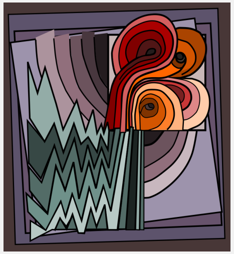
CMP Colloquium · Leipzig · May 19th, 2023

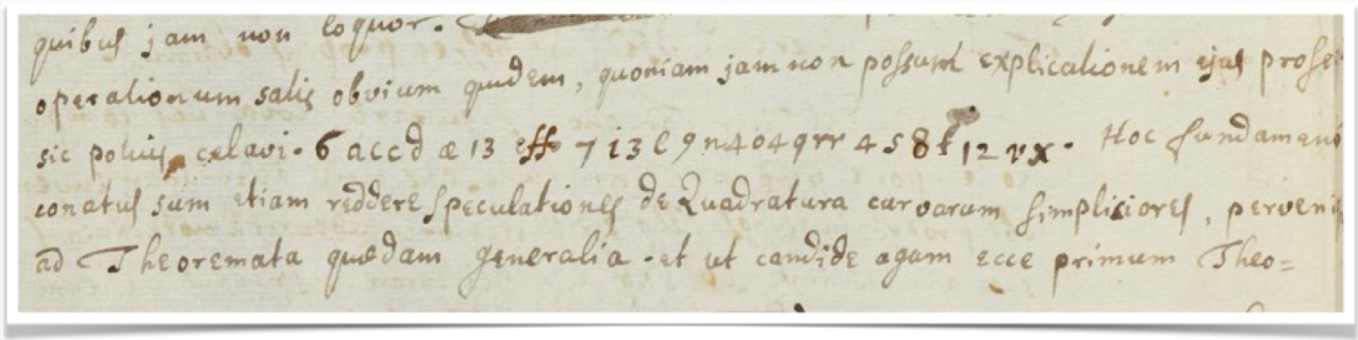
New directions & open problems in stochastic analysis



Massimiliano Gubinelli · University of Oxford

[made with TeXmacs]

what is stochastic analysis?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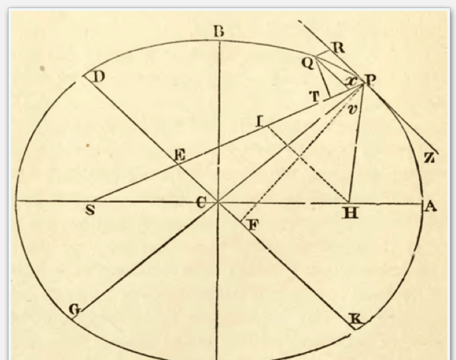
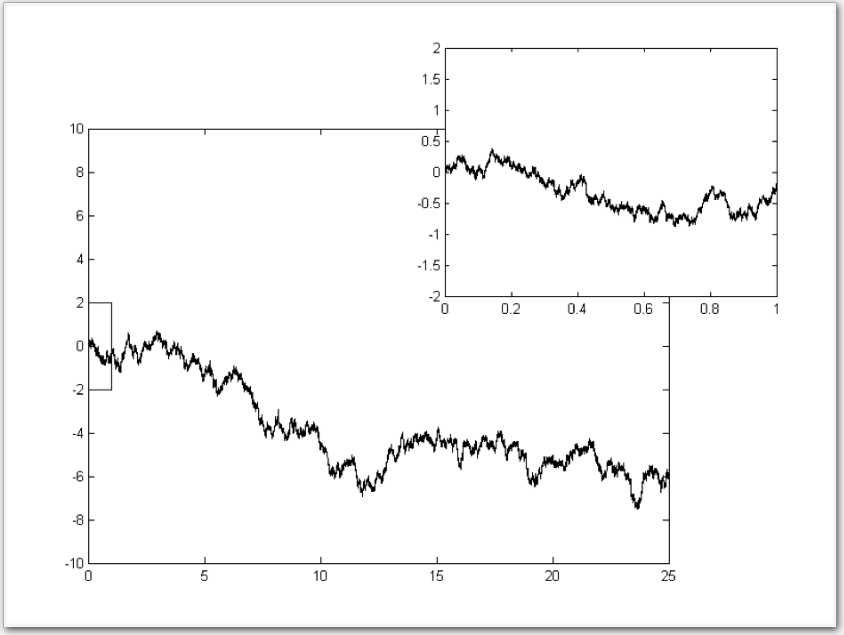
Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa (Newton)



[Given an equation involving any number of fluent quantities to find the fluxions, and vice versa]

