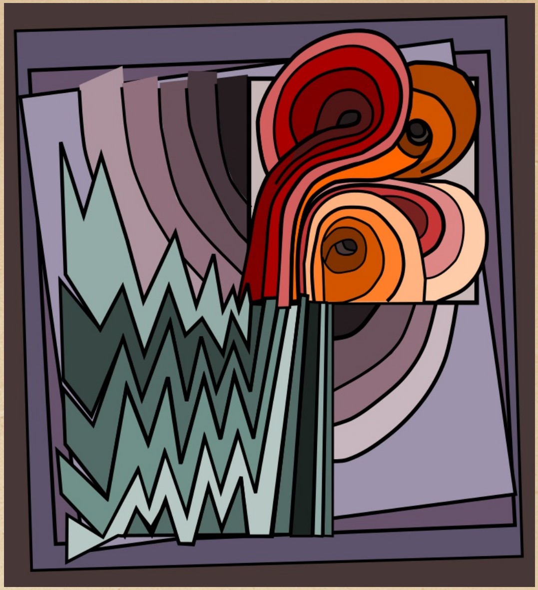
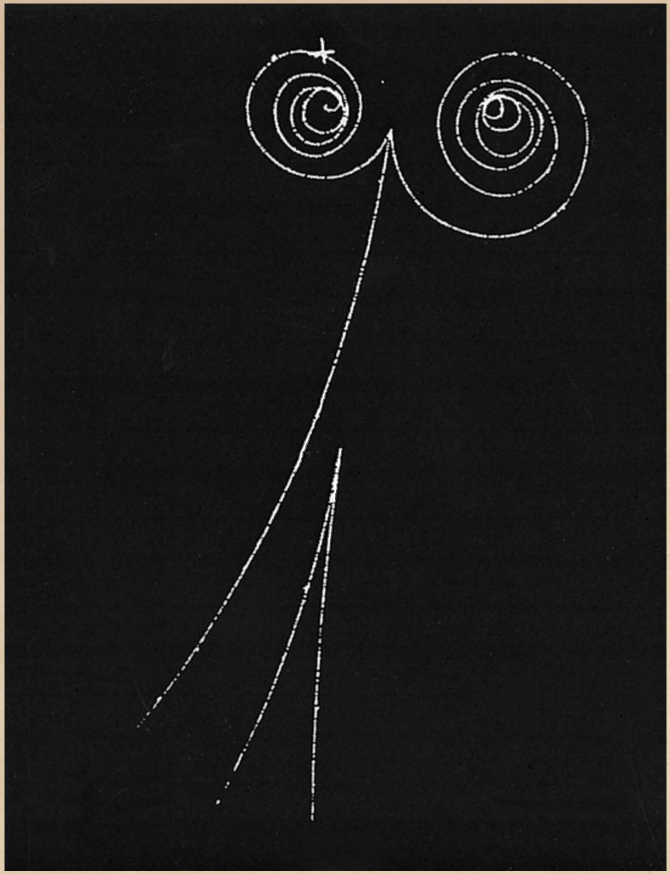
Singular SPDEs and Related Topics - HIM, Bonn - October 23rd 2019

