

# Paracontrolled distributions (with applications to singular SPDEs)

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## Some problems in singular SPDEs /I

Define and solve (locally) the following SPDEs:

- ▶ Stochastic differential equations (1+0):  $u \in [0, T] \rightarrow \mathbb{R}^n$

$$\partial_t u(t) = \sum_i f_i(u(t)) \xi^i(t)$$

with  $\xi : \mathbb{R} \rightarrow \mathbb{R}^m$   $m$ -dimensional white noise in time.

- ▶ Burgers equations (1+1):  $u \in [0, T] \times \mathbb{T} \rightarrow \mathbb{R}^n$

$$\partial_t u(t, x) = \Delta u(t, x) + f(u(t, x)) Du(t, x) + \xi(t, x)$$

with  $\xi : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}^n$  space-time white noise.

- ▶ Generalized Parabolic Anderson model (1+2):  $u \in [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$

$$\partial_t u(t, x) = \Delta u(t, x) + f(u(t, x)) \xi(x)$$

with  $\xi : \mathbb{T}^2 \rightarrow \mathbb{R}$  space white noise.

Recall that

$$\xi \in \mathcal{C}^{-d/2-}$$

## Some problems in singular SPDEs /II

Define and solve (locally) the following SPDEs:

- ▶ Kardar-Parisi-Zhang equation (1+1)

$$\partial_t h(t, x) = \Delta h(t, x) + "(Du(t, x))^2 - \infty" + \xi(t, x)$$

with  $\xi : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$  space-time white noise.

- ▶ Stochastic quantization equation (1+3)

$$\partial_t u(t, x) = \Delta u(t, x) + "u(t, x)^3" + \xi(t, x)$$

with  $\xi : \mathbb{R} \times \mathbb{T}^3 \rightarrow \mathbb{R}$  space-time white noise.

- ▶ But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u(t, x) = \Delta u(t, x) + f(u(t, x))\xi(t, x)$$

with  $\xi : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$  space-time white noise.

Joint work with P. Imkeller and N. Perkowski.  
(Also K. Chouk and R. Catellier for  $(\Phi)_3^4$ ).

## Rough differential equation

Consider the simple controlled ODE ( $\eta$  smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^m f_i(u(t)) \eta^i(t)$$

$u : \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $\eta : \mathbb{R} \rightarrow \mathbb{R}^d$  and smooth vectorfields  $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ .

### Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for  $\eta \in \mathcal{C}^{\gamma-1}$  with  $\gamma < 1/2$ .

Reason:  $u \in \mathcal{C}^\gamma$  and  $\eta \in \mathcal{C}^{\gamma-1}$  cannot be multiplied when  $2\gamma - 1 \leq 0$ . The r.h.s. of the equation is not well defined.

Here  $\mathcal{C}^\alpha = B_{\infty, \infty}^\alpha$  is the Holder–Besov space (or a local version).

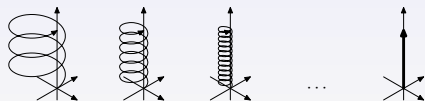
## What can go wrong?

Consider the sequence of functions  $x^n : \mathbb{R} \rightarrow \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then  $x^n(\cdot) \rightarrow 0$  in  $C^\gamma([0, T]; \mathbb{R}^2)$  for any  $\gamma < 1/2$ . But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s) \partial_t x^{n,2}(s) ds \rightarrow \frac{t}{2}$$



$$I(x^{n,1}, x^{n,2})(t) \not\rightarrow I(0,0)(t) = 0$$

The definite integral  $I(\cdot, \cdot)(t)$  is not a continuous map  $C^\gamma \times C^\gamma \rightarrow \mathbb{R}$  for  $\gamma < 1/2$ .

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

## Concept of solution

**Goal:** Show that  $\Psi$  factorizes as

$$\eta \xrightarrow{J} (\eta, \theta \circ \eta) \xrightarrow{\Phi} u$$

(here  $\partial_t \theta = \eta$  and  $\theta \circ \eta = X^2(\eta)$  will be described later)

▷ *Analytic step:* show that when  $\gamma > 1/3$ :

$$\Phi : \mathcal{X} \rightarrow \mathcal{C}^\gamma$$

is continuous.  $\mathcal{X} = \overline{\text{Im} J} \subseteq \mathcal{C}^{\gamma-1} \times \mathcal{C}^{2\gamma-1}$  is the space of *enhanced signals* (or rough paths, or models).