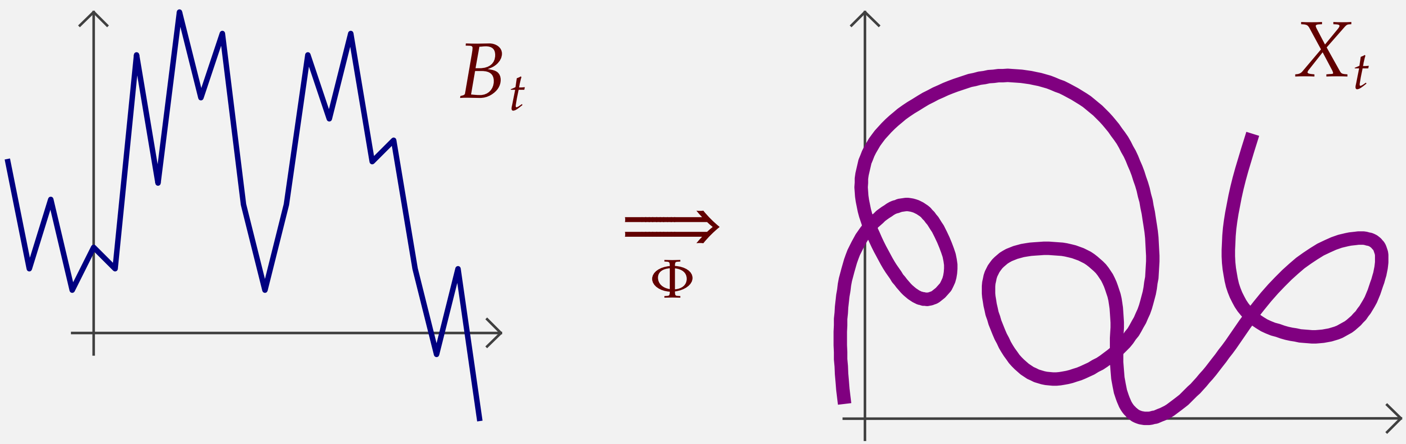
diffusion processes

The word "*random*" comes from a French hunting term: "*randon*" designates the erratic course of the deer which zigzags trying to escape the dogs. The word also gave "*randonnée*" (hiking) in French.



Ito's brillant idea

Ito arrived to his calculus while trying to understand Feller's theory of diffusions an evolution in the space of probability measures and he introduced stochastic differential equations to define a map (**the Itô map**) which send Wiener measure to the law of a diffusion.



Itô's original paper

(Japanese version 1942, M.A.M.S. 1951)

Differential Equations Determining a Markoff Process*

KIYOSI ITÔ

More generally, for a simple Markoff process with its states being represented by the real numbers and having continuous parameter, the problem of determining quantities corresponding to $p_{ij}^{(k)}$ mentioned above and of constructing the corresponding Markoff process once these quantities are given has been investigated systematically by Kolmogoroff[3], who reduced the problem to the study of differential equations or integro-differential equations satisfied by the transition probability function.

W. Feller[4] has proved under fairly strong assumptions that these equations possess a unique solution and furthermore that the solution exhibits the properties of transition probability function.

However, if we adopt more strict point of view such as the one J. Doob[5] has applied toward his investigation of stochastic processes, it seems to us that the aforementioned work done by Feller is not quite adequate. For example, even though the differential equation determining the transition probability function of a continuous stochastic process was solved in §3 of that paper, no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is:

1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

§1. Definition of Differentiation of a Markoff Process

Let $\{y_t\}$ be a (simple) Markoff process and denote by $F_{t_0 t}$ the conditional probability distribution[1] of $y_t - y_{t_0}$ given that "y_{t_0} is determined". $F_{t_0 t}$ is clearly a P_{y_t}-measurable (ρ) function[2] of y_t, where ρ denotes the Lévy distance among probability distributions.

Definition 1.1.[3]
If

$$(1) \qquad F_{t_0 t}^{*[1/t-t_0]}$$

(here $[a]$ is the integer part of the number a, and "$*k$" denotes the k-fold convolution) converges in probability with respect to the Lévy distance ρ as $t \to t_0 + 0$, then we call the limit random variable (taking values in the space of probability distributions) the derivative of $\{y_t\}$ at t_0 and denote it by

$$(2) \qquad D_{t_0}\{y_t\} \text{ or } Dy_{t_0}.$$

Corollary 1.1. *Dy_{t_0} is an infinitely divisible probability distribution.*[4]

bability distribution.

Dy_{t_0} obtained above is a function of t_0 as well as of y_{t_0}, and so, we denote it by $L(t_0, y_{t_0})$ corresponds precisely to the "basic transition probability" discussed in the Introduction.

The precise formulation of the problem of Kolmogoroff, then, is to solve the equation

$$(4) \qquad Dy_t = L(t, y_t)$$

when the quantity $L(t, y)$ is given.

§2. A Comparison Theorem

Let us prove a comparison theorem for Dy_{t_0} which we shall make use of later.

Theorem 2.1. *Let $\{y_t\}$, $\{z_t\}$ be simple Markoff processes satisfying the following conditions:*

(1) $y_{t_0} = z_{t_0}$.
(2) $E(y_t - z_t \mid y_{t_0}) = o(t - t_0)$, *where o is the Landau symbol.*
(3) $\sigma(y_t - z_t \mid y_{t_0}) = o(\sqrt{t - t_0})$.

