Energy solutions and weak KPZ universality

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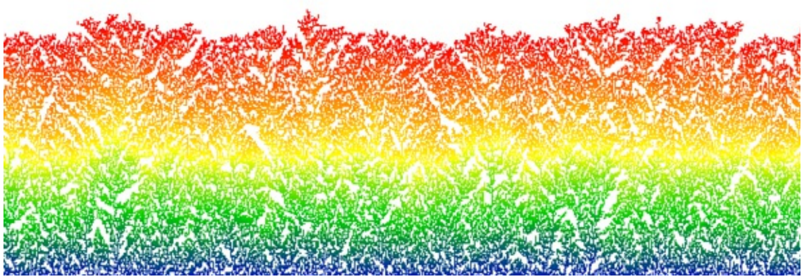
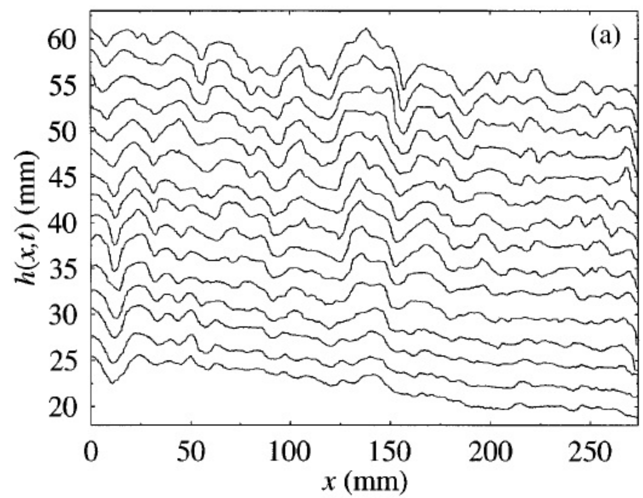
KPZ [Kardar–Parisi–Zhang '86] is the following SPDE

$$\partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{diffusion}} + \underbrace{\chi\,(\partial_x h(t,x))^2}_{\text{growth}} + \underbrace{\xi(t,x)}_{\text{noise}}, \qquad t \geqslant 0, x \in \mathbb{R}, \mathbb{T}$$

with ξ space–time white noise.





▹ KPZ introduced the equation in order to capture the universal macroscopic behaviour of the fluctuations h of growing interfaces.

Talk based on joint work with: M. Jara, N. Perkowski and J. Diehl.

▷ *Strong conjecture*: the KPZ equation is just an element of a wider universality class:

$$\varepsilon^1[h(t\varepsilon^{-3}, x\varepsilon^{-2}) - \varphi(t,x)] \to \mathcal{H}(t,x) \qquad \textbf{(KPZ fixpoint)}$$

as $\varepsilon \to 0$. Difficult problem. Only known for fixed t and special $h(0,\cdot)$

[Borodin, Corwin, Ferrari, Quastel, Sasamoto, Spohn and *many* others]

▷ *Gaussian fluctuations:* when asymmetric non–linear effects are "small"

$$\varepsilon^1 h_\varepsilon(t\varepsilon^{-4}, x\varepsilon^{-2}) \longrightarrow \text{Gaussian random field}$$

[Spohn '86, Kipnis-Olla-Varadhan '89, Zhu '92, Chang-Yau '92,...]

▷ *Weak conjecture:* the KPZ equation itself describes height fluctuations for interface models where the asymmetric non–linear effects are of order ε:

$$\varepsilon^1(h_\varepsilon(t\varepsilon^{-4}, x\varepsilon^{-2}) - \varphi(t,x)) \longrightarrow h(t,x)$$

▹ **Weakly asymmetric simple exclusion process.** Particles jumps with Poisson clocks on sites of the lattice $\mathbb{Z}$, no two particles at the same site. Leftward with rate $1/2+\alpha$ and rightward with rate $1/2-\alpha$. Number of particle is locally conserved.

