Two controlled paths to the KPZ equation

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KPZ is the following SPDE

$$\mathscr{L}h(t,x) = \chi(\mathrm{D}h(t,x))^2 - \infty + \xi(t,x), \qquad t \geqslant 0, x \in \mathbb{R}, \mathbb{T}$$

with ξ space–time white noise and $\mathscr{L} = \partial_t - \Delta$.

▷ KPZ introduced (in the 80's) the equation in order to capture the universal macroscopic behaviour of the fluctuations h of growing interfaces. In this respect KPZ is just an element of a wider universality class.

▷ The KPZ equation is believed to describe also these fluctuations in a certain asymptotic regime where the non–linear effects are weak in the microscopic scale.

▷ Bertini–Giacomin ('96) proved that existence of random function h describing the scaling limit of the fluctuation of WASEP for which $\varphi = e^h$ satisfies the Stochastic Heat Equation (SHE)

$$\mathscr{L}\varphi(t,x) = \varphi(t,x)\xi(t,x), \qquad t \geqslant 0, x \in \mathbb{R}$$

where the r.h.s. is defined as an Ito integral with respect to the Brownian sheet (à la Walsh).

▷ Cole–Hopf transformation (ξ_ε a regularisation of ξ) , $\varphi_\varepsilon = e^{h_\varepsilon}$

$$\mathscr{L}\varphi_\varepsilon(t,x) = \varphi_\varepsilon(t,x)\xi_\varepsilon(t,x) - C_\varepsilon\varphi_\varepsilon(t,x)$$
$$\Updownarrow$$
$$\mathscr{L}h_\varepsilon(t,x) = (\mathrm{D}h_\varepsilon(t,x))^2 - C_\varepsilon + \xi_\varepsilon(t,x)$$

Not a general approach to universality, needs a specific structure, especially at the microscopic level.

▷ Intrinsic notions of solution:

- Energy solutions (Jara–Gonçalves) : weak notion, global in time solutions, easy to estabilish, no uniqueness.
- Rough paths (Hairer) : strong notion, local solutions, uniqueness / stability.

First part of the talk is about Hairer's approach, reloaded in the paracontrolled setting.

Second part is about a new result of uniqueness for energy solutions.

Talk based on joint work with: N. Perkowski.

KPZ is ok, but Burgers is more convenient. Let $u = \mathrm{D}h$

$$\mathscr{L}u = \mathrm{D}(u^2) + \mathrm{D}\xi, \qquad t \geqslant 0, x \in \mathbb{T}.$$

Hairer's approach is based on a partial expansion of the solution. Make the change of variables

$$u^Q = u - (X + X^{\vee} + 2X^{\curlyvee})$$

where $X, X^{\vee}, X^{\curlyvee}$ solve

$$\mathscr{L}X = \xi, \quad \mathscr{L}X^{\vee} = \mathrm{D}(X^2), \quad \mathscr{L}X^{\curlyvee} = \mathrm{D}(XX^{\vee}), \quad \mathscr{L}X^{\vee\vee} = \mathrm{D}(X^{\vee})^2, \quad \mathscr{L}X^{\curlyvee\!\!\vee} = \mathrm{D}\big(XX^{\curlyvee}\big).$$

Then

$$\mathscr{L}u^Q = \underbrace{2\mathrm{D}[(X + X^{\vee})(2X^{\curlyvee} + u^Q)]}_{\text{not defined}} + \mathrm{D}X^{\vee} + \mathrm{D}(2X^{\curlyvee} + u^Q)^2$$

Regularity $(C\mathscr{C}^{\alpha} = C([0,T]; B^{\alpha}_{\infty,\infty})$.

$$X \in C\mathscr{C}^{-1/2-}, \quad X^{\vee} \in C\mathscr{C}^{-0-}, \quad X^{\curlyvee}, X^{\curlyvee\!\!\vee} \in C\mathscr{C}^{1/2-}, \quad X^{\vee\vee} \in C\mathscr{C}^{1-}$$