(Here $E(x \mid y)$ denotes the conditional expectation of x given y and $\sigma(x \mid y)$ denotes the conditional standard deviation of x given y. Also, the quantity o may depend on t_0 or y_{t_0}). Then, whenever Dz_{t_0} exists, Dy_{t_0} exists also, and $Dy_{t_0} = Dz_{t_0}$ holds.

H. Föllmer, "On Kiyosi Itô's Work and its Impact" (Gauss prize laudatio 2006)

In 1987 Kiyosi Itô received the Wolf Prize in Mathematics. The laudatio states that "he has given us a full understanding of the infinitesimal development of Markov sample paths. This may be viewed as Newton's law in the stochastic realm, providing a direct translation between the governing partial differential equation and the underlying probabilistic mechanism. Its main ingredient is the differential and integral calculus of functions of Brownian motion. The resulting theory is a cornerstone of modern probability, both pure and applied".

The reference to Newton stresses the fundamental character of Itô's contribution to the theory of Markov processes.

Let us also mention Leibniz in order to emphasize the fundamental importance of Itô's work from another point of view. In fact Itô's approach can be seen as a natural extension of Leibniz's algorithmic formulation of the differential calculus. In a manuscript written in 1675 Leibniz argues that the whole differential calculus can be developed out of the basic product rule $\mathrm{d}(\mathrm{XY}) = X\,\mathrm{d}Y + Y\,\mathrm{d}X$ and he writes:

"Quod theorema sane memorabile omnibus curvis commune est".

But when Kiyosi Itô came to Princeton in 1954, at that time a stronghold of probability theory with William Feller as the central figure, his new approach to diffusion theory did not attract much attention. Feller was mainly interested in the general structure of one-dimensional diffusions with local generator

$$F = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$$

motivated by his intuition that a “one-dimensional diffusion traveler makes a trip in accor- dance with the road map indicated by the scale function s and with the speed indicated by the measure m” [...]

The first systematic exposition in Germany was the book Stochastische Differentialgleichungen [2] by Ludwig Arnold, with the motion of satellites as a prime example. It was based on seminars and lectures at the Technical University Stuttgart which he was urged to give by his colleagues in Engineering.

In the seventies the relevance of Itô's work was also recognized in physics and in particular in quantum field theory. When I came to ETH Zurich in 1977, Barry Simon gave a series of lectures for Swiss physicists on path integral techniques which included the construction of Itô's integral for Brownian motion, an introduction to stochastic calculus, and applications to Schrödinger operators with magnetic fields; see chapter V in [45]. When Kiyosi Itô was awarded a honorary degree by ETH Zurich in 1987, this was in fact due to a joint initiative of mathematicians and physicists.

stochastic analysis today

> [...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.
>
> [D. Stroock, “Elements of stochastic calculus and analysis ”, Springer, 2018]

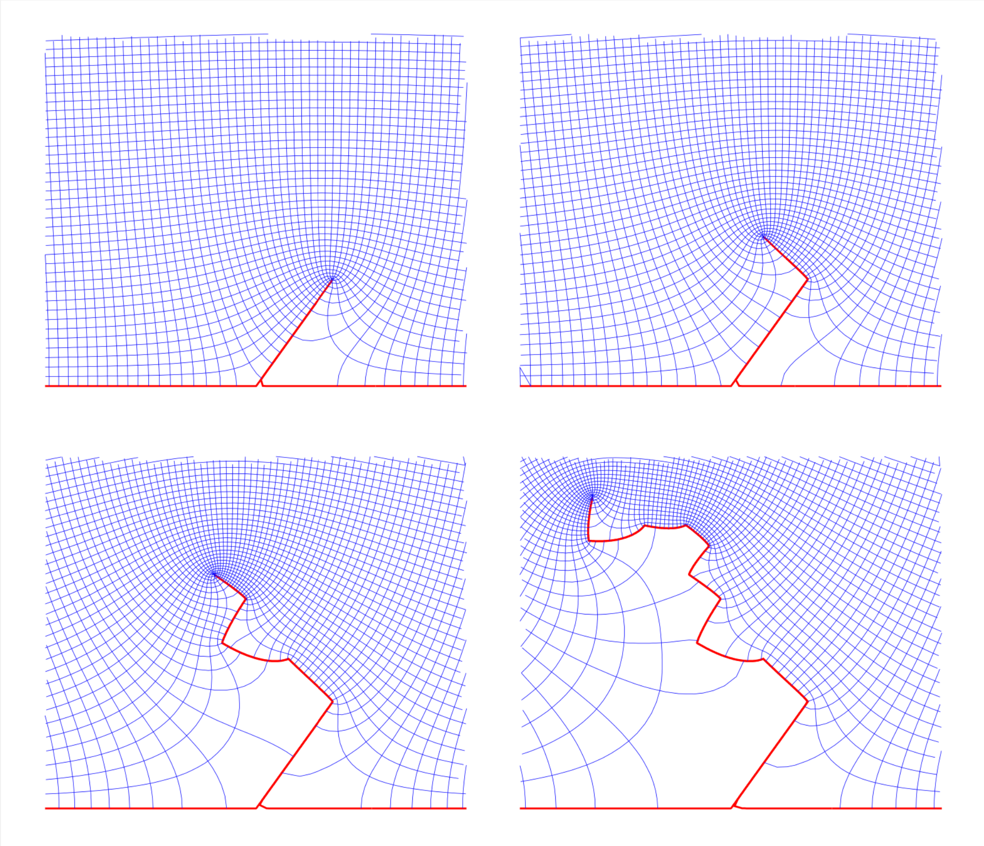
Nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…

