Weak universality of the stationary Kardar–Parisi–Zhang equation

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[Birthday cake for KPZ, Workshop "New approaches to non-equilibrium and random systems: KPZ integrability, universality, applications and experiments", Kavli Institute for Theoretical Physics, March 3rd 2016]

KPZ is the following SPDE

$$\partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{diffusion}} + \underbrace{\chi\,(\partial_x h(t,x))^2}_{\text{growth}} + \underbrace{\xi(t,x)}_{\text{noise}}, \qquad t \geqslant 0, x \in \mathbb{R}, \mathbb{T}$$

with ξ space–time white noise.

▷ KPZ introduced (30 years ago!) the equation in order to capture the universal macroscopic behaviour of the fluctuations h of growing interfaces.

▷ **KPZ fixpoint:** the KPZ equation is just an element of a wider universality class:

$$\varepsilon^1[h(t\varepsilon^{-3}, x\varepsilon^{-2}) - \varphi(t,x)] \to \mathcal{H}(t,x)$$

as $\varepsilon \to 0$. Difficult problem. Only known for fixed t and special $h(0,\cdot)$

[Amir–Corwin–Quastel 2011, Sasamoto–Spohn 2010, Borodin–Corwin 2014]

Talk based on joint work with: M. Jara, N. Perkowski and J. Diehl.

▷ The KPZ equation describes also fluctuations in a certain asymptotic regime where the non–linear effects are weak in the microscopic scale.

Basic ingredients:

- One conservation law for a quantity u_ε
- Tunable asymmetry of order ε

Universal limit:

$$\varepsilon^{-1}(u_\varepsilon(t\varepsilon^{-4}, x\varepsilon^{-2}) - \varphi(t,x)) \longrightarrow u(t,x)$$

where u solves the Stochastic Burgers equation (SBE)

$$\partial_t u(t,x) = \Delta u(t,x) + \chi \partial_x u(t,x)^2 + \partial_x \xi(t,x)$$

Equivalent to KPZ with $u = \partial_x h$: u_ε represent height gradient for an interface.

▷ **Weakly asymmetric simple exclusion process.** Particles jumps with Poisson clocks on sites of the lattice $\mathbb{Z}$, no two particles at the same site. Leftward with rate $1/2+\alpha$ and rightward with rate $1/2-\alpha$. Number of particle is locally conserved.

▷ **Ginzburg–Landau $\nabla\varphi$ model.** Interacting Brownian motions on $\mathbb{Z}$:

$$\mathrm{d}X^i = ((1/2+\alpha)\,V'(X^{i+1}-X^i) - (1/2-\alpha)V'(X^i - X^{i-1}))\mathrm{d}t + \mathrm{d}B^i_t, \qquad i\in\mathbb{Z}$$

▷ **Hairer–Quastel model.** SPDE:

$$\partial_t g(t,x) = \Delta g(t,x) + \alpha F(\partial_x g(t,x)) + \eta(t,x) \qquad x\in\mathbb{R}$$

where η is a short range/short memory gaussian process.

$\alpha = 0 \Rightarrow$ convergence to Gaussian fluctuations

$\alpha = \varepsilon \Rightarrow$ convergence to KPZ

[Bertini–Giacomin 1996 (WASEP), Hairer–Quastel 2015]

Energy solutions: a good notion of martingale solutions to SBE which allows to prove the weak KPZ universality conjecture for a large class of **stationary** models.

▷ [Gonçalves–Jara 2010/2014] Initial notion of energy solutions

▷ [Jara–G. 2013] Refined notion of energy solutions

▷ [G.–Perkowski 2015] Uniqueness for the refined notion

Proofs of weak universality from energy solutions:

▷ General exclusion processes. [Gonçalves–Jara 2014, Gonçalves–Jara–Simon 2016, Franco-Gonçalves-Simon 2016]

▷ Zero–range processes and many other particle systems. [Gonçalves-Jara-Sethuraman 2015]

▷ Ginzburg–Landau $\nabla\varphi$ model. [Diehl-G.-Perkowski 2016?]

▷ Hairer–Quastel model. [G.-Perkowski 2016]

▷ [Bertini–Giacomin 1996]: existence of random function h describing the scaling limit of the fluctuation of WASEP for which $\varphi = e^h$ satisfies the Stochastic Heat Equation (SHE)

$$\mathscr{L}\varphi(t,x) = \varphi(t,x)\xi(t,x), \qquad t \geqslant 0, x \in \mathbb{R}.$$

▷ **Cole–Hopf:** transformation (ξ_ε a regularisation of ξ) , $\varphi_\varepsilon = e^{h_\varepsilon}$

$$\mathscr{L}\varphi_\varepsilon(t,x) = \varphi_\varepsilon(t,x)\xi_\varepsilon(t,x) - C_\varepsilon\varphi_\varepsilon(t,x)$$
$$\Updownarrow$$
$$\mathscr{L}h_\varepsilon(t,x) = (\partial_x h_\varepsilon(t,x))^2 - C_\varepsilon + \xi_\varepsilon(t,x)$$

No equation!