Decomposition of a product into *paraproducts* and *resonant term*

$$fg = f \prec g + f \circ g + f \succ g$$

Theorem (Bony, Meyer)

$$(f,g) \in \mathscr{C}^{\alpha} \times \mathscr{C}^{\beta} \to f \prec g = g \succ f \in \mathscr{C}^{\beta+\alpha\wedge 0}, \qquad \alpha,\beta \in \mathbb{R}\setminus\mathbb{N}$$

$$(f,g) \in \mathscr{C}^{\alpha} \times \mathscr{C}^{\beta} \to f \circ g \in \mathscr{C}^{\alpha+\beta}, \qquad \alpha+\beta > 0$$

Paralinearization:

$$f \in \mathscr{C}^{\alpha} \to R(f) = G(f) - G'(f) \prec f \in \mathscr{C}^{2\alpha}, \qquad \alpha > 0$$

A single new *key* ingredient:

Lemma (G-Imkeller-Perkowski 2012)

$$(f,g,h) \in \mathscr{C}^{\alpha} \times \mathscr{C}^{\beta} \times \mathscr{C}^{\gamma} \to C(f,g,h) = (f \prec g) \circ h - f(g \circ h) \in \mathscr{C}^{\alpha+\beta+\gamma}, \qquad \alpha+\beta+\gamma > 0$$

$$\mathscr{L}u^Q = 2\mathrm{D}[X(2X^{\curlyvee}+u^Q)] + 2\mathrm{D}[X^{\vee}(2X^{\curlyvee}+u^Q)] + \mathrm{D}(X^{\vee})^2 + \mathrm{D}(2X^{\curlyvee}+u^Q)^2$$

$$X(2X^{\curlyvee}+u^Q) = (2X^{\curlyvee}+u^Q) \prec X + X \circ (2X^{\curlyvee}+u^Q) + X \prec (2X^{\curlyvee}+u^Q)$$

Note that

$$\mathscr{L}u^Q = 2(2X^{\curlyvee}+u^Q) \prec \mathrm{D}X + C\mathscr{C}^{-1-}$$

from which we can deduce that u^Q has a paracontrolled structure

$$u^Q = 2(2X^{\curlyvee}+u^Q) \prec Q + C\mathscr{C}^{1-}, \qquad \mathscr{L}Q = \mathrm{D}X.$$

With $Q \in C\mathscr{C}^{1/2-}$. The commutator lemma then gives

$$X \circ (2X^{\curlyvee}+u^Q) = 2X \circ X^{\curlyvee} + 2((2X^{\curlyvee}+u^Q) \prec Q) \circ X + C\mathscr{C}^{1-} \circ X$$

$$= 2X \circ X^{\curlyvee} + 2(2X^{\curlyvee}+u^Q)(Q \circ X) + 2C(2X^{\curlyvee}+u^Q, Q, X) + C\mathscr{C}^{1-} \circ X$$

We *assume* that $Q \circ X \in C\mathscr{C}^{0-}$. Then

$$\mathscr{L} u^{Q} = 2(2X^{\curlyvee} + u^{Q}) \prec X + \mathscr{L}(4X^{\curlyvee_{\circ}} + X^{\vee\vee}) + 4(2X^{\curlyvee} + u^{Q})(Q \circ X) + C\mathscr{C}^{-1-}$$

where $\mathscr{L} X^{\curlyvee_{\circ}} = \mathrm{D}(X \circ X^{\curlyvee})$ with $X^{\curlyvee_{\circ}} \in C\mathscr{C}^{1-}$. Let

$$\mathbb{X} = (X, X^{\vee}, X^{\curlyvee}, 4X^{\curlyvee_{\circ}} + X^{\vee\vee}, Q \circ X)$$

the extendend noise (rough path / model). A final change of variables gives

$$u = X + X^{\vee} + 2X^{\curlyvee} + u^{Q}, \qquad u^{Q} = 2(2X^{\curlyvee} + u^{Q}) \prec Q + u^{\#}$$

with $u^{\#} \in C\mathscr{C}^{1-}$ and satisfying

$$\mathscr{L} u^{\#} = \mathscr{L}(4X^{\curlyvee_{\circ}} + X^{\vee\vee}) + 4(2X^{\curlyvee} + u^{Q})(Q \circ X) + C\mathscr{C}^{-1-}$$

▷ Fixpoint equation for $u^{\#}$.