analysis vs. stochastic analysis

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y) \in \mathcal{O} \subseteq \mathbb{R}^2$	global description	$P_t(x,\mathrm{d}y)$
$\alpha(x-x_0)^2+\beta(y-y_0)^2=\gamma$	·	$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t)\approx x(t)+a\delta t+o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at+bt^2+\cdots$	building block	$(W_t)_t$
$(\ddot{x}(t),\ddot{y}(t))=F(x(t),y(t))$	local/global link	$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$

▷ other examples: rough paths, regularity structures, SLE,...

SLE



[Kennedy, Tom. `Numerical Computations for the Schramm-Loewner Evolution'. JSP 137 (2009) 10.1007/s10955-009-9866-2]

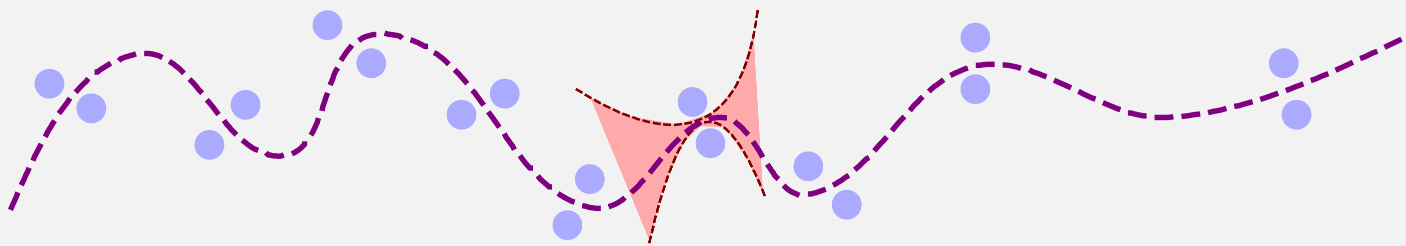
sewing lemma: controlled paths & regularity structures

integration of a *germ* $A\colon [0,T]^2 \to \mathbb{R}$ · when can we have

$$A(s,t) = y(t) - y(s) + O(|t-s|^{\zeta}), \qquad 0 \leqslant s \leqslant t \leqslant T$$

for some $y\colon [0,T] \to \mathbb{R}$? · unique for $\zeta > 1$ · sufficient condition: *coherence* of A for $\zeta > 1$

$$A(s,t) = A(s,u) + A(u,t) + O(|t-s|^{\zeta}), \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T$$



two-parameter stochastic calculus

Hajek, Bruce. `Stochastic Equations of Hyperbolic Type and a Two-Parameter Stratonovich Calculus'. *Ann. Prob.* 10 (1982): 451–63.

Norris, J.R. `Twisted Sheets'. *Jour. Func. Anal.* 132 (1995): 273–334. 10.1006/jfan.1995.1107.

and differential equations in Sect. 3. The basic rules of two-parameter stochastic calculus may be guessed: one has the usual Itô formula for C^2 functions, together with two further Itô product rules which work like partial differentiation

$$d_s(f(x_{st})) = f'(x_{st})\, d_s x_{st} + \tfrac{1}{2} f''(x_{st})\, d_s x_{st}\, d_s x_{st},$$

$$d_s(a_{st}\, d_t x_{st}) = d_s a_{st}\, d_t x_{st} + a_{st}\, d_s d_t x_{st} + d_s a_{st}\, d_s d_t x_{st},$$

$$d_s(d_t x_{st}\, d_t y_{st}) = d_s d_t x_{st}\, d_t y_{st} + d_t x_{st}\, d_s d_t y_{st} + d_s d_t x_{st}\, d_s d_t y_{st}.$$

There is also a Stratonovich calculus, which is related in the usual way to the Itô calculus, and which transforms the above formulas into those one

in a manifold.

