questions

- Uniqueness??
- Γ-convergence of the variational description of $\mathscr{W}_\Lambda(f)$?

not clear. We lack sufficient knowledge of the dependence on f of the solutions to the EL equations above.

For any $f:\mathscr{S}'(\mathbb{R}^d)\to\mathbb{R}$ (non necessarily linear) let $\mathscr{W}_\Lambda^\hbar(f)$ be defined by:

$$e^{-\frac{1}{\hbar}\mathscr{W}_\Lambda^\hbar(f)}:=\int e^{-f(\phi)}\,\theta_\Lambda^\hbar(\mathrm{d}\phi).$$

where

$$\mathrm{d}\theta_\Lambda^\hbar(\phi)=\exp\left(-\frac{1}{\hbar}V_\Lambda^\hbar(\phi)\right)\mathrm{d}\mu^\hbar(\phi)=\exp\left(-\frac{\lambda}{\hbar}\int_\Lambda [\![\phi^4]\!]\right)\mathrm{d}\mu^\hbar(\phi)$$

and $\mu^\hbar$, is the Gaussian measure with covariance $\hbar(m^2-\Delta)^{-1}$.

Theorem. *Any accumulation point $\theta^\hbar$ of $\theta_\Lambda^\hbar$ satisfies a Laplace principle with rate function*

$$J(\phi)=\lambda\int_{\mathbb{R}^2}\phi^4\mathrm{d}x+\int_{\mathbb{R}^2}\phi(m^2-\Delta)\phi\,\mathrm{d}x.$$

That is

$$\lim_{\hbar\to 0}\mathscr{W}^\hbar(f)=\inf_\psi\{f(\psi)+J(\psi)\}.$$

Exponential interaction

we can study similarly the model with

$$V^{\xi}(\varphi) = \int_{\mathbb{R}^2} \xi(x) [\![\exp(\beta\varphi(x))]\!] \mathrm{d}x$$

for $\beta^2 < 8\pi$ and $\xi: \mathbb{R}^2 \to [0,1]$ a spatial cutoff function.

$$V^{\xi}(W+Z) = \int_{\mathbb{R}^2} \xi(x) \exp(\beta Z(x)) \underbrace{[\![\exp(\beta W(x))]\!] \mathrm{d}x}_{M^{\beta}(\mathrm{d}x)}$$

$$= \int_{\mathbb{R}^2} \xi(x) \exp(\beta Z(x)) M^{\beta}(\mathrm{d}x), \qquad \text{[Gaussian multiplicative chaos]}$$

BD formula

$$\begin{aligned} \mathcal{W}^{\xi,\exp}(f) &= -\log \int \exp(-f(\phi)) \mathrm{d}\nu^{\xi} \\ &= \inf_{Z \in \mathfrak{H}_a} \mathbb{E}\left[f(W+Z) + \int \xi \exp(\beta Z) \mathrm{d}M^{\beta} + \frac{1}{2} \int_0^1 \int ((m^2 - \Delta)^{1/2} \dot{Z}_t)^2 \mathrm{d}t \right] \end{aligned}$$

▷ the function $Z \mapsto V^{\xi}(W+Z)$ is convex!

variational description of the infinite volume limit

▷ thanks to convexity the EL equations have a unique limit Z in the ∞ volume limit

▷ moreover we have the Γ-convergence of the variational description:

$$\mathcal{W}_{\mathbb{R}^2}(f) = \lim_{n\to\infty}\left[-\log\int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp}\right]$$

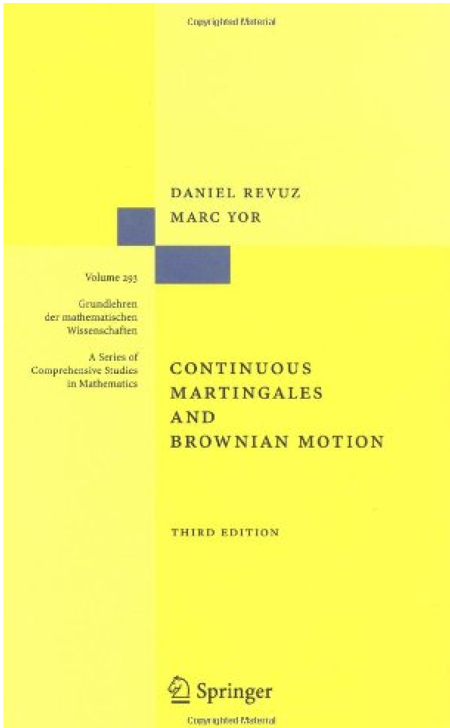
$$= \lim_{n\to\infty}[\mathcal{W}_{\xi_n}(f) - \mathcal{W}_{\xi_n}(0)] = \inf_K G^{f,\infty,\exp}(K)$$