But in general  $J$  is not a continuous map  $\mathcal{C}^{\gamma-1} \rightarrow \mathcal{C}^{\gamma-1} \times \mathcal{C}^{2\gamma-1}$ .

▷ *Probabilistic step:* prove that there exists a "reasonable definition" of  $J(\xi)$  when  $\xi$  is a white noise.  $J(\xi)$  is an explicit polynomial in  $\xi$  so direct computations are possible.

## Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces  $\mathcal{C}^\gamma = B_{\infty,\infty}^\gamma$ .

$f \in \mathcal{C}^\gamma, \gamma \in \mathbb{R}$  iff

$$\|\Delta_i f\|_{L^\infty} \leq \|f\|_\gamma 2^{-i\gamma}, \quad i \geq -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi) \hat{f}(\xi)$$

where  $\rho_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$  are smooth functions with support in annuli  $\simeq 2^i \mathcal{A}$  when  $i \geq 0$  and form a partition of unity

$$\sum_{i \geq -1} \rho_i(\xi) = 1$$

for all  $\xi \neq 0$  so that

$$f = \sum_{i \geq -1} \Delta_i f$$

in  $\mathcal{S}'$ .

# Paraproducts

Deconstruction of a product:  $f \in \mathcal{C}^\rho, g \in \mathcal{C}^\gamma$

$$fg = \sum_{i,j \geq -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i < j-1} \Delta_i f \Delta_j g \quad f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$\pi_{<}(f, g) \in \mathcal{C}^{\min(\gamma+\rho, \gamma)}$$

$$\pi_{\circ}(f, g) \in \mathcal{C}^{\gamma+\rho} \quad \text{if } \gamma + \rho > 0$$



**Proof.** Recall  $f \in \mathcal{C}^\rho, g \in \mathcal{C}^\gamma$ .

$$i \ll j \Rightarrow \text{supp } \mathcal{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathcal{A}$$

$$i \sim j \Rightarrow \text{supp } \mathcal{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathcal{B}$$

So if  $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j-q} \sum_{i:i < j-1} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i:i < j-1} O(2^{-i\rho-j\gamma}) = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathcal{C}^\gamma,$$

while if  $\rho < 0$

$$\Delta_q(f \prec g) = \sum_{i:i < j-1} O(2^{-i\rho-j\gamma}) = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathcal{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i \sim j \geq q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i \geq q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathcal{C}^{\gamma+\rho}$$

but *only if* the sum converges.

## Small detour : Young integral

Take  $f \in \mathcal{C}^\rho, g \in \mathcal{C}^\gamma$  with  $\gamma, \rho \in (0, 1)$

$$fDg = \underbrace{f \prec Dg}_{\mathcal{C}^{\gamma-1}} + \underbrace{f \circ Dg + f \succ Dg}_{\mathcal{C}^{\gamma+\rho-1}}$$

then

$$\begin{aligned} \int fDg &= \int \underbrace{f \prec Dg}_{\mathcal{C}^\gamma} + \int \underbrace{(f \circ Dg + f \succ Dg)}_{\mathcal{C}^{\gamma+\rho}} \\ &= f \prec g + \mathcal{C}^{\gamma+\rho}. \end{aligned}$$

Compare with standard estimate for the Young integral in Hölder spaces (valid when  $\gamma + \rho > 1$ ):

$$\int_s^t f_u dg_u = f_s(g_t - g_s) + O(|t - s|^{\gamma+\rho}).$$

Expansion in smallness of increments vs. Expansion in regularity

## The main commutator

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

which however "is" smoother than  $f \prec g$  if  $f$  or  $g$  has positive regularity.

Paraproducts decouple the problem from the source of the problem.

### Commutator

The trilinear operator  $C(f, g, h) = (f \prec g) \circ h - f(g \circ h)$  satisfies

$$\|C(f, g, h)\|_{\beta+\gamma} \lesssim \|f\|_{\alpha} \|g\|_{\beta} \|h\|_{\gamma}$$

when  $\beta + \gamma < 0$  and  $\alpha + \beta + \gamma > 0$ ,  $\alpha < 1$ .

# The Good, the Ugly and the Bad

*Concrete example.* Let  $B$  be a  $d$ -dimensional Brownian motion (or a regularisation  $B^\varepsilon$ ) and  $\varphi$  a smooth function. Then  $B \in C^\gamma$  for  $\gamma < 1/2$ .