Stochastic analysis of quantum fields



M. Gubinelli, University of Bonn



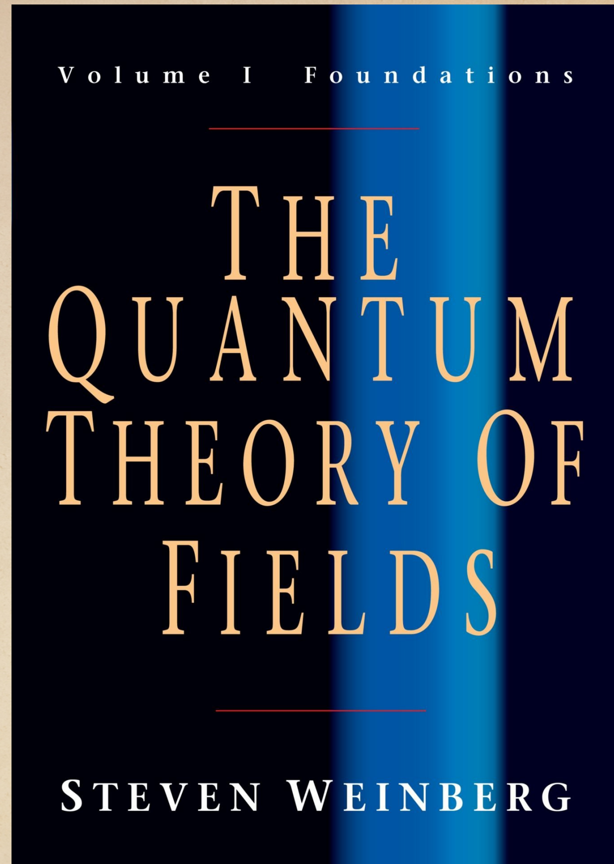
Quantum mechanics

+

Special relativity

=

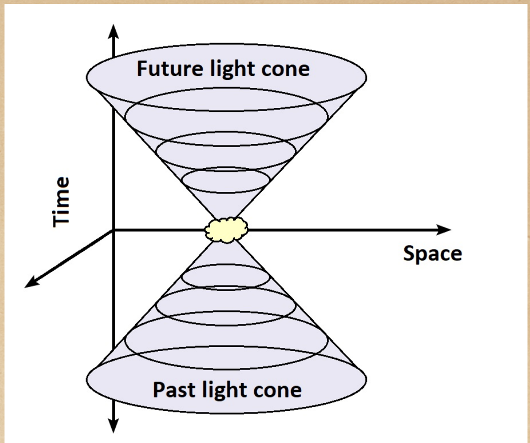
Quantum field theory


Volume I Foundations
THE QUANTUM THEORY OF FIELDS
STEVEN WEINBERG

QFT ≈ "noncommutative PDE theory"

$$(\partial_t^2 - \Delta)\Phi(z) = F(\Phi(z)), \qquad z \in \mathbb{R}^{3+1}$$

$$[\Phi(t,x), \dot{\Phi}(t,y)] = \delta(x-y)$$



Φ is an operator valued distribution on an Hilbert space:

$\Phi(f)$ is an unbounded self-adjoint operator for all test functions $f : \mathbb{R}^{3+1} \to \mathbb{R}$

Does a consistent theory of quantum fields exists?

- Axiomatic approach ('60): Wightman, Haag-Kastler.

  i. Hilbert space $\mathscr{H}$,

  ii. Positive energy representation $(U(\Lambda,a))_{(\Lambda,a)}$ of Poincaré group $G=\{(\Lambda,a)\}$

  iii. Fields $\varphi(x)$ transforming as $U(\Lambda,a)^*\varphi(x)U(\Lambda,a)=\varphi(\Lambda x+a)$, $x\in\mathbb{R}^{3+1}$

  iv. Vacuum vector $\Omega\in\mathscr{H}$: $U(\Lambda,a)\Omega=\Omega$

- Examples?

Progress in $\mathbb{R}^{1+1}$ (1965-1976): Nelson, Glimm, Jaffe, Segal…

- Schwinger

$$t \to it \qquad \Box \longrightarrow \Delta$$

- Symanzik – functional integral representation of Euclidean correlation function
- Nelson – reconstruction of the Hilbert space from Euclidean data and spatial Markov property
- Schwinger functions

$$S_n(f_1 \otimes \cdots \otimes f_n) := \int_{\mathscr{S}'(\mathbb{R}^{d+1})} \varphi(f_1)\cdots\varphi(f_n)\,\nu(\mathrm{d}\varphi).$$

Osterwalder-Schrader axioms

Distribution property – $\beta > 0$,

$$|S_n(f_1 \otimes \ldots \otimes f_n)| \leqslant (n!)^\beta \prod_{i=1}^{n} \|f_i\|_s. \qquad \forall n \geqslant 0, f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^3).$$

Euclidean invariance – $(a, R).f_n(x) = f_n(a + Rx)$, $(a, R) \in \mathbb{R}^3 \times \mathrm{O}(3)$