▷ Intrinsic notions of solution:

- Energy solutions [Gonçalves–Jara 10/14] : weak notion, global in time solutions. Uniqueness established only recently [G.–Perkowski 2015].
- Rough paths [Hairer 2013] : strong notion, local solutions, uniqueness / stability.
- Regularity structures [Hairer 2014], paracontrolled distributions [G.–Perkowski 2015].

▷ **Cole–Hopf:** Not a general approach to universality, needs a specific structure, especially at the microscopic level. Only OK for specific models. [Bertini-Giacomin 1997, Dembo-Tsai 2013, Corwin-Tsai 2015, Corwin–Shen–Tsai 2016].

▷ **Rough–paths:** (but also Regularity structures or Paracontrolled distributions) need control of regularity, universality so far only for semilinear SPDEs (and on the torus). [Hairer–Quastel 2015, Hairer–Shen 2015, G.–Perkowski 2015]

▷ **Energy solutions:** requires precise knowledge of the invariant measure but otherwise quite flexible and powerful (and works easily on $\mathbb{R}$).

Approach to weak universality:

- tightness of fluctuations
- martingale characterization of limit points;
- uniqueness $\Rightarrow$ convergence.

Definition 1 (Jara–Gonçalves, 2010) *u is an **energy solution** of SBE if*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathcal{B}_t(\varphi)$$

is a martingale with bracket $[M(\varphi)]_t = t\|\partial_x\varphi\|_{L^2}^2$ and if

$$\mathbb{E}|\mathcal{B}_{s,t}(\varphi) - \mathcal{B}^{\varepsilon}_{s,t}(\varphi)|^2 \leqslant C\varepsilon|t-s|\|\partial_x\varphi\|_{L^2}^2 \qquad \textit{(energy condition)}$$

where

$$\mathcal{B}^{\varepsilon}_{s,t}(\varphi) = \int_s^t \partial_x(\rho_\varepsilon * u_s)^2(\varphi)\mathrm{d}s$$

and $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\varepsilon^{-1}x)$.

▷ An energy solution is given by a **pair** $(u, \mathcal{B})$. Very little information about $\mathcal{B}$. As a consequence, energy solutions are too weak to be compared meaningfully.

Jara–G. introduced another notion of energy solution

Definition 2 (Jara–G. 2013) $(u, \mathcal{A})$ *is a* ***controlled process*** *if*

1. *(Dirichlet)* $u_t(\varphi)$ *is a Dirichlet process with*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathcal{A}_t(\varphi)$$

is a martingale with bracket $[M(\varphi)]_t = t\|\partial_x\varphi\|_{L^2}^2$ *and* $[\mathcal{A}(\varphi)] = 0$.

2. *(Stationarity)* u_t *is a white noise for all* t*;*
3. *(Time–reversal)* $\overleftarrow{u}_t = u_{T-t}$ *satisfies* $1.$ *with* $\overleftarrow{\mathcal{A}}_t(\varphi) = \mathcal{A}_T(\varphi) - \mathcal{A}_{T-t}(\varphi)$.

▷ **Key property:** For controlled processes we can define and estimate efficiently additive functionals of the form

$$\int_0^t f(u_s)\mathrm{d}s.$$

Let $\mathscr{L}_{\mathrm{OU}}$ be the generator of the OU process X given by $\mathscr{L}X = \mathrm{D}\xi$. Itô formula for Dirichlet processes [Russo–Vallois], forward first:

$$F(u_T) = F(u_0) + \int_0^T \nabla F(u_s)\mathrm{d}M_s + \int_0^T \nabla F(u_s)\mathrm{d}\mathcal{A}_s + \int_0^T \mathscr{L}_{\mathrm{OU}} F(u_s)\mathrm{d}s$$

and then backward:

$$F(\overleftarrow{u}_T) = F(\overleftarrow{u}_0) + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{\mathcal{A}}_s + \int_0^T \mathscr{L}_{\mathrm{OU}} F(\overleftarrow{u}_s)\mathrm{d}s.$$

Summing and using BDG inequalities :

$$2\int_0^T \mathscr{L}_{\mathrm{OU}} F(u_s)\mathrm{d}s = -\int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s - \int_0^T \nabla F(u_s)\mathrm{d}M_s$$

$$\mathbb{E}\left|\int_0^T \mathscr{L}_{\mathrm{OU}} F(u_s)\mathrm{d}s\right|^p \lesssim_p T^{p/2}\,\mathbb{E}[\mathcal{E}_{\mathrm{OU}}(F)^{p/2}]$$

Result: powerful control of additive functionals of controlled processes.