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathcal{C}^{2\gamma-1}}$$

and recall the parolinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathcal{C}^{2\gamma}$$

Then

$$\begin{aligned}\varphi(B) \circ DB &= (\varphi'(B) \prec B) \circ DB + \underbrace{\mathcal{C}^{2\gamma} \circ DB}_{\text{OK}} \\ &= \varphi'(B)(B \circ DB) + \mathcal{C}^{3\gamma-1}\end{aligned}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B) \underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathcal{C}^{3\gamma-1}$$

## The Besov area

The Besov area  $B \circ DB$  can be defined and studied efficiently using Gaussian arguments:

$$B^\varepsilon \circ DB^\varepsilon \rightarrow B \circ DB$$

almost surely in  $\mathcal{C}_{\text{loc}}^{2\gamma-1}$  as  $\varepsilon \rightarrow 0$ .

Remark. If  $d = 1$  (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2}D(B \circ B)$$

which is well defined and belongs indeed to  $\mathcal{C}^{2\gamma-1}$ .

**Tools:** Besov embeddings  $L^p(\Omega; C^\theta) \rightarrow L^p(\Omega; B_{p,p}^{\theta'}) \simeq B_{p,p}^{\theta'}(L^p(\Omega))$ , Gaussian hypercontractivity  $L^p(\Omega) \rightarrow L^2(\Omega)$ , explicit  $L^2$  computations.

## Controlled paths/distributions

Controlled paths are paths which “looks like” a *given* path which often is random (but not necessarily).

A “good” quantification of this proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

A mix of functional analytic arguments and probabilistic ones.

### Basic analogies

- ▶ Itô processes

$$dX_t = f_t dM_t + g_t dt$$

- ▶ Amplitude modulation

$$f(t) = g(t) \sin(\omega t)$$

with  $|\text{supp } \hat{g}| \ll \omega$ .

## (Para)controlled structure

### Idea

Use the paraproduct to *define* a controlled structure. We say  $y \in \mathcal{D}_x^\rho$  if  $x \in \mathcal{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with  $y^x \in C^{\rho-\gamma}$  and  $y^\sharp \in C^\rho$ .

**Paralinearization.** Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a sufficiently smooth function and  $x \in \mathcal{C}^\gamma$ ,  $\gamma > 0$ . Then

$$\varphi(x) = \varphi'(x) \prec x + \mathcal{C}^{2\gamma}$$

▷ Another commutator:  $f, g \in \mathcal{C}^{\rho-\gamma}$ ,  $x \in \mathcal{C}^\gamma$

$$f \prec (g \prec h) = (fg) \prec h + \mathcal{C}^\rho$$

**Stability.** ( $\rho \leq 2\gamma$ )

$$\varphi(y) = (\varphi'(y)y^x) \prec x + \mathcal{C}^\rho$$

so we can take  $\varphi(y)^x = \varphi'(y)y^x$ .

## RDEs - I - the r.h.s.

$u : \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $\xi \in \mathcal{C}^{-1/2-}$  is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz. Take  $\partial_t X = \xi$ ,  $X \in \mathcal{C}^{1/2-}$  and assume that  $u \in \mathcal{D}_X^{1-}$ :

$$u = u^X \prec X + u^\sharp$$

with  $u^\sharp \in \mathcal{C}^{1-}$  and  $u^X \in \mathcal{C}^{1/2-}$ .

▷ Paralinearization:

$$f(u) = f'(u) \prec u + \mathcal{C}^{1-} = (f'(u)u^X) \prec X + \mathcal{C}^{1-}$$

▷ Commutator lemma:

$$\begin{aligned} f(u) \circ \xi &= ((f'(u)u^X) \prec X) \circ \xi + \mathcal{C}^{1-} \circ \xi \\ &= \underbrace{(f'(u)u^X)(X \circ \xi)}_{\in \mathcal{C}^{0-}} + \underbrace{C(f'(u)u^X, X, \xi)}_{\in \mathcal{C}^{1/2-}} + \mathcal{C}^{1-} \circ \xi \end{aligned}$$

if we assume that  $(X \circ \xi) \in \mathcal{C}^{0-}$ .