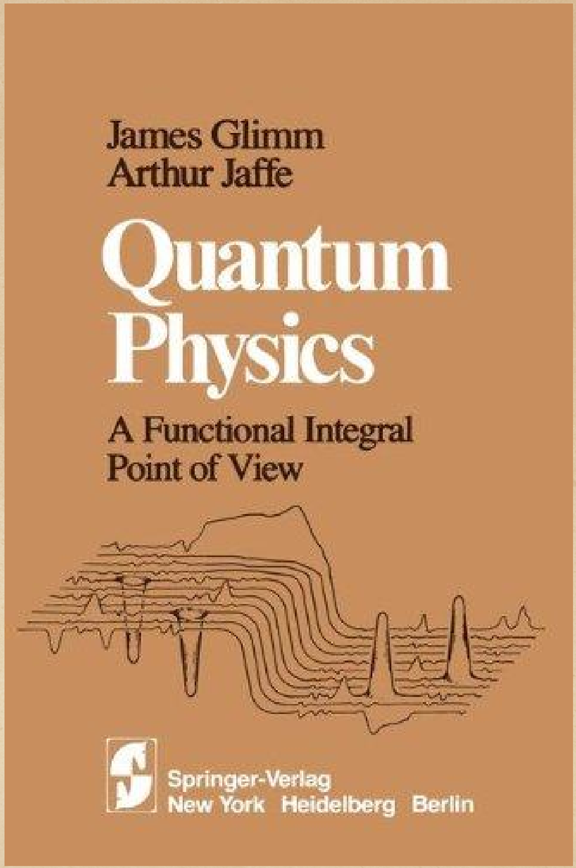
$$S_n((a,R).f_1 \otimes \ldots \otimes (a,R).f_n) = S_n(f_1 \otimes \ldots \otimes f_n),$$

Reflection positivity – $(f_n \in \mathscr{S}_{\mathbb{C}}(\mathbb{R}^{3n}_+))_{n \in \mathbb{N}_0}$ (with finitely many nonzero elements)

$$\sum_{n,m \in \mathbb{N}_0} S_{n+m}(\overline{\theta f_n} \otimes f_m) \geqslant 0,$$

Symmetry – $S_n(f_1 \otimes \cdots \otimes f_n) = S_n(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)})$

- ('70-'80) Glimm, Jaffe. Nelson. Segal. Guerra, Rosen, Simon, and many others…
- Construction of theories in $2+1$ dimensions
- $(\Phi^4_3)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75)
- $(\Phi^4_3)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)



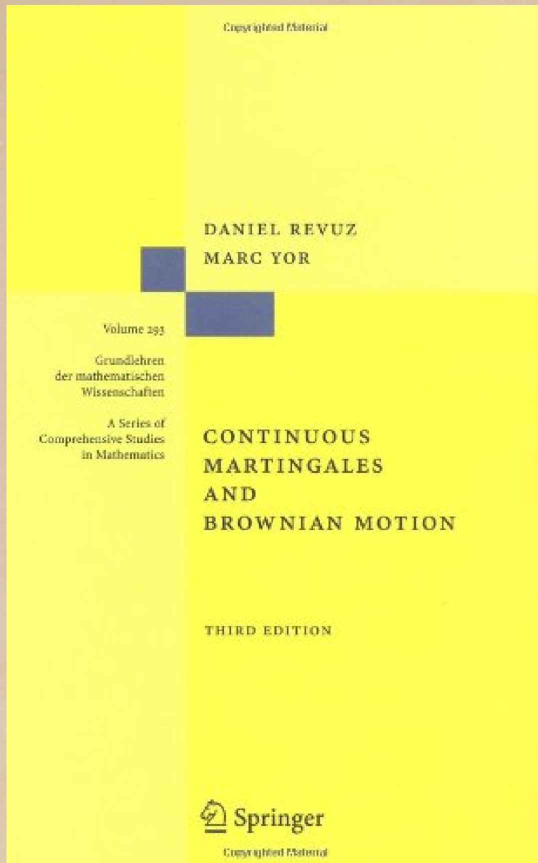
- *Other constructions of* Φ^4_3. Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)

Ito and Dœblin want to study diffusion processes via their *sample paths*

Measures	*Samples*
$(\mu_t)_t \subseteq \Pi(S)$	$X : \Omega \to C(\mathbb{R}_+, S)$
$\mu_t(\mathrm{d}y) = \int P_{t-s}(x, \mathrm{d}y)\mu_s(\mathrm{d}x)$	$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t$