with functional

$$G^{f,\infty,\exp}(K) = \mathbb{E}\left[f(W+Z+K) + \underbrace{\int \exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^{\beta} + \mathcal{E}(K)}_{\geqslant 0}\right]$$

which depends via Z on the infinite volume measure for the exp interaction.

thanks!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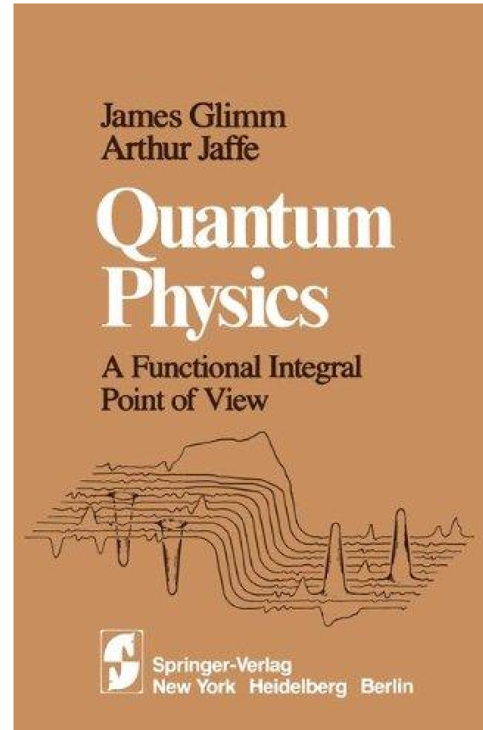
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(additional slides)

- **parabolic stochastic quantisation.** the parameter is an additional "fictious" coordinate $t \in \mathbb{R}$, playing the röle of a simulation time. The EQF is viewed as the invariant measure of a Markov process (SDE). Building block is a space-time white noise. [Parisi/Wu, Nelson, Jona-Lasinio/Mitter, Albeverio/Röckner, Da Prato/Debbusche, Hairer, Catellier/Chouk, Mourrat/Weber, G./Hofmanova, Albeverio/Kusuoka, Chandra/Moinat/Weber, Shen, Garban, many others...]

$$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **canonical stochastic quantisation.** same as for parabolic, but the evolution takes place in "phase space" and the SDE is second order in time, giving rise to a stochastic wave equation. [G./Koch/Oh, Tolomeo, Oh/Robert/Wang]

$$\partial_t^2 \phi + \partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **elliptic stochastic quantisation.** the parameter is a coordinate $z \in \mathbb{R}^2$. Building block is a white noise in $\mathbb{R}^{d+2}$. An elliptic stochastic partial differential equation describes the EQF as a function of the white noise. Link with supersymmetry.

  [Parisi/Sourlas, Klein/Landau/Perez, Albeverio/De Vecchi/G., Barashkov/De Vecchi]

$$-\Delta_z \phi(z,x) = \frac{1}{2}[(\Delta_x - m^2)\phi(z,x) - p'(\phi(z,x))] + 2^{1/2}\xi(z,x)$$

- **variational method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The EQF is described as the solution of a stochastic optimal control problem. [Barashkov/G.]
- **rg method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The effective action of the EQF satisfies an Hamilton–Jacobi–Bellmann equation. [Wilson, Wegner, Polchinski, Salmhofer, Brydges/Kennedy, Mitter, Gawedzki/Kupiainen, Brydges/Bauerschmidt/Slade, Bauerschmidt/Bodineau, Bauerschmidt/Hofstetter, also many others…]