Paracontrolled distributions. Let $(f, f^x) \in \mathcal{Q}_{\mathrm{rbe}}(\mathbb{X}) \subseteq C\mathscr{C}^{-1/2-} \times C\mathscr{C}^{1/2-}$ if

$$f = X + X^{\vee} + 2X^{\vee\!\vee} + f^Q, \qquad f^\# = f^Q - f^x \prec Q \in C\mathscr{C}^{1-}$$

▷ The non–linear term

$$\mathrm{D}f^2 = \mathscr{L}(X^{\vee} + 2X^{\vee\!\vee} + 4X^{\vee\!\vee\!\vee} + X^{\vee\vee}) + 2(2X^{\vee\!\vee} + f^Q) \prec Q + 2f^x(Q \circ X) + C\mathscr{C}^{-1-}$$

is well defined for all $(f, f^x) \in \mathcal{Q}_{\mathrm{rbe}}(\mathbb{X})$.

Theorem *(Local) existence, uniqueness and stability of* $(u, u^x) \in \mathcal{Q}_{\mathrm{rbe}}(\mathbb{X})$ *satisfying*

$$\mathscr{L}u = \mathrm{D}u^2 + \xi$$

with $u^x = 2(2X^{\vee\!\vee} + u^Q)$. *Continuous solution map* $\Psi: (u_0, \mathbb{X}) \mapsto (u, u^x)$, *in particular if* $\mathbb{X}_\varepsilon \to \mathbb{X}$ *then*

$$u_\varepsilon \to u.$$

▷ Related results about paracontrolled solutions to KPZ and RHE (SHE).

Let h be a solution to KPZ with smooth noise θ and let $\overleftarrow{h}(t) = h(T-t)$ and B a Brownian motion of variance 2, then

$$(\partial_t + \Delta)\overleftarrow{h} = -(\mathrm{D}\overleftarrow{h})^2 + c(\theta) - \overleftarrow{\theta}, \qquad \overleftarrow{h}(T) = h_0$$

▹ Ito formula gives

$$\overleftarrow{h}(0,x) - \int_0^T \mathrm{D}\overleftarrow{h}(s, x+B_s)\mathrm{d}B_s - \int_0^T (\mathrm{D}\overleftarrow{h})^2(s, x+B_s)\mathrm{d}s$$

$$= \overleftarrow{h}(T, x+B_T) + \int_0^T [\overleftarrow{\theta}(s, x+B_s) - c(\theta)]\mathrm{d}s = -F(B)$$

Theorem (Dubué–Dupuis, Üstunel) *Let* $\gamma_t^v = x + B_t + \int_0^t v_s \,\mathrm{d}s$ *then*

$$-\log \mathbb{E}[e^{-F(B)}] = \inf_v \mathbb{E}\left[F(\gamma_\cdot^v) + \frac{1}{4}\int_0^T |v_s|^2 \,\mathrm{d}s\right].$$

$$-h(T,x) = -\overleftarrow{h}(0,x) = -\log \mathbb{E}[e^{-F(B)}] = \inf_v \mathbb{E}\left\{-h_0(\gamma_T^v) + \int_0^T [-\overleftarrow{\theta}(s, \gamma_s^v) + c(\theta) + \frac{|v_s|^2}{4}]\mathrm{d}s\right\}$$

$$h(T,x)=\sup_v \mathbb{E}[\underbrace{h_0(\gamma_T^v)+\int_0^T(\overleftarrow{\theta}(s,\gamma_s^v)-c(\theta)-\frac{|v_s|^2}{4})\mathrm{d}s]}_{\Phi(\gamma^v)}$$