[forward–backward Itô trick, Kipnis–Varadhan 1986, Chang–Landim–Olla 2001].

Lemma 3 *If $(u,\mathcal{A})$ is controlled then*

$$\mathcal{B}_t(\varphi) := \lim_{\varepsilon\to 0}\mathcal{B}_t^\varepsilon(\varphi) = \lim_{\varepsilon\to 0}\int_s^t \partial_x(\rho_\varepsilon * u_s)^2(\varphi)\mathrm{d}s$$

with good estimates on space–time regularity (e.g. zero quadratic variation).

Definition 4 (Jara–G. 2013) *A controlled process $(u,\mathcal{A})$ is a stationary solution to SBE if*

$$\mathcal{A} = \mathcal{B}.$$

▷ Existence is proved via stationary Galerkin approximations u^N. The forward–backward Itô trick gives tightness for the approximate drift $\mathcal{B}^N$.

▷ Not difficult to show that particle systems converge to limits satisfying this alternative notion.

▷ This notion of solution is more powerful since brings along all the information about estimations of additive functionals, not only of $\mathcal{B}$.

Theorem 5 (G.–Perkowski, 2015) *There exists only one controlled energy solution, in particular it coincides with the Cole–Hopf solution.*

The proof uses a key estimate from [Funaki–Quastel 2014].

Let $(u,\mathcal{A})$ be an energy solution and let $u^\varepsilon = \rho_\varepsilon * u$. Then u^ε satisfies

$$\mathrm{d}u_t^\varepsilon(x) = \Delta u_t^\varepsilon(x)\mathrm{d}t + (\rho_\varepsilon * \mathrm{d}\mathcal{A}_t)(x) + (\rho_\varepsilon * \mathrm{d}M_t)(x)$$

Consider $\varphi_t^\varepsilon(x) = e^{h_t^\varepsilon(x)}$ where $\partial_x h_t^\varepsilon(x) = u_t^\varepsilon(x)$. Then

$$\mathrm{d}\varphi_t^\varepsilon(x) = e^{h_t^\varepsilon(x)}(\Delta h_t^\varepsilon(x)\mathrm{d}t + c_\varepsilon \mathrm{d}t + \partial_x^{-1}(\rho_\varepsilon * \mathrm{d}\mathcal{A}_t)(x) + \partial_x^{-1}(\rho_\varepsilon * \mathrm{d}M_t)(x))$$

$$= \Delta\varphi_t^\varepsilon(x)\mathrm{d}t + \varphi_t^\varepsilon(x)(Q_t^\varepsilon + K^\varepsilon)\mathrm{d}t + \varphi_t^\varepsilon(x)(\rho_\varepsilon * \mathrm{d}W_t)(x) + \mathrm{d}R_t^\varepsilon(\varphi)$$

$$R_t^\varepsilon(\varphi) = \int_0^t (\varphi_s^\varepsilon(x)\partial_x^{-1}(\rho_\varepsilon * \mathrm{d}\mathcal{A}_s)(x) - \varphi_s^\varepsilon(x)\Pi_0(u_s^\varepsilon(x))^2 \mathrm{d}s - K^\varepsilon \mathrm{d}s), \quad Q_t^\varepsilon = \int_{\mathbb{T}} ((u_s^\varepsilon(x))^2 - c_\varepsilon)\mathrm{d}x.$$

If we show that $R_t^\varepsilon(\varphi) \to 0$ then $\varphi^\varepsilon \to \varphi$ solution to a tilted SHE which is unique.

We approximate R^ε as

$$R_t^{\varepsilon,\delta}(\varphi) = \int_0^t (-K_\varepsilon \mathrm{d}s + \varphi_s^\varepsilon(x)\partial_x^{-1}(\rho_\varepsilon * \mathrm{d}\mathcal{B}_s^\delta)(x) - \varphi_s^\varepsilon(x)(u_s^\varepsilon(x))^2 \mathrm{d}s)$$

$$= \int_0^t \{-K_\varepsilon + e^{\partial_x^{-1} u_s^\varepsilon(x)}[\rho_\varepsilon * (\rho_\delta * u_s)^2 - (\rho_\varepsilon * u_s)^2](x)\}\mathrm{d}t = \int_0^t f_{\varepsilon,\delta}(u_s)\mathrm{d}s$$

So we use the forward–backward Itô trick to get an L^2 estimate

$$\mathbb{E}|R_t^{\varepsilon,\delta}(\varphi)|^2 \lesssim t\, \| f_{\varepsilon,\delta} \|_{\mathcal{H}^{-1}}^2$$

where $\mathcal{H}^{-1}$ is the Sobolev space associated to the OU generator.