▹ **Ginzburg–Landau $\nabla\varphi$ model.** Interacting Brownian motions on $\mathbb{Z}$:

$$\mathrm{d}X^i = ((1/2+\alpha)V'(X^{i+1}-X^i) - (1/2-\alpha)V'(X^i - X^{i-1}))\mathrm{d}t + \mathrm{d}B^i_t, \qquad i \in \mathbb{Z}$$

▹ **Hairer–Quastel model.**

$$\partial_t g(t,x) = \Delta g(t,x) + \alpha F(\partial_x g(t,x)) + \eta(t,x) \qquad x \in \mathbb{R}$$

where η is a short range/short memory gaussian process.

$\alpha = 0 \Rightarrow$ convergence to Gaussian fluctuations

$\alpha = \varepsilon \Rightarrow$ convergence to KPZ

[Bertini–Giacomin '96 (WASEP), Hairer–Quastel '15]

Stochastic Burgers equation (SBE)

$$u = \partial_x h \qquad \partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi$$

Energy solutions: a good notion of martingale solutions to SBE which allows to prove the weak KPZ universality conjecture for a large class of **stationary** models.

- [Assing '02] *Generalized* martingale problem for SBE ($u = \partial_x h$)
- [Gonçalves–Jara '10/'14] Initial notion of energy solutions (also [Assing '11])
- [Jara–G. '13] Refined notion of energy solutions (forward–backward trick)
- [G.–Perkowski '15] Uniqueness for the refined notion

Proofs of weak universality from energy solutions:

- General exclusion processes. [Gonçalves–Jara '14, Gonçalves–Jara–Simon '16, Franco-Gonçalves-Simon '16]
- Zero–range processes and many other particle systems. [Gonçalves-Jara-Sethuraman '15]
- Ginzburg–Landau $\nabla\varphi$ model. [Diehl-G.-Perkowski '16]
- Hairer–Quastel model. [G.-Perkowski '16]

▷ [Bertini–Giacomin '96]: existence of random function h describing the scaling limit of the fluctuation of WASEP for which $\varphi = e^h$ satisfies the Stochastic Heat Equation (SHE)

$$\mathscr{L}\varphi(t,x) = \varphi(t,x)\xi(t,x), \qquad t \geqslant 0, x \in \mathbb{R}.$$

▷ **Cole–Hopf:** transformation ($\xi_\varepsilon = \rho_\varepsilon * \xi$ a regularisation of ξ) , $\varphi_\varepsilon = e^{h_\varepsilon}$

$$\mathscr{L}\varphi_\varepsilon(t,x) = \varphi_\varepsilon(t,x)\xi_\varepsilon(t,x) - C_\varepsilon\varphi_\varepsilon(t,x)$$
$$\Updownarrow$$
$$\mathscr{L}h_\varepsilon(t,x) = (\partial_x h_\varepsilon(t,x))^2 - C_\varepsilon + \xi_\varepsilon(t,x)$$

No equation!

▷ Intrinsic notions of solution:

- Martingale/Energy solutions [Assing '04, Gonçalves–Jara '10] : weak notion, global in time solutions. Uniqueness established only recently [G.–Perkowski '15].
- Rough paths [Hairer '13] : strong notion, local solutions, uniqueness / stability.
- Regularity structures [Hairer '14], paracontrolled distributions [G.–Imkeller–Perkowski '14, G.–Perkowski '15].

▷ **Cole–Hopf:** Not a general approach to universality, needs a specific structure, especially at the microscopic level. Only OK for specific models.

[Bertini-Giacomin ’97, Dembo-Tsai ’13, Corwin-Tsai ’15, Corwin–Shen–Tsai ’16].

▷ **Rough–paths:** (but also Regularity structures or Paracontrolled distributions) need control of regularity, universality so far only for semilinear SPDEs (and on the torus).

[Hairer–Quastel ’15, Hairer–Shen ’15, G.–Perkowski ’15]

▷ **Energy solutions:** requires precise knowledge of the invariant measure but otherwise quite flexible and powerful (and works easily on $\mathbb{R}$).

Approach to weak universality:

- tightness of fluctuations
- martingale characterization of limit points;
- uniqueness $\Rightarrow$ convergence.