▷ Use that $(\partial_t+\Delta)\overleftarrow{Y}=-\overleftarrow{\theta}$ and Itô formula to have

$$\overleftarrow{Y}(T,\gamma_T)=\overleftarrow{Y}(0,\gamma_0)+\int_0^T(v_s\mathrm{D}\overleftarrow{Y}-\overleftarrow{\theta})(s,\gamma_s^v)\mathrm{d}s+\mathrm{mart}$$

$$\Phi(\gamma^v)=h_0(\gamma_T^v)-\overleftarrow{Y}(T,\gamma_T)+\overleftarrow{Y}(0,\gamma_0)-\int_0^T[-v_s\mathrm{D}\overleftarrow{Y}+c(\theta)+\frac{|v_s|^2}{4}](s,\gamma_s^v)\mathrm{d}s+\mathrm{mart}$$

$$=h_0(\gamma_T^v)-\overleftarrow{Y}(T,\gamma_T^v)+\overleftarrow{Y}(0,\gamma_0^v)-\int_0^T[-(\mathrm{D}\overleftarrow{Y})^2+c(\theta)+\frac{|v_s-2\mathrm{D}\overleftarrow{Y}|^2}{4}](s,\gamma_s^v)\mathrm{d}s+\mathrm{mart}$$

▷ We obtain a new form of the optimization problem (with $v_s=2\mathrm{D}\overleftarrow{Y}+v_s^1$) :

$$h(T,x)=\sup_{v^1}\mathbb{E}\left[h_0(\gamma_T^v)-\overleftarrow{Y}(T,\gamma_T^v)+\overleftarrow{Y}(0,\gamma_0^v)+\int_0^T[(\mathrm{D}\overleftarrow{Y})^2-c(\theta)-\frac{|v_s^1|^2}{4}](s,\gamma_s^v)\mathrm{d}s\right]$$

Note that θ disappeared. Define now $(\partial_t+\Delta)\overleftarrow{Y}^{\vee}=-(\mathrm{D}\overleftarrow{Y})^2-c(\theta)$ and iterate...

Theorem *For smooth θ we have*

$$(h - Y - Y^{\vee} - Y^R)(T,x) = \sup_v \mathbb{E}\left[h_0(\zeta_T^v) - Y(0,\zeta_T^v) + \int_0^T (|\mathrm{D}\overleftarrow{Y}^R|^2 - \frac{1}{4}|v - 2\mathrm{D}\overleftarrow{Y}^R|^2)(s,\zeta_s^v)\mathrm{d}s \right]$$

where $\mathrm{D}Y^{\tau} = X^{\tau}$, $\zeta_t^v = x + \int_0^t (2\overleftarrow{X} + 2\overleftarrow{X}^{\vee} + v)(v,\zeta_s^v)\mathrm{d}s + B_t$ *and*

$$\mathscr{L}Y^R = (X^{\vee})^2 + 2XX^{\vee} + 2(X + X^{\vee})\mathrm{D}Y^R, \qquad Y^R(0) = 0.$$

▷ The equation for Y^R is a linear paracontrolled equation with solution $Y^R \in C\mathscr{C}^{1/2-}$

$$Y^R = Y^{\vee\!\!\vee} + Y^x \prec\!\!\prec P + Y^{\#}, \qquad \mathscr{L}P = \theta.$$

▷ In particular, since $Y, Y^{\vee}, Y^R$ are bounded in $[0,T]\times\mathbb{T}$ we get uniform bounds for h:

$$h(T,x) \leqslant 2\|Y\|_{C_T L^\infty} + \|Y^{\vee}\|_{C_T L^\infty} + \|Y^R\|_{C_T L^\infty} + \|\mathrm{D}Y^R\|_{C_T L^\infty}^2 + \|h_0\|_{L^\infty}$$

and

$$-h(T,x) \leqslant 2\|Y\|_{C_T L^\infty} + \|Y^{\vee}\|_{C_T L^\infty} + \|Y^R\|_{C_T L^\infty} + \|h_0\|_{L^\infty}$$

▷ The uniform bound

$$\|h(T,\cdot)\|_{L^\infty} \leqslant K_T(\mathbb{Y}) + \|h_0\|_{L^\infty}$$

ensures global in time existence for solutions of KPZ. And solutions to the RHE

$$\mathscr{L}\varphi = \varphi\theta - c(\theta)\varphi, \qquad \varphi(0) = e^{h_0}$$

have a uniform *lower bound* which is strictly away from zero

$$\varphi(t,x) \geqslant e^{-\|h(t,\cdot)\|_{L^\infty}} \geqslant e^{-K_T(\mathbb{Y}) - \|h_0\|_{L^\infty}} > 0.$$

▷ *In particular* alternative proof of the strict positivity of the SHE started from strictly positive initial data (cfr. Müller).