Advantages

- lower dimensional problem
- more tools (e.g. fixpoint theorems)
- more intuition
- ***canonical*** *reference object* $(B_t)_t$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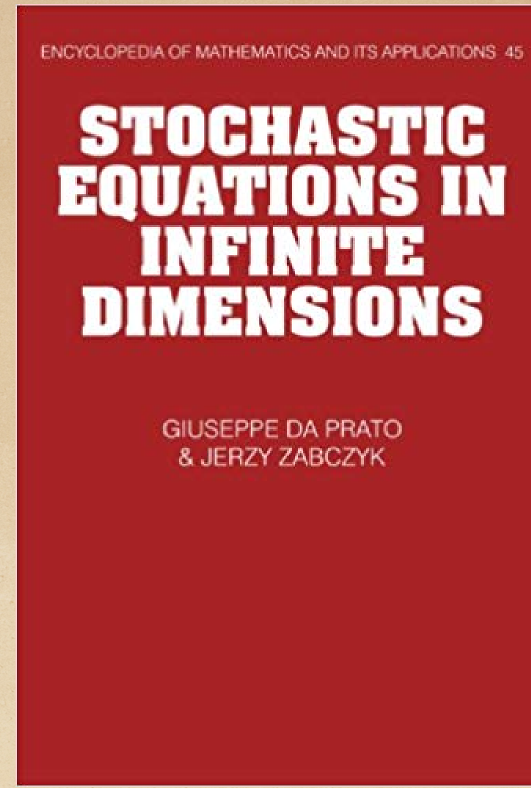

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 45
STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS
GIUSEPPE DA PRATO
& JERZY ZABCZYK

Relation between a stochastic differential equation and a probability measure

(broadly speaking)

- Nelson and Parisi–Wu ('84) advocated the *constructive* use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for the use of Euclidean quantum field theory (in particular gauge theories)
- Theoretical version of MCMC methods

(parabolic) stochastic quantisation

$\Lambda =$ finite set, $\mathbb{T}^d, \mathbb{R}^d$

equation	$\partial_t \phi(t) = -\frac{\delta V(\phi(t))}{\delta \phi} + \sqrt{2}\,\xi(t), \qquad \phi : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$
measure	$\phi(t) \sim \nu(\mathrm{d}\varphi) = \frac{e^{-V(\varphi)}}{Z}\mathrm{d}\varphi \in \mathrm{Prob}(\Lambda \to \mathbb{R})$

- ▷ The measure ν is described via *white noise*
- ▷ Markov process, invariant measures, ergodicity

dynamic Φ^4_d

$$V(\varphi)=\int \frac{1}{2}|\nabla \varphi|^2+\frac{m^2-\infty}{2}\varphi^2+\frac{\lambda}{4}\varphi^4.$$

$$\partial_t\varphi=\Delta\varphi-\lambda(\varphi^3-\infty\varphi)-m^2\varphi+\sqrt{2}\xi \qquad \mathbb{R}^3\times\mathbb{R}_+$$

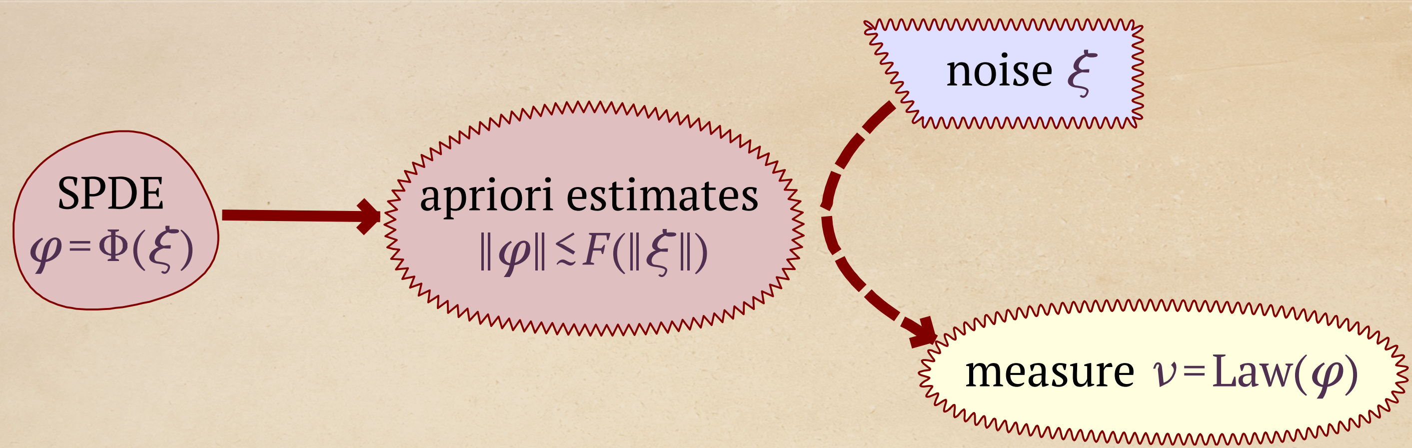
(d=2) Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Tsatsoulis, Weber ('16) Röckner, R.Zhu, X.Zhu ('17)

▹ $d=3$ is more singular: regularity structures (Hairer), paracontrolled distributions (G. Imkeller, Perkowski)

Hairer ('14) Kupiainen ('16) Catellier, Chouk ('17) Mourrat, Weber ('17) Hairer, Mattingly ('18) R.Zhu, X.Zhu ('18) Albeverio, Kusuoka ('18) G, Hofmanová ('18) Moinat, Weber ('18)

Reflection positivity + Euclidean invariance $\Rightarrow$ singularities, infinite volume limit

G. Hofmanová ('18) – construction of Φ^4_3 on $\mathbb{R}^3$ via stochastic quantisation and verification of (most of) the axioms.