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi$$

Definition (Jara–Gonçalves, '10) *u is an **energy solution** of SBE if*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathscr{B}_t(\varphi)$$

is a martingale with bracket $[M(\varphi)]_t = t\|\partial_x\varphi\|_{L^2}^2$ *and if*

$$\mathbb{E}\,|\,\mathscr{B}_{s,t}(\varphi) - \mathscr{B}^{\varepsilon}_{s,t}(\varphi)\,|^2 \leqslant C\,\varepsilon\,|\,t-s\,|\,\|\partial_x\varphi\|_{L^2}^2 \qquad \textit{(energy condition)}$$

where

$$\mathscr{B}^{\varepsilon}_{s,t}(\varphi) = \int_s^t \partial_x(\rho_\varepsilon * u_s)^2(\varphi)\mathrm{d}s$$

and $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\varepsilon^{-1}x)$.

▹ An energy solution is given by a **pair** $(u, \mathscr{B})$. Very little information about $\mathscr{B}$. As a consequence, energy solutions are too weak to be compared meaningfully.

Jara–G. introduced another notion of energy solution

Definition (Jara–G. '13) $(u, \mathcal{A})$ *is a* ***controlled process*** *if*

1. *(Dirichlet)* $u_t(\varphi)$ *is a Dirichlet process with*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathcal{A}_t(\varphi)$$

   is a martingale with bracket $[M(\varphi)]_t = t\|\partial_x\varphi\|_{L^2}^2$ *and* $[\mathcal{A}(\varphi)] = 0$.

2. *(Stationarity)* u_t *is a white noise for all* t*;*
3. *(Time–reversal)* $\overleftarrow{u}_t = u_{T-t}$ *satisfies* 1. *with* $\overleftarrow{\mathcal{A}}_t(\varphi) = \mathcal{A}_T(\varphi) - \mathcal{A}_{T-t}(\varphi)$.

▹ **Key property:** For controlled processes we can define and estimate efficiently additive functionals of the form

$$\int_0^t f(u_s)\mathrm{d}s.$$

Let $\mathscr{L}_0$ be the generator of the OU process X given by $\mathscr{L}_0 X = \mathrm{D}\xi$. Itô formula for Dirichlet processes [Russo–Vallois], forward first:

$$F(u_T) = F(u_0) + \int_0^T \nabla F(u_s)\mathrm{d}M_s + \int_0^T \nabla F(u_s)\mathrm{d}\mathscr{A}_s + \int_0^T \mathscr{L}_0 F(u_s)\mathrm{d}s$$

and then backward:

$$F(\overleftarrow{u}_T) = F(\overleftarrow{u}_0) + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{\mathscr{A}}_s + \int_0^T \mathscr{L}_0 F(\overleftarrow{u}_s)\mathrm{d}s.$$

Summing and using BDG inequalities :

$$2\int_0^T \mathscr{L}_0 F(u_s)\mathrm{d}s = -\int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s - \int_0^T \nabla F(u_s)\mathrm{d}M_s$$

$$\mathbb{E}\left|\int_0^T \mathscr{L}_0 F(u_s)\mathrm{d}s\right|^p \lesssim_p T^{p/2}\,\mathbb{E}[\mathscr{E}_0(F)(u_0)^{p/2}]$$

$$\mathbb{E}\left|\int_0^T G(u_s)\mathrm{d}s\right|^2 \lesssim T\,\|G\|^2_{\mathscr{H}^{-1}} \qquad \|G\|_{\mathscr{H}^{-1}} = \sup_F\,[2\mathbb{E}[F(u_0)\,G(u_0)] - \mathbb{E}[\mathscr{E}_0(F)(u_0)]]$$

Result: control of additive functionals of controlled processes.

[Forward–backward Itô trick, Kipnis–Varadhan ’86, Chang–Landim–Olla ’01]

Lemma *If $(u, \mathscr{A})$ is controlled then*

$$\mathscr{B}_t(\varphi) := \lim_{\varepsilon \to 0} \mathscr{B}^{\varepsilon}_t(\varphi) = \lim_{\varepsilon \to 0} \int_s^t \partial_x(\rho_\varepsilon * u_s)^2(\varphi) \mathrm{d}s$$

with good estimates on space–time regularity (e.g. zero quadratic variation).