▷ This property does not depends on the law of the driving noise θ (but on its regularity and enhancement $\mathbb{Y}$).

▷ Also a comparison principle for KPZ holds: for all $T \geqslant 0$

$$\|h^1 - h^2\|_{C_T L^\infty} \leqslant \|h_0^1 - h_0^2\|_{L^\infty}.$$

▷ The variational representation holds a priori only for smooth θ.

▷ Following Delarue–Diel we can however define the controlled diffusion

$$\zeta_t^v = x + \int_0^t (2\overleftarrow{X} + 2\overleftarrow{X}^{\vee} + v)(v, \zeta_s^v)\mathrm{d}s + B_t$$

as a solution of a martingale problem for more irregular θ (in particular ξ).

▷ The generator of ζ^0 is $\mathscr{G} = \Delta + 2(\overleftarrow{X} + \overleftarrow{X}^{\vee})\mathrm{D}$ and is possible to solve

$$(\partial_t + \mathscr{G})F = f, \qquad F(T)\,\text{given.}$$

as a paracontrolled equation for a large class of $(F(T), f)$ with $F \in C_T\mathscr{C}^{3/2-}$.

▷ A martingale solution of the controlled SDE is then a measure on trajectories $(\gamma_t)_{t\in[0,T]}$ such that

$$M_t^f = F(t, \gamma_t) - \int_0^t (f(s, \gamma_s) + v_s \mathrm{D}F(s, \gamma_s))\mathrm{d}s$$

is a martingale. This is enough to make the optimization problem work at the limit.

A general weakly asymmetric interface model

$$\begin{aligned} \mathrm{d}\,\varphi_N(t,x) &= \Delta_{\mathbb{Z}_N}\varphi_N(t,x)\,\mathrm{d}\,t + \sqrt{\varepsilon}\,(B_{\mathbb{Z}_N}(\mathrm{D}_{\mathbb{Z}_N}\varphi_N(t),\mathrm{D}_{\mathbb{Z}_N}\varphi_N(t)))(x)\,\mathrm{d}\,t + \mathrm{d}\,W_N(t,x),\\ \varphi_N(0,x) &= \varphi_0^N(x). \end{aligned}$$

Diffusive rescaling

$$u_N(t,x) = \varepsilon^{-1/2}\,\mathrm{D}_{\mathbb{Z}_N}\varphi_N\,(t/\varepsilon^2, x/\varepsilon).$$

This is a stochastic process on $\mathbb{R}_+ \times \mathbb{T}_N$ with $\mathbb{T}_N = (\varepsilon\mathbb{Z})/(2\pi\mathbb{Z})$ which solves the SDE

$$\begin{aligned} \mathrm{d}\,u_N(t,x) &= \Delta_N u_N(t,x)\,\mathrm{d}\,t + (\mathrm{D}_N B_N(u_N(t),u_N(t)))(x)\,\mathrm{d}\,t + \mathrm{d}\,(\mathrm{D}_N\,\varepsilon^{-1/2}\,W_N(t,x))\\ u_N(0) &= u_0^N. \end{aligned}$$

where

$$\Delta_N\,\varphi(x) = \varepsilon^{-2}\int_{\mathbb{Z}}\varphi(x+\varepsilon y)\,\pi(\mathrm{d}\,y), \qquad \mathrm{D}_N\,\varphi(x) = \varepsilon^{-1}\int_{\mathbb{Z}}\varphi(x+\varepsilon y)\,\nu(\mathrm{d}\,y),$$
$$B_N(\varphi,\psi)(x) = \int_{\mathbb{Z}^2}\varphi(x+\varepsilon y)\,\psi(x+\varepsilon z)\,\mu(\mathrm{d}\,y,\mathrm{d}\,z).$$

+ some moment conditions on π, ν, μ.