In Sect. 3 we turn to two-parameter stochastic differential equations. The equations we study are stochastic generalizations of hyperbolic equations of the form

$$\frac{\partial^2 x}{\partial s\, \partial t} = a(x) + b(x)\left(\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right). \qquad (1.1)$$

Euclidean quantum fields

conceptually: stationary Markovian d dimensional fields / Gibbsian stochastic fields

prob. measures ν on $\mathscr{S}'(\mathbb{R}^d)$ · (Feynman–Kac) path integral formalism

$$\nu(\mathrm{d}\varphi) \approx \frac{e^{-S(\varphi)}}{Z}\mathscr{D}\varphi \approx \frac{e^{-\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}}{Z'}\mu(\mathrm{d}\varphi), \qquad \mu(\mathrm{d}\varphi) \approx \frac{e^{-S_0(\varphi)}}{Z}\mathscr{D}\varphi$$

$$S(\varphi) = \underbrace{\int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 + m^2\varphi(x)^2}_{S_0(\varphi)} + \underbrace{\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}_{\mathscr{V}(\varphi)}$$

heuristic description · large scale & small scale problems · renormalization

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they are natural probabilistic objects

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stochastic equations for EQFTs

Gaussian free field $\mu : \mathbb{E}[\varphi(x)\varphi(y)] = (m^2 - \Delta)^{-1}(x-y)$ · ξ white noise

❶ "Gaussian map":

$$\varphi(x) = (m^2 - \Delta)^{-1/2}\xi(x), \qquad (m^2-\Delta)\varphi(x) = (m^2-\Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❶ Stochastic mechanics (Nelson):

$$\partial_{x_0}\varphi(x_0,\bar{x}) = -(m^2-\Delta_{\bar{x}})^{1/2}\varphi(x_0,\bar{x}) + \xi(x_0,\bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu):

$$\varphi(x) \sim \phi(t,x) \qquad \partial_t\phi(t,x) = -(m^2-\Delta_x)\phi(t,x) + \xi(t,x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x) \sim \phi(z,x) \qquad (-\Delta_z)\phi(t,x) = -(m^2-\Delta_x)\phi(z,x) + \xi(z,x), \qquad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

stochastic equations for EQFTs in general $(V \neq 0)$

❶ Shifted Gaussian map (Albeverio/Yoshida)

$$[\varphi(x) = V'(\varphi) + (m^2 - \Delta)^{-1/2}\xi(x), \qquad (m^2 - \Delta)\varphi(x) + (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d]$$

❷ Stochastic mechanics (Nelson):

$$\partial_{x_0}\varphi(x_0, \bar{x}) = -(m^2 - \Delta_{\bar{x}})^{1/2}\varphi(x_0, \bar{x}) - V'(\varphi(x_0, \bar{x})) + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu):

$$\varphi(x) \sim \phi(t, x) \qquad \partial_t \phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) - V'(\varphi(x_0, \bar{x})) + \xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x) \sim \phi(z, x) \qquad (-\Delta_z)\phi(t, x) = -(m^2 - \Delta_x)\phi(z, x) - V'(\varphi(x_0, \bar{x})) + \xi(z, x), \quad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

Nelson's Markov field equations

11. Remarks on Markov field equations

E. Nelson

11.1. Introduction

We have no new existence theorems in constructive quantum field theory to present here, but we wish to indicate a new direction which looks promising and which certainly poses many interesting questions.

Only the theory of a neutral scalar field with a quartic self-interaction will be considered. This theory has been much studied in dimension $d=2$ and Glimm and Jaffe have pioneered the study in dimension $d=3$. (For references, see [4]—in particular, see the first article by Glimm, Jaffe, and Spencer and the reference listed there.)

We wish to stress field equations, and so we will begin with a heuristic discussion from that point of view. The simplest non-linear relativistic field equation with good formal properties is

$$(\Box + m^2)A = -gA^3 + \alpha A, \tag{11.1}$$

corresponding to the interaction Lagrangian density $-(g/4)A^4 + (\alpha/2)A^2$. We could of course absorb the linear term αA in the term $m^2 A$, but we prefer not to. Here m^2 and g are positive.

The Euclidean approach to the problem of quantized solutions to (11.1) is, in rough outline, as follows. The Wightman distributions (vacuum expe-

11.3. The Markov field equation

Let us compute:

$$\begin{aligned}
\hat{E}\phi(x) - \mu &= \frac{1}{N}\int_{-\infty}^{\infty} (\xi-\mu)\exp\left[\left(-\frac{g}{4}\xi^4 + \frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]d\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{g}{4}\xi^2 + \frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\left(-\sigma^2\frac{d}{d\xi}\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]\right)d\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty} \left(\sigma^2\frac{d}{d\xi}\exp\left[\left(-\frac{g}{4}\xi^4 + \frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\right)\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]d\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty} \sigma^2(-g\xi^3 + \alpha\xi)\varepsilon^d \exp\left[\left(-\frac{g}{4}\xi^4 + \frac{\alpha}{2}\xi^2\right)\varepsilon^d\right] \times \\
&\qquad\qquad \times \exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]d\xi \\
&= \sigma^2\hat{E}[-g\phi(x)^3 + \alpha\phi(x)]\varepsilon^d.
\end{aligned}$$