Definition (Jara–G. 2013) *A controlled process $(u, \mathscr{A})$ is a stationary solution to SBE if*

$$\mathscr{A} = \mathscr{B}.$$

▹ Existence is proved via stationary Galerkin approximations u^N. The forward–backward Itô trick gives tightness for the approximate drift $\mathscr{B}^N$.

▹ Not difficult to show that particle systems converge to limits satisfying this alternative notion.

▹ This notion of solution is more powerful since brings along all the information about estimations of additive functionals, not only of $\mathscr{B}$.

Theorem (G.–Perkowski, '15) *There exists only one controlled energy solution, in particular it coincides with the Cole–Hopf solution.*

The proof uses a key estimate from [Funaki–Quastel '14]. Works on $\mathbb{R}, \mathbb{T}$.

Let $(u, \mathcal{A})$ be an energy solution and let $u^{\varepsilon} = \rho_{\varepsilon} * u$. Then u^{ε} satisfies

$$\mathrm{d}u_t^{\varepsilon}(x) = \Delta u_t^{\varepsilon}(x)\mathrm{d}t + (\rho_{\varepsilon} * \mathrm{d}\mathcal{A}_t)(x) + (\rho_{\varepsilon} * \mathrm{d}M_t)(x)$$

Consider $\varphi_t^{\varepsilon}(x) = e^{h_t^{\varepsilon}(x)}$ where $\partial_x h_t^{\varepsilon}(x) = u_t^{\varepsilon}(x)$. Then

$$\mathrm{d}\varphi_t^{\varepsilon}(x) = e^{h_t^{\varepsilon}(x)}(\Delta h_t^{\varepsilon}(x)\mathrm{d}t + c_{\varepsilon}\mathrm{d}t + \partial_x^{-1}(\rho_{\varepsilon} * \mathrm{d}\mathcal{A}_t)(x) + \partial_x^{-1}(\rho_{\varepsilon} * \mathrm{d}M_t)(x))$$

$$= \Delta\varphi_t^{\varepsilon}(x)\mathrm{d}t + \varphi_t^{\varepsilon}(x)(Q_t^{\varepsilon} + K^{\varepsilon})\mathrm{d}t + \varphi_t^{\varepsilon}(x)(\rho_{\varepsilon} * \mathrm{d}W_t)(x) + \mathrm{d}R_t^{\varepsilon}(\varphi)$$

$$R_t^{\varepsilon}(\varphi) = \int_0^t (\varphi_s^{\varepsilon}(x)\partial_x^{-1}(\rho_{\varepsilon} * \mathrm{d}\mathcal{A}_s)(x) - \varphi_s^{\varepsilon}(x)\Pi_0(u_s^{\varepsilon}(x))^2\mathrm{d}s - K^{\varepsilon}\mathrm{d}s), \quad Q_t^{\varepsilon} = \int_{\mathbb{T}}((u_s^{\varepsilon}(x))^2 - c_{\varepsilon})\mathrm{d}x.$$

If we show that $R_t^{\varepsilon}(\varphi) \to 0$ then $\varphi^{\varepsilon} \to \varphi$ solution to a tilted SHE which is unique.

We approximate R^ε as

$$R_t^{\varepsilon,\delta}(\varphi) = \int_0^t (-K_\varepsilon \mathrm{d}s + \varphi_s^\varepsilon(x)\partial_x^{-1}(\rho_\varepsilon * \mathrm{d}\mathscr{B}_s^\delta)(x) - \varphi_s^\varepsilon(x)(u_s^\varepsilon(x))^2 \mathrm{d}s)$$

$$= \int_0^t \{-K_\varepsilon + e^{\partial_x^{-1} u_s^\varepsilon(x)}[\rho_\varepsilon * (\rho_\delta * u_s)^2 - (\rho_\varepsilon * u_s)^2](x)\}\mathrm{d}t = \int_0^t f_{\varepsilon,\delta}(u_s)\mathrm{d}s$$

So we use the forward–backward Itô trick to get an L^2 estimate

$$\mathbb{E}\,|R_t^{\varepsilon,\delta}(\varphi)|^2 \lesssim t\,\|f_{\varepsilon,\delta}\|_{\mathscr{H}^{-1}}^2$$

where $\mathscr{H}^{-1}$ is the Sobolev space associated to the OU generator.