Theorem 1 *u_N converges in distribution in $C\mathscr{C}^{-1/2-}$ to the unique paracontrolled solution u of*

$$\mathscr{L}u = \mathrm{D}u^2 + 4c\,\mathrm{D}u + \mathrm{D}\xi, \qquad u(0) = u_0, \tag{1}$$

where ξ is a space-time white noise which is independent of u_0, and where

$$c = -\frac{1}{4\pi}\int_0^{\pi} \frac{\mathrm{Im}(g(x)\bar{h}(x))}{x} \frac{h(x,-x)|g(x)|^2}{|f(x)|^2}\mathrm{d}x \in \mathbb{R}.$$

Here

$$f(x) = \frac{\int_{\mathbb{Z}} e^{ixy}\pi(\mathrm{d}y)}{-x^2}, \quad g(x) = \frac{\int_{\mathbb{Z}} e^{ixy}\nu(\mathrm{d}y)}{ix}, \quad h(x_1,x_2) = \int_{\mathbb{Z}^2} e^{i(x_1 z_1 + x_2 z_2)}\mu(\mathrm{d}z_1,\mathrm{d}z_2).$$

If

$$\Delta_N f(x) = \varepsilon^{-2}(f(x+\varepsilon) + f(x-\varepsilon) - 2f(x)), \qquad \mathrm{D}_N f(x) = \varepsilon^{-1}(f(x) - f(x-\varepsilon))$$

$$B_N(\varphi,\psi)(x) = \varphi(x)\psi(x)$$

then $c = 1/8$.

▷ The less obvious choice

$$B_N(\varphi,\psi)(x) = \frac{1}{2(\kappa+\lambda)}(\kappa\,\varphi(x)\,\psi(x) + \lambda\,(\varphi(x)\,\psi(x+\varepsilon) + \varphi(x+\varepsilon)\,\psi(x)) + \kappa\,\varphi(x+\varepsilon)\,\psi(x+\varepsilon))$$

for some $\kappa,\lambda\in[0,\infty)$ with $\kappa+\lambda>0$ gives $c=0$.

▷ The Sasamoto–Spohn (2009) discretization corresponds to $\kappa = 1$, $\lambda = 1/2$. In that case one furthermore has

$$\sum_{x\in\mathbb{T}_N}\varphi(x)\,\mathrm{D}_N B_N(\varphi,\varphi)(x) = 0,$$

⇒ the existence of a family of stationary measures for u_N of the form

$$\mu^{\varepsilon,m}(\mathrm{d}x) = \prod_{j=0}^{N-1}\frac{\exp(-\varepsilon x_j^2 + m\,x_j)}{Z_m^{\varepsilon}}\mathrm{d}x_j.$$

⇒ the white noise is an invariant distribution for the stochastic Burgers equation. [To the best of our knowledge, ours is the first proof which does not rely on the Cole–Hopf transform, see Bertini–Giacomin (1996), Funaki–Quastel (2014)]

Another notion of solution for the SBE (but not only).

Definition 2 **(Jara–Gonçalves, 2010)** *u is an **energy solution** of SBE if*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathscr{B}_t(\varphi)$$

is a martingale with bracket $[M(\varphi)]_t = t\|\mathrm{D}\varphi\|_{L^2}^2$ *and if*

$$\mathbb{E}|\mathscr{B}_{s,t}(\varphi) - \mathscr{B}_{s,t}^{\varepsilon}(\varphi)|^2 \leqslant C\,\varepsilon|t-s|\|\mathrm{D}\varphi\|_{L^2}^2 \qquad \textit{(energy condition)}$$

where $\mathscr{B}_{s,t}^{\varepsilon}(\varphi) = \int_s^t \mathrm{D}(\rho_\varepsilon * u_s)^2(\varphi)\mathrm{d}s$ *and* $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\varepsilon^{-1}x)$.