So if  $u$  is paracontrolled by  $X$ :

$$u = u^X \prec X + u^\sharp$$

and if  $X \circ \xi \in \mathcal{C}^{0-}$  we have a control on the r.h.s. of the equation:

$$f(u)\xi = \underline{f(u) \prec \xi} + f'(u)u^X(X \circ \xi) + \mathcal{C}^{1/2-}$$

What about the l.h.s.?

$$\partial_t u = \partial_t u^X \prec X + \underline{u^X \prec \xi} + \partial_t u^\sharp$$

so letting  $u^X = f(u)$  we have

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f'(u)f(u)(X \circ \xi) + \mathcal{C}^{1/2-}$$

## RDEs - III - the paracontrolled fixed point.

The RDE

$$\partial_t u = f(u)\xi,$$

is equivalent to the system

$$\begin{aligned}\partial_t X &= \xi, \\ \partial_t u^\sharp &= (f'(u)f(u))(X \circ \xi) - \underbrace{\partial_t f(u) \prec X}_{\in \mathcal{C}^{0-}} + \underbrace{R(f, u, X, \xi)}_{\in \mathcal{C}^{1/2-}} \circ \xi, \\ u &= f(u) \prec X + u^\sharp\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space  $\mathcal{D}_X^{1-}$  if we assume that

$$X \in \mathcal{C}^{1/2-}, \quad (X \circ \xi) \in \mathcal{C}^{0-}.$$

## Structure of the solution

▷ When  $\xi$  smooth, the solution to

$$\partial_t u = f(u)\xi, \quad u(0) = u_0$$

is given by  $u = \Phi(u_0, \xi, X \circ \xi)$  where

$$\Phi : \mathbb{R}^d \times \mathcal{C}^{\gamma-1} \times \mathcal{C}^{2\gamma-1} \rightarrow \mathcal{C}^\gamma$$

is continuous for any  $\gamma > 1/3$  and  $z = \Phi(u_0, \xi, \varphi)$  is given by the unique solution in  $\mathcal{D}_X^{2\gamma}$  to

$$\begin{cases} z = f(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (f'(z)f(z))\varphi - \underbrace{\partial_t f(z) \prec X}_{\in \mathcal{C}^{0-}} + \underbrace{R(f, z, X, \xi) \circ \xi}_{\in \mathcal{C}^{1/2-}} \end{cases}$$

▷ If  $(\xi^n, X^n \circ \xi^n) \rightarrow (\xi, \eta)$  in  $\mathcal{C}^{\gamma-1} \times \mathcal{C}^{2\gamma-1}$  and

$$\partial_t u^n = f(u^n)\xi^n, \quad u(0) = u_0$$

then

$$u^n \rightarrow u = \Phi(u_0, \xi, \eta).$$

## Relaxed form of the RDE

▷ Note that in general we can have  $\xi^{1,n} \rightarrow \xi$ ,  $\xi^{2,n} \rightarrow \xi$  and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take  $\xi^n, \xi$  smooth but  $\xi^n \rightarrow \xi$  in  $\mathcal{C}^{\gamma-1}$ . It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathcal{C}^{2\gamma-1}$$

In this case  $u^n \rightarrow u$  and  $u = \Phi(\xi, X \circ \xi + \varphi)$  solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

## "Itô" form of the RDE

In the smooth setting

$$\begin{aligned}u &= \Phi(\xi, X \circ \xi + \varphi) \\ \partial_t u &= f(u)\xi + f'(u)f(u)\varphi.\end{aligned}$$

If we choose  $\varphi = -X \circ \xi$  then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\partial_t v = f(v)\xi - f'(v)f(v)X \circ \xi,$$

and has the particular property of being a continuous map of  $\xi \in \mathcal{C}^{\gamma-1}$  alone.

## Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - D^2$ ,  $u : \mathbb{R} \times \mathbb{T}^2 \rightarrow \mathbb{R}$ ,  $\xi \in \mathcal{C}^{-1-}(\mathbb{T}^2)$  space white noise.

$$\mathcal{L}u = f(u)\xi$$

▷ Paracontrolled ansatz

$$\mathcal{L}X = \xi, \text{ so } X \in C([0, T], \mathcal{C}^{1-})$$

$$u = f(u) \prec X + u^\sharp$$

▷ Paralinearization:

$$f(u) = (f'(u)f(u)) \prec X + R(f, u, X)$$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ A problem: if  $\xi$  is the white noise

$$\begin{aligned} X \circ \xi &= X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX) \\ &= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2 = c + \mathcal{C}^{0-} \end{aligned}$$

with  $c = +\infty$ .