Following the strategy in Funaki–Quastel a detailed computation shows that there exists a choice for $K_\varepsilon \to K = -1/12$ for which

$$\|f_{\varepsilon,\delta}\|_{\mathscr{H}^{-1}}^2 = \sup_\Phi [2\mathbb{E}(f_{\varepsilon,\delta}\Phi) - \|\Phi\|_{\mathscr{H}^1}^2] \to 0.$$

It is enough to show that $|\mathbb{E}(f_{\varepsilon,\delta}\Phi)| \leqslant o(1)\|\Phi\|_{\mathscr{H}^1}$.

Consider the stochastic PDE

$$\partial_t v = \Delta v + \varepsilon^{1/2} \partial_x (\rho^{*2} * F(v)) + \partial_x (\rho * \xi)$$

on $[0, \infty) \times \mathbb{T}_\varepsilon$ with $\mathbb{T}_\varepsilon = \varepsilon^{-1}\mathbb{T}$. Rescaling $u^\varepsilon(t,x) = \varepsilon^{-1/2} v(t\varepsilon^{-2}, x\varepsilon^{-1})$ solves

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-1} \partial_x [\rho_\varepsilon^{*2} * F(\varepsilon^{1/2} u^\varepsilon)] + \partial_x (\rho_\varepsilon * \xi)$$

Theorem (G.-Perkowski 2015) *Assume that $F, F' \in L^2(\nu)$ where ν is the standard normal distribution. Then*

$$\tilde{u}_t^\varepsilon(x) = u_t^\varepsilon(x - \varepsilon^{-1/2} c_1(F) t),$$

converges in distribution to the unique stationary energy solution u of

$$\partial_t u = \Delta u + c_2(F) \partial_x u^2 + \partial_x \xi,$$

where $c_k(F) = \frac{1}{k!} E[F(U) H_k(U)]$ for $U \sim \nu$.

Galileian transformation. Itô formula shows that $\tilde{u}^{\varepsilon}_t(x) = u^{\varepsilon}_t(x - \varepsilon^{-1/2} c_1(F) t)$ solves

$$\partial_t \tilde{u}^{\varepsilon} = \Delta \tilde{u}^{\varepsilon} + \varepsilon^{-1} \partial_x \rho_{\varepsilon}^{*2} * (F(\varepsilon^{1/2} \tilde{u}^{\varepsilon}) - c_1(F) \varepsilon^{1/2} \tilde{u}^{\varepsilon}) + \partial_x \rho_{\varepsilon} * \xi$$

so we replaced the function F by $\tilde{F}(x) = F(x) - c_1(F) x$.

Proposition **(Boltzmann–Gibbs principle)** *Let $G, G' \in L^2(\nu)$. For $\kappa > 0$*

$$\mathbb{E}[\,|\int_s^t \langle \varepsilon^{-1} \partial_x \rho_{\varepsilon}^{*2} * [G(\varepsilon^{1/2} u^{\varepsilon}_r) - c_1(G) \varepsilon^{1/2} \tilde{u}^{\varepsilon}_r], \varphi \rangle \, \mathrm{d}r\,|^2]$$

$$\lesssim_G |t-s|^{3/2-\kappa} \|\varphi'\|^2$$

uniformly in $\varepsilon > 0$, and for all $\delta > \varepsilon$

$$\mathbb{E}[\,|\int_s^t \langle \varepsilon^{-1} \partial_x \rho_{\varepsilon}^{*2} * [G(\varepsilon^{1/2} u^{\varepsilon}_r) - c_1(G) \varepsilon^{1/2} u^{\varepsilon}_r - c_2(G)(\varepsilon^{1/2} \rho_{\delta} * u^{\varepsilon}_r)^2], \varphi \rangle \mathrm{d}r\,|^2]$$
$$\lesssim_G |t-s| \, \|\varphi'\|^2 (\delta + \varepsilon \log^2(1/\varepsilon))$$

Easy proof based on Itô trick and chaos expansion. For other models the proof goes via a multiscale strategy developed by Jara and Gonçalves ('14).

Thanks!