▷ Jara and Gonçalves proved that a large class of weakly asymmetric simple exclusion models have fluctuations which converge to **stationary** energy solutions of SBE with fixed time marginal given by white noise.

▷ An energy solution is given by a **pair** $(u, \mathscr{B})$. Very little information about $\mathscr{A}$. As a result energy solutions are to weak to be compared meaningfully.

▷ Uniqueness is not obvious. Proved only recently (GP 2015).

Jara–G. introduced another notion of energy solution

Definition 3 (Jara–G. 2013) $(u,\mathscr{A})$ *is a* ***controlled process*** *if*

1. *(Dirichlet)* $u_t(\varphi)$ *is a Dirichlet process with*

$$M_t(\varphi) = u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \mathscr{A}_t(\varphi)$$

   is a martingale with bracket $[M(\varphi)]_t = t\|\mathrm{D}\varphi\|_{L^2}^2$ *and* $[\mathscr{A}(\varphi)] = 0$.

2. *(Stationarity)* u_t *is a white noise for all* t*;*
3. *(Time–reversal)* $\overleftarrow{u}_t = u_{T-t}$ *satisfies* 1. *with* $\overleftarrow{\mathscr{A}}_t(\varphi) = \mathscr{A}_T(\varphi) - \mathscr{A}_{T-t}(\varphi)$.

▷ For controlled processes we can define and control functionals of the form

$$\int_0^t f(u_s)\mathrm{d}s.$$

Assume that F solves the Poisson equation $\mathscr{L}_{\mathrm{OU}}F = f$ where $\mathscr{L}_{\mathrm{OU}}$ is the generator of the OU process X given by $\mathscr{L}X = \mathrm{D}\xi$. Then by the Itô formula for Dirichlet processes

$$F(u_t) = F(u_0) + \int_0^t \nabla F(u_s)\mathrm{d}M_s + \int_0^t \nabla F(u_s)\mathrm{d}\mathscr{A}_s + \int_0^t \mathscr{L}_{\mathrm{OU}}F(u_s)\mathrm{d}s$$

and

$$F(\overleftarrow{u}_T) = F(\overleftarrow{u}_0) + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s + \int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{\mathscr{A}}_s + \int_0^T \mathscr{L}_{\mathrm{OU}}F(\overleftarrow{u}_s)\mathrm{d}s$$

Summing we get

$$2\int_0^t \mathscr{L}_{\mathrm{OU}}F(u_s)\mathrm{d}s = -\int_0^T \nabla F(\overleftarrow{u}_s)\mathrm{d}\overleftarrow{M}_s - \int_0^t \nabla F(u_s)\mathrm{d}M_s$$

Then BDG inequalities give

$$\mathbb{E}\left|\int_0^T f(u_s)\mathrm{d}s\right|^p \lesssim_p T^{p/2}\,\mathbb{E}[\mathscr{E}_{\mathrm{OU}}(F)^{p/2}]$$

Gives a powerful control of additive functionals (Itô trick, Kipnis–Varadhan).

Lemma 4 *If $(u, \mathscr{A})$ is controlled then*

$$\mathscr{B}_t(\varphi) := \lim_{\varepsilon \to 0} \mathscr{B}_t^{\varepsilon}(\varphi)$$

with useful estimates.

Definition 5 (Jara,G. 2013) *A controlled process $(u, \mathscr{A})$ is a stationary solution to SBE if*

$$\mathscr{A} = \mathscr{B}.$$

▷ Existence is proved via stationary Galerkin approximations u^N. The Itô trick gives tightness for the approximate drift $\mathscr{B}^N$.

▷ Not difficult to show that particle systems converge to limits satisfying this notion too.

▷ This notion of solution is more powerful since brings along all the information about estimations of additive functionals, not only of $\mathscr{B}$.

Theorem 6 (G. Perkowski, 2015) *There exists only one controlled energy solutions are unique, in particular it coincides with the Cole–Hopf solution.*

The proof is quite easy, it uses a key estimate from Funaki–Quastel (2014).