## Renormalization

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denote by  $\diamond$ . Now

$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\begin{aligned} \mathcal{L}u^\sharp &= -\mathcal{L}f(u) \prec X + Df(u) \prec DX + (f'(u)f(u))(X \circ \xi - c) \\ &\quad + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi \end{aligned}$$

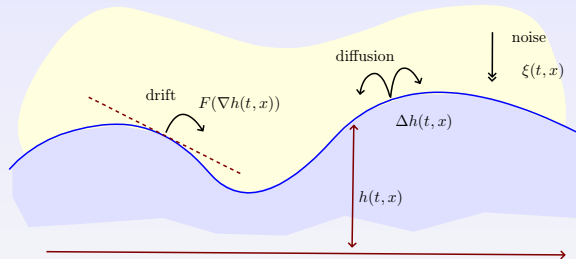
together with

$$u = f(u) \prec X + u^\sharp$$

and where

$$X \in \mathcal{C}^{1-}, \quad (X \circ \xi - c) \in \mathcal{C}^{0-}, \quad u^\sharp \in \mathcal{C}^{2-}.$$

# The Kardar–Parisi–Zhang equation



Large scale dynamics of the height  $h : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$  of an interface

$$\partial_t h \simeq \Delta h + F(Dh) + \xi$$

The universal limit should coincide with the large scale fluctuations of the KPZ equation

$$\partial_t h = \Delta h + [(Dh)^2 - \infty] + \xi$$

with  $\xi : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$  space-time white noise.



## Stochastic Burgers equation

Take  $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if  $u_0$  has a Gaussian distribution with covariance  $\mathbb{E}[u_0(x)u_0(y)] = \delta(x - y)$  then for all  $t \geq 0$  the random function  $u(t, \cdot)$  has a Gaussian law with the same covariance.

▷ **First order approximation:** Let  $X(t, x)$  be the solution of the linear equation

$$\partial_t X(t, x) = \partial_x^2 X(t, x) + \partial_x \xi(t, x), \quad x \in \mathbb{T}, t \geq 0$$

$X$  is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t, x)X(s, y)] = p_{|t-s|}(x - y).$$

Almost surely  $X(t, \cdot) \in \mathcal{C}^\gamma$  for any  $\gamma < -1/2$  and any  $t \in \mathbb{R}$ . For any  $t \in \mathbb{R}$   $X(t, \cdot)$  has the law of the white noise over  $\mathbb{T}$ .

## Expansion /I

▷ Let  $u = X + u_1$  then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▷ Let  $X^\vee$  be the solution to

$$\mathcal{L}X^\vee = \partial_x X^2 \quad \Rightarrow \quad X^\vee \in \mathcal{C}^{0-}$$

and decompose further  $u_1 = X^\vee + u_2$ . Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^\vee X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^\vee X^\vee)}_{-1-} + 2\partial_x(u_2 X^\vee) + \partial_x(u_2)^2$$

▷ Define  $\mathcal{L}X^\vee = 2\partial_x(X^\vee X)$  and  $u_2 = X^\vee + u_3$  then  $X^\vee \in \mathcal{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^\vee X)}_{-3/2-} + \underbrace{\partial_x(X^\vee X^\vee)}_{-1-} + 2\partial_x(u_2 X^\vee) + \partial_x(u_2)^2$$

## Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\forall} + U$$

$$\begin{aligned}\mathcal{L}U &= 2\partial_x(UX) + 2\partial_x(X^{\forall}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\forall} + U)X^{\vee}) + \partial_x(2X^{\forall} + U)^2 \\ &= 2\partial_x(UX) + \mathcal{L}(2X^{\forall} + X^{\forall}) + 2\partial_x((2X^{\forall} + U)X^{\vee}) + \partial_x(2X^{\forall} + U)^2\end{aligned}$$

and the regularities for the driving terms

$X$	$X^{\vee}$	$X^{\forall}$	$X^{\forall\forall}$	$X^{\forall\forall\forall}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume  $U \in \mathcal{C}^{1/2-}$  so that the terms

$$2\partial_x((2X^{\forall} + U)X^{\vee}) + \partial_x(2X^{\forall} + U)^2$$

are well defined.

The remaining problem is to deal with  $2\partial_x(UX)$ .