We may write this as

$$\phi(x) - \mu = \sigma^2[-g\phi(x)^3 + \alpha\phi(x)]\varepsilon^d + \sigma^2\omega(x)\varepsilon^d, \tag{11.8}$$

where $\omega(x)$ is a function of $\phi(x)$ and $\phi(y)$ for the nearest neighbours y of x (because (11.8) is the definition of $\omega(x)$) and

$$\hat{E}\omega(x) = 0. \tag{11.9}$$

Nelson, E. `Remarks on Markov Field Equations'. *Functional Integration and Its Applications (Proc. Internat. Conf., London, 1974)*, 1975, 136–43.

Wilson's parameter

The renormalization group and critical phenomena*

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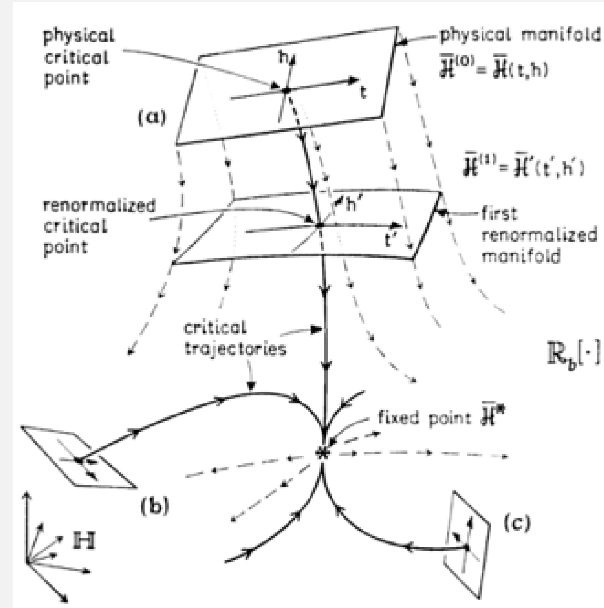
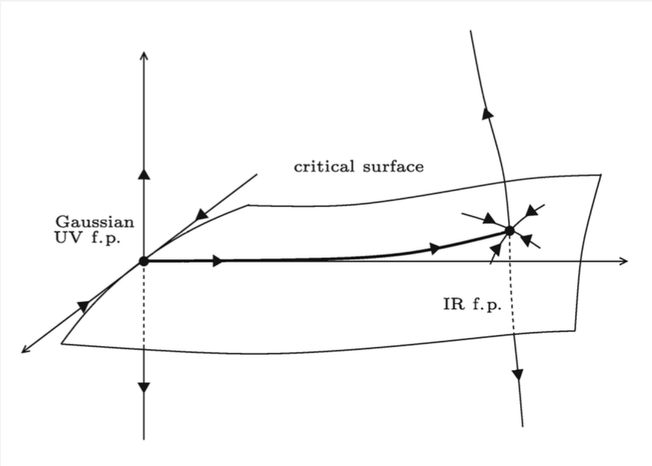
The possible types of cooperative behavior, in the renormalization group picture, are determined by the possible fixed points $\mathcal{H}^*$ of τ. Suppose for example that there are three fixed points $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, and $\mathcal{H}^*_C$. Then one would have three possible forms of cooperative behavior. If a particular system has an initial interaction $\mathcal{H}_0$, one has to construct the sequence $\mathcal{H}_1$, $\mathcal{H}_2$, etc. in order to find out which of $\mathcal{H}^*_A$, $\mathcal{H}^*_B$, or $\mathcal{H}^*_C$ gives the limit of the sequence. If $\mathcal{H}^*_A$ is the limit of the sequence, then the cooperative behavior resulting from $\mathcal{H}_0$ will be the cooperative behavior determined by $\mathcal{H}^*_A$. In this example the set of all possible initial interactions $\mathcal{H}_0$ would divide into three subsets (called "domains"), one for each fixed point. Universality would now hold separately for each domain. See section 12 for further discussion.