Let $(u, \mathscr{A})$ be an energy solution and let $u^\varepsilon = \rho_\varepsilon * u$. Then u^ε satisfies

$$\mathrm{d}u_t^\varepsilon(x) = \Delta u_t^\varepsilon(x)\mathrm{d}t + (\rho_\varepsilon * \mathrm{d}\mathscr{A}_t)(x) + (\rho_\varepsilon * \mathrm{d}M_t)(x)$$

Consider $\varphi_t^\varepsilon(x) = e^{h_t^\varepsilon(x)}$ where $\mathrm{D}h_t^\varepsilon(x) = u_t^\varepsilon(x)$. Then

$$\mathrm{d}\varphi_t^\varepsilon(x) = e^{h_t^\varepsilon(x)}(\Delta h_t^\varepsilon(x)\mathrm{d}t + c_\varepsilon \mathrm{d}t + \mathrm{D}^{-1}(\rho_\varepsilon * \mathrm{d}\mathscr{A}_t)(x) + \mathrm{D}^{-1}(\rho_\varepsilon * \mathrm{d}M_t)(x))$$

$$= \Delta\varphi_t^\varepsilon(x)\mathrm{d}t + \varphi_t^\varepsilon(x)(Q_t^\varepsilon + K^\varepsilon)\mathrm{d}t + \varphi_t^\varepsilon(x)(\rho_\varepsilon * \mathrm{d}W_t)(x) + \mathrm{d}R_t^\varepsilon(\varphi)$$

$$R_t^\varepsilon(\varphi) = \int_0^t (\varphi_s^\varepsilon(x)\mathrm{D}^{-1}(\rho_\varepsilon * \mathrm{d}\mathscr{A}_s)(x) - \varphi_s^\varepsilon(x)\Pi_0(u_s^\varepsilon(x))^2\mathrm{d}s - K^\varepsilon\mathrm{d}s), \quad Q_t^\varepsilon = \int_{\mathbb{T}}((u_s^\varepsilon(x))^2 - c_\varepsilon)\mathrm{d}x.$$

If we show that $R_t^\varepsilon(\varphi) \to 0$ then $\varphi^\varepsilon \to \varphi$ solution to a tilted SHE which is unique.

We approximate R^ε as

$$R_t^{\varepsilon,\delta}(\varphi) = \int_0^t (-K_\varepsilon \mathrm{d}s + \varphi_s^\varepsilon(x) \mathrm{D}^{-1}(\rho_\varepsilon * \mathrm{d}\mathscr{B}_s^\delta)(x) - \varphi_s^\varepsilon(x)(u_s^\varepsilon(x))^2 \mathrm{d}s)$$

$$= \int_0^t (-K_\varepsilon + e^{\mathrm{D}^{-1} u_s^\varepsilon(x)} (\rho_\varepsilon * (\rho_\delta * u_s)^2 - (\rho_\varepsilon * u_s)^2)(x)) \mathrm{d}t = \int_0^t f_{\varepsilon,\delta}(u_s) \mathrm{d}s$$

So we use the forward–backward Itô trick to get an L^2 estimate

$$\mathbb{E}|R_t^{\varepsilon,\delta}(\varphi)|^2 \lesssim t\, \|f_{\varepsilon,\delta}\|_{\mathscr{H}^{-1}}^2$$

where $\mathscr{H}^{-1}$ is the Sobolev space associated to the OU generator.

Following the strategy in Funaki–Quastel a detailed computation shows that there exists a choiche for $K_\varepsilon \to K = -1/2$ for which

$$\|f_{\varepsilon,\delta}\|_{\mathscr{H}^{-1}}^2 = \sup_\Phi [2\mathbb{E}(f_{\varepsilon,\delta}\Phi) - \|\Phi\|_{\mathscr{H}^1}^2] \to 0.$$

It is enough to show that $|\mathbb{E}(f_{\varepsilon,\delta}\Phi)| \leqslant o(1) \|\Phi\|_{\mathscr{H}^1}$.

Thanks!