## Paracontrolled ansatz for SBE

▷ Make the following ansatz  $U = U' \prec Y + U^\sharp$ . Then

$$\mathcal{L}U = \mathcal{L}U' \prec Y + U' \prec \mathcal{L}Y - \partial_x U' \prec \partial_x Y + LU^\sharp$$

while

$$\begin{aligned}\mathcal{L}U &= 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\vee} + X^{\vee}) + 2\partial_x((2X^{\vee} + U)X^{\vee}) + \partial_x(2X^{\vee} + U)^2}_{Q(U)} \\ &= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U) \\ &= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)\end{aligned}$$

so we can set  $U' = 2U$  and  $\mathcal{L}Y = \partial_x X$  and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Y + \partial_x U' \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

▷ Observe that  $Y, U, U' \in \mathcal{C}^{1/2-}$  and we can assume that  $U^\sharp \in \mathcal{C}^{1-}$ .

## Commutator

- ▷ The difficulty is now concentrated in the resonant term  $U \circ X$  which is not well defined.
- ▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Y) \circ X + U^\sharp \circ X = 2U(Y \circ X) + \underbrace{C(2U, Y, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

- ▷ A stochastic estimate shows that  $Y \circ X \in \mathcal{C}^{0-}$
- ▷ The final fixed point equation reads

$$\begin{aligned} \mathcal{L}U^\sharp &= 4\partial_x(U \circ X) + 4\partial_x C(U, Y, X) + 2\partial_x(U^\sharp \circ X) - 2LU \prec Y \\ &\quad + 2\partial_x U \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + Q(U) \end{aligned}$$

- ▷ This equation has a (local in time) solution  $U = \Phi(J(\xi))$  which is a continuous function of the data  $J(\xi)$  given by a collection of multilinear functions of  $\xi$ :

$$J(\xi) = (X, X^\vee, X^\psi, X^\psi, X^\psi, X \circ Y)$$

# Stochastic Quantization

Stochastic quantization of  $(\Phi^4)_3$ :  $\xi \in C^{-5/2-}$ ,  $u \in C^{-1/2-}$ ,  $u = u_1 + u_2 + u_{\geq 3}$ .

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1u - c_2u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geq 2} = \xi + \lambda(u_1^3 - 3c_1u_1) + 3\lambda(u_{\geq 2}(u_1^2 - c_1)) + 3\lambda(u_{\geq 2}^2u_1) + \lambda u_{\geq 2}^3 - \lambda c_2u$$

$$\triangleright \mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}, \mathcal{L}u_2 = \lambda(u_1^3 - 3c_1u_1) \Rightarrow u_2 \in C^{1/2-}$$

$$\mathcal{L}u_{\geq 3} = 3\lambda(u_{\geq 2}(u_1^2 - c_1)) + 3\lambda(u_2^2u_1) + 6\lambda(u_{\geq 3}u_2u_1) + 3\lambda(u_{\geq 3}^2u_1) + \lambda u_{\geq 3}^3 - \lambda c_2u$$

$$\triangleright \text{Ansatz: } u_{\geq 3} = 3\lambda u_{\geq 2} \prec X + u^\sharp, \text{ with } \mathcal{L}X = (u_1^2 - c_1)$$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geq 2} \prec X + 3\lambda D u_{\geq 2} \prec DX + 3\lambda(u_{\geq 2} \circ (u_1^2 - c_1) - c_2u) + 3\lambda(u_{\geq 2} \succ (u_1^2 - c_1))$$

$$+ 3\lambda(u_2^2u_1) + 6\lambda(u_{\geq 3}(u_2u_1)) + 3\lambda(u_{\geq 3}^2u_1) + \lambda u_{\geq 3}^3$$

$$u_{\geq 2} \circ (u_1^2 - c_1) - c_2u = (u_2 \circ (u_1^2 - c_1) - c_2u_1) + (u_{\geq 3} \circ (u_1^2 - c_1) - c_2u_{\geq 2})$$

$$(u_{\geq 3} \circ (u_1^2 - c_1) - c_2u_{\geq 2}) = (3\lambda(u_{\geq 2} \prec X) \circ (u_1^2 - c_1) - c_2u_{\geq 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geq 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda \mathcal{C}(u_{\geq 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

$$\triangleright \text{Basic objects: } (u_1^2 - c_1), (u_1^3 - 3c_1u_1), (3\lambda(X \circ (u_1^2 - c_1)) - c_2), (u_2u_1), (u_2^2u_1)$$

Thanks