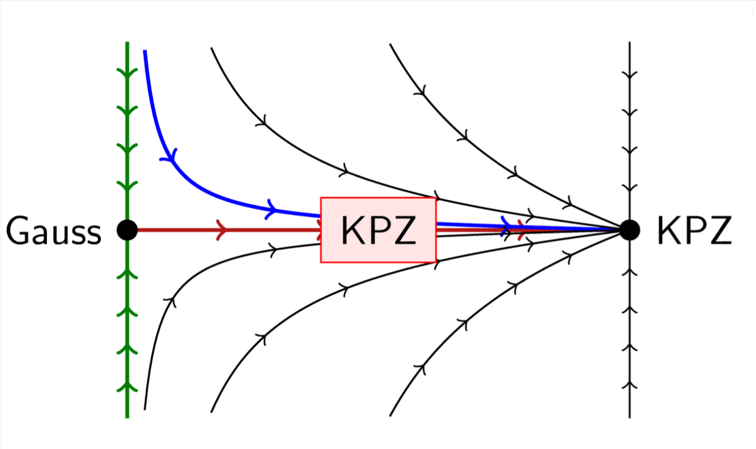
This is how one derives a form of universality in the renormalization group picture. It is not so bold as previous formulations [9]. Experience with soluble examples of the renormalization group transformation for critical phenomena shows that it generally has a number of fixed points, so one has to define domains of initial Hamiltonians associated with each fixed point, and only within a given domain is the critical behavior independent of the initial interaction.



random fields can be decomposed along the parameter given by a typical scale of fluctuations · "dynamical" picture of spatially extended fluctuations

subcritical & asymptotically free theories (UV)





KPZ: $\partial_t h_\varepsilon = \Delta h_\varepsilon + \lambda \varepsilon^{-1/2} (\nabla h_\varepsilon)^2 + \xi$ (non-reversible)

$\Phi^4_{2,3}$: $\partial_t \varphi_\varepsilon = \Delta \varphi_\varepsilon + \lambda \varepsilon^{-\alpha} \varphi^3_\varepsilon + \xi$ (reversible)

the small scales “looks like” GFF

Wilson–Polchinski continuous RG

▷ stochastic analysis along the scale decomposition

$$C_{t\wedge s} := \mathbb{E}[X_t \otimes X_s] = (m^2 - \Delta)^{-1} \int_0^{t\wedge s} \sigma_u^2(-\Delta)\mathrm{d}u, \qquad X_t = \int_0^t \dot{C}_u^{1/2} \mathrm{d}W_u$$

$$e^{-\mathscr{V}_t(\varphi_t)} := \mathbb{E}[e^{-\mathscr{V}_T(X_T)} | \mathscr{F}_t] \qquad \partial_t \mathscr{V}_t(\psi) - \frac{1}{2} D^2_{\dot{C}_t} \mathscr{V}_t(\psi) + |D\mathscr{V}_t(\psi)|^2_{\dot{C}_t} = 0$$

$$\mathrm{d}\Phi_t = \dot{C}_t D\mathscr{V}_t(\Phi_t)\mathrm{d}t + \mathrm{d}X_t, \qquad \Phi_0 = 0$$

$$\mathbb{E}[f(\Phi_t)] = \frac{\mathbb{E}[f(X_t)e^{-\mathscr{V}_T(X_T)}]}{\mathbb{E}[e^{-\mathscr{V}_T(X_T)}]} = \frac{\mathbb{E}[f(X_t)e^{-\mathscr{V}_t(X_t)}]}{\mathbb{E}[e^{-\mathscr{V}_t(X_t)}]}$$

☞ $(\Phi_t)_{t\geqslant 0}$ is the solution of an optimal control problem (“the variational method”)

$$-\log \mathbb{E}[e^{-h(X_t) - \mathscr{V}_T(X_T)}] = \inf_u \mathbb{E}\left\{ h(\Phi_t^u) + \mathscr{V}_T(\Phi_T^u) + \frac{1}{2}\int_0^\infty \|u_s\|^2 \mathrm{d}s \right\}$$

scale parameter $t \in [0, \infty] \cdot \Phi_t^u = X_t + \int_0^t \dot{C}_u^{1/2} u_s \mathrm{d}s, \qquad u_s^{\text{opt}} = \dot{C}_t^{1/2} D\mathscr{V}_t(\Phi_t)\mathrm{d}t$

stochastic quantisation as a stochastic analysis

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x, \mathrm{d}y)$	global description	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-S(\varphi)} \mathrm{d}\varphi$
$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$		$\left\langle F(\varphi)\frac{\delta S(\varphi)}{\delta\varphi} + \frac{\delta F(\varphi)}{\delta\varphi}\right\rangle = 0$
t	change parameter	t
$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t) a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$	local description	$\phi(t+\delta t) \approx \alpha\phi(t) + \beta\delta X(t) + \cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X = \frac{1}{2}[(\Delta_x - m^2)X] + \xi$
$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$	local/global link	$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - V'(\phi)] + \xi$

some open problems

global solutions of several SPDEs

“large field problem” · global in time and space · well-posedness of the FBSDE

OK for $\Phi^4_{2,3}$ and $\exp(\varphi)_2$ but open for all the other models:

- Sine–Gordon (above the second threshold) [elliptic, parabolic, FBSDEs]

$$V'(\varphi) = \lambda \sin(\beta\varphi)$$

- σ-models, even in $d = 1$ (dynamics of a loop in a manifold) [parabolic]

$$\partial_t u = \Delta u + g(u)(\nabla u \otimes \nabla u) + h(u)\xi, \quad u: \mathbb{R}_+ \times \mathbb{T} \to \mathscr{M}$$