Following the strategy in Funaki–Quastel a detailed computation shows that there exists a choice for $K_\varepsilon \to K = -1/2$ for which

$$\| f_{\varepsilon,\delta} \|_{\mathcal{H}^{-1}}^2 = \sup_{\Phi} [2\mathbb{E}(f_{\varepsilon,\delta}\Phi) - \|\Phi\|_{\mathcal{H}^1}^2] \to 0.$$

It is enough to show that $|\mathbb{E}(f_{\varepsilon,\delta}\Phi)| \leqslant o(1)\|\Phi\|_{\mathcal{H}^1}$.

Consider the stochastic PDE

$$\partial_t v = \Delta v + \varepsilon^{1/2} \partial_x F(v) + \partial_x \chi^\varepsilon$$

on $[0,\infty) \times \mathbb{T}_\varepsilon$ with $\mathbb{T}_\varepsilon = \mathbb{R}/(2\pi\varepsilon^{-1}\mathbb{Z})$, where χ^ε is a Gaussian noise that is white in time and spatially smooth. We modify the equation such that after rescaling $\tilde{u}_t^\varepsilon(x) = \varepsilon^{-1/2} v_{t\varepsilon^{-2}}(x\varepsilon^{-1})$ we have

$$\partial_t \tilde{u}^\varepsilon = \Delta \tilde{u}^\varepsilon + \varepsilon^{-1} \partial_x \Pi_0^N F(\varepsilon^{1/2} \tilde{u}^\varepsilon) + \partial_x \Pi_0^N \tilde{\xi}, \qquad \tilde{u}_0^\varepsilon = \Pi_0^N \eta, \tag{1}$$

where $\tilde{\xi}$ is a space-time white noise on $[0,\infty) \times \mathbb{T}$ (where $\mathbb{T} = \mathbb{T}_1$) with variance 2, η is a space white noise which is independent of $\tilde{\xi}$, Π_0^N denotes the projection onto the Fourier modes $0 < |k| \leqslant N$, and $N = \pi/\varepsilon$.

Theorem *Assume that $F, F' \in L^2(\nu)$ where ν is the standard normal distribution. Then $u_t^\varepsilon(x) := \tilde{u}_t^\varepsilon(x - \varepsilon^{-1/2} c_1(F)\, t)$, $(t,x) \in [0,T] \times \mathbb{T}$, converges in distribution to the unique stationary energy solution u of*

$$\partial_t u = \Delta u + c_2(F)\, \partial_x u^2 + \partial_x \xi,$$

where ξ is a space-time white noise and for $U \sim \nu$ and $c_k(F) = \frac{1}{k!} \mathbb{E}[F(U) H_k(U)]$.

Galileian transformation. Performing the change of variables $u_t^{\varepsilon}(x) = \tilde{u}_t^{\varepsilon}(x - \varepsilon^{-1/2} c_1(F)\, t)$ Itô formula shows that u^{ε} solves

$$\partial_t u^{\varepsilon} = \Delta u^{\varepsilon} + \varepsilon^{-1} \partial_x \Pi_0^N (F(\varepsilon^{1/2} u^{\varepsilon}) - c_1(F)\, \varepsilon^{1/2} u^{\varepsilon}) + \partial_x \Pi_0^N \xi, \qquad u_0^{\varepsilon} = \Pi_0^N \eta, \tag{2}$$

so we replaced the function F by $\tilde{F}(x) = F(x) - c_1(F)\, x$.

Proposition **(Boltzmann–Gibbs principle)** *Let $G, G' \in L^2(\nu)$ Then for all $\ell \in \mathbb{Z}$ and $0 \leqslant s < t \leqslant s+1$ and all $\kappa > 0$*

$$\mathbb{E}\left[\left| \int_s^t \langle \varepsilon^{-1} \partial_x \Pi_0^N G(\varepsilon^{1/2} u_r^{\varepsilon}) - \varepsilon^{-1/2} c_1(G)\, \partial_x \Pi_0^N u_r^{\varepsilon}, e_{-\ell} \rangle \, \mathrm{d} r \right|^2\right]$$
$$\lesssim |t-s|^{3/2-\kappa} \ell^2 \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x)$$

uniformly in $N \in \mathbb{N}$, and for all $M \leqslant N/2$

$$\mathbb{E}\left[\left| \int_s^t \langle \varepsilon^{-1} \partial_x \Pi_0^N G(\varepsilon^{1/2} u_r^{\varepsilon}) - \varepsilon^{-1/2} c_1(G)\, \partial_x \Pi_0^N u_r^{\varepsilon} - c_2(G)\, \partial_x (\Pi_0^M u_r^{\varepsilon})^2, e_{-\ell} \rangle \, \mathrm{d} r \right|^2\right]$$
$$\lesssim |t-s|\, \ell^2 (M^{-1} + \varepsilon \log^2 N) \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x).$$

Thanks!