- ▷ Much like Ito's approach to diffusions
- ▷ Markovianity does not play any role

canonical stochastic quantisation

equation

$$\begin{cases} \partial_t \phi(t) = -\dfrac{\delta H(\phi(t),\dot\phi(t))}{\delta \dot\phi} \\ \partial_t \dot\phi(t) = \underbrace{-\dfrac{\delta H(\phi(t),\dot\phi(t))}{\delta \phi}}_{\text{Hamiltonian dynamics}} \underbrace{-\gamma\dot\phi(t) + \sqrt{2}\,\xi(t),}_{\text{linear Langevin dynamics}} \end{cases} \quad \phi,\dot\phi : \mathbb{R}\times\Lambda \to \mathbb{R}$$

$$H(\varphi,\dot\varphi) := V(\varphi) + \frac{\gamma}{2}\dot\varphi^2$$

measure

$$(\phi(t),\dot\phi(t)) \sim \nu(\mathrm{d}\varphi\,\mathrm{d}\dot\varphi) = \frac{e^{-H(\varphi,\dot\varphi)}}{Z}\mathrm{d}\varphi\,\mathrm{d}\dot\varphi \in \mathrm{Prob}(\Lambda \to \mathbb{R}^2)$$

▷ Introduced by Ryang, Saito and Shigemoto ('85).

For Φ^4_d, $d=1,2,3$

$$V(\varphi)=\int\frac{1}{2}|\nabla\varphi|^2+\frac{m^2-\infty}{2}\varphi^2+\frac{\lambda}{4}\varphi^4,$$

$$\partial_t^2\phi=\Delta\phi+(m^2-\infty)\phi+\lambda\phi^3-\gamma\partial_t\phi+\sqrt{2}\,\xi,$$

Problem: no Schauder estimates, scaling arguments less clear.

Conjecture: *same* renormalization constants of the static measure!

▹ $d=1$. Tolomeo ('18) unique ergodicity.

▹ $d=2$. G, Koch, Oh ('18) local well-posedness (any polynomial), G, Koch, Oh, Tolomeo (in preparation) global well-posedness.

▹ $d=3$. G, Koch, Oh ('18) only quadratic nonlinearity.

elliptic stochastic quantisation

equation	$\Delta_z \phi(z) = -\dfrac{\delta V(\phi(z))}{\delta \phi} + \xi(z), \qquad \phi : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$
measure	$\phi(z) \sim \nu(\mathrm{d}\varphi) = \dfrac{e^{-4\pi V(\varphi)}}{Z} \mathrm{d}\varphi \in \mathrm{Prob}(\Lambda \to \mathbb{R})$

Discovered perturbatively by Imry, Ma ('75), Young ('77). Non–perturbative “proof” by Parisi and Sourlas ('79-'82) using *supersymmetry*

$$(\mathrm{SPDE})_{d+2} \xrightarrow[\text{“Girsanov”}]{} (\mathrm{SUSY\ EQFT})_{d+2} \xrightarrow[\text{dimensional reduction}]{} (\mathrm{measure})_d$$

the Gaussian case

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 \qquad \Delta_z \varphi(z) = -m^2 \varphi(z) + \xi(z), \qquad z \in \mathbb{R}^2$$

$$\varphi(z) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot z}}{|k|^2 + m^2} \frac{\eta(\mathrm{d}k)}{2\pi}$$

$$\mathbb{E}[\varphi(0)^2] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}k}{(|k|^2 + m^2)^2} = \frac{1}{(2\pi)^2 m^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}k}{(|k|^2 + 1)^2} = \frac{1}{4\pi m^2} \int_0^\infty \frac{\mathrm{d}\rho^2}{(\rho^2 + 1)^2} = \frac{1}{4\pi m^2}$$

$$\varphi(0) \sim e^{-4\pi \frac{m^2}{2} \phi^2} \mathrm{d}\phi \sim e^{-4\pi V(\phi)} \mathrm{d}\phi$$

- Rigorous proof of dimensional reduction by Klein, Landau and Perez ('84)

- Recently complete proof by Albeverio, G. and De Vecchi ('18). First for Λ finite dimensional + technical conditions. Then extended to (some) renormalized EQFT.

Stochastic quantisation of Liouville action up to the critical value of $\sigma^2 < 8\pi$ in $\Lambda = \mathbb{T}^2$

$$V(\varphi) = \int_{\mathbb{T}^2} \frac{1}{2} |\nabla \varphi|^2 + \alpha e^{\sigma\varphi - \sigma^2 \infty}$$

An alternative approach to induce an equation for Φ^4_3 in $\mathbb{T}^3$ [Barashkov, G. ('18)]

$$e^{-\mathscr{W}(f)} = \int_{\mathscr{S}'(\mathbb{T}^3)} e^{\langle f, \phi\rangle - \lambda\int \phi^4} \underbrace{e^{-\frac{1}{2}\int |\nabla \phi|^2 + a\phi^2} \mathrm{d}\phi}_{\text{Gaussian free field}} = \lim_{T\to\infty} \mathbb{E}[e^{\langle f, W_T\rangle - \lambda\int (W_T^4 - a_T W_T^2)}]$$

▹ W_T is a regularisation of the Gaussian free field at scale T:

$$\mathbb{E}[W_T(f) W_S(f)] = (T \wedge S)\langle f, \rho((-\Delta)^{1/2}/S)(1-\Delta)^{-1} g\rangle$$

$$W_t = \int_0^t J_t \mathrm{d}X_t$$

with $(X_t)_t$ cylindrical Wiener process in $L^2(\mathbb{T}^3)$.

Take $V_T(\phi) = \lambda \int (\phi^4 - a_T \phi^2 - b_T)$

$$\mathcal{W}_T(f) = -\log \mathbb{E}[e^{\langle f, W_T \rangle - V_T(W_T)}]$$

$$= \inf_u \mathbb{E}\left[\langle f, W_T + I_T(u) \rangle - V_T(W_T + I_T(u)) + \frac{1}{2} \int_0^\infty \|u_s^2\|_{L^2(\mathbb{T}^3)} \mathrm{d}s \right]$$

where

$$I_T(u) = \int_0^T J_t u_t \mathrm{d}t$$

and the infimum is over processes u adapted to the Wiener filtration.

$\exists a_T, b_T$ with $a_T, b_T \to \infty$ as $T \to \infty$, such that

$$\mathcal{W}_T(f) = \mathbb{E}\left[-\langle f, W_T + I_T(u)\rangle + \lambda V_T(W_T + I_T(u)) + \frac{1}{2}\|u\|^2_{\mathcal{H}}\right]$$
$$= \mathbb{E}\left[-f(W_T + I_T(u)) + \Phi_T(W, u) + \lambda \int (I_T(u))^4 + \frac{1}{2}\|l^T(u)\|^2_{\mathcal{H}}\right]$$

$$l_t^T(u) := u_t + \lambda \mathbb{1}_{t \leqslant T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t \leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(u)), \qquad \mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{T}^3),$$

$$|\mathbb{E}\Phi_T(W, u)| \leqslant \mathbb{E}Q(W) + \frac{1}{4}(\lambda\|I_T(u)\|^4_{L^4} + \|l^T(u)\|^2_{\mathcal{H}})$$

Barashkov, G. ('18)

Theorem

$$\mathcal{W}_\infty(f) := \liminf_{T\to\infty} \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[-f(W_T + I_T(u)) + \Phi_T(W,u) + \lambda\int (I_T(u))^4 + \frac{1}{2}\|l^T(u)\|^2_{\mathcal{H}}\right]$$

$$= \inf_{u\in\mathbb{H}_a^{-1/2-\varepsilon}} \mathbb{E}\left[-f(W_\infty + I_\infty(u)) + \Phi_\infty(W,u) + \lambda\int (I_\infty(u))^4 + \frac{1}{2}\|l^\infty(u)\|^2_{\mathcal{H}}\right]$$

$$\int e^f \,\mathrm{d}\Phi^4_3 = e^{\mathcal{W}_\infty(0) - \mathcal{W}_\infty(f)} = \lim_{T\to\infty} \int e^f \,\mathrm{d}\Phi^4_{3,T}.$$

▷ First “explicit” description of Φ^4_3

Thanks.