- Abelian and non-Abelian gauge theories (and Higgs) [parabolic]

$$\begin{cases} \partial_t A = \Delta A + g A\nabla A + g AAA + e\varphi\nabla\varphi + \xi \\ \partial_t \varphi = \Delta\varphi + e A\nabla\varphi + e AA\varphi + \lambda|\varphi|^2\varphi + \xi \end{cases}$$

What about elliptic & FBSDEs for σ-models?

uniqueness for $\Phi^4_{2,3}$

e.g. parabolic SQ:

$$\partial_t \varphi + m^2 \varphi - \Delta \varphi + \lambda [\![\varphi^3]\!] = \xi,$$

would like to show that this equation has a unique stationary strong solution $\lambda > 0$ small (wrt m^2). And maybe two solutions for λ large.

▷ **main difficulty**: non-convexity of the potential. $\psi := \varphi - \tilde{\varphi}$

$$\partial_t \psi + m^2 \psi - \Delta \psi + \lambda [\![\varphi^2 + \tilde{\varphi}^2]\!] \psi = 0$$

$$[\![\varphi^2 + \tilde{\varphi}^2]\!] \ngeqslant 0!$$

would need to prove that local perturbations do not propagate. This information would also allow to prove decay of correlations.

Grassmann SPDEs

we have now a stochastic analysis of Grassmann valued random variables

$$\psi^{\alpha}\psi^{\beta}+\psi^{\beta}\psi^{\alpha}=0$$

it can be used to describe Gibbisan Grassmann fields (Fermionic EQFTs), non-commutative analog of EQFTs.

▷ we have concepts of "a.s." or L^p spaces but we are not able to solve singular SPDEs globally, lack of coercive estimates

▷ same for equations involving classical fields and Grassmann fields, relevant also for supersymmetric EQFTs

supersymmetry (SUSY) and supersymmetric EQFTs

for example: parabolic SQ of SUSY Φ_d^4

$$\partial_t \Phi(t,X) + m^2 \Phi(t,X) - \Delta_X \Phi(t,X) + \lambda [\![\Phi(t,X)^3]\!] = \Xi(t,X),$$

with $X = (x_1, \ldots, x_d, \theta, \bar{\theta}) \in \mathbb{R}^{d|2}$, $(\theta_i)_i$ Grassmann coordinates

$$\Phi(X) = \varphi(x) + c(x)\theta + \bar{c}(x)\bar{\theta} + \omega(x)\theta\bar{\theta}$$

$$f(\Phi(X)) = f(\varphi(x)) + f'(\varphi(x))(c(x)\theta + \bar{c}(x)\bar{\theta} + \omega(x)\theta\bar{\theta}) + f''(\varphi(x))c(x)\bar{c}(x)\theta\bar{\theta}$$

$$\Delta_X = \Delta_x + \partial_\theta \partial_{\bar{\theta}}$$

SUSY-GFF

$$\mathbb{E}[\Phi(X)\Phi(Y)] = (m^2 - \Delta_X)^{-1}(X - Y)$$

▷ existence? uniqueness?

a remark on non-commutative probability (for probabilists)

I look at non-comm probability as I would look at complex numbers : a larger structure which contains some object of interest and which allow “more mathematics”.

as complex number reveal a **deeper structure** of algebraic equations, non-comm prob do for probabilistic problems:

- Onsager's solution of the Ising model is a theory of a Gaussian non-commutative field

  Schultz, T. D., D. C. Mattis, and E. H. Lieb. `Two-Dimensional Ising Model as a Soluble Problem of Many Fermions'. *Rev of Mod. Phys.* 36 (1964) 10.1103/RevModPhys.36.856

- Determinantal point processes are related to free Fermions:

  G. Olshanski, Grigori. `Determinantal Point Processes and Fermion Quasifree States'. *Comm. Math. Phys.* 378 (2020) 10.1007/s00220-020-03716-1

- Elliptic stochastic quantisation is proven via (Parisi–Sourlas) SUSY dimensional reduction

  S. Albeverio, F. C. De Vecchi, and M. Gubinelli. `Elliptic Stochastic Quantization'. *Ann. Prob.* 48 (2020) 10.1214/19-AOP1404

- + some others examples…

non-equilibrium reaction-diffusion systems & NESSs

stochastic dynamics with conservation laws

$$\partial_t \varphi + \Delta(m^2 \varphi - \Delta\varphi + \lambda\varphi^3) = \nabla \cdot \xi,$$

this should have φ_d^4 as invariant measure, however we can make another model:

$$\partial_t \varphi + (\Delta_\parallel + \alpha \Delta_\perp)(m^2 \varphi - \Delta\varphi + \lambda\varphi^3) = \nabla \cdot \xi, \qquad x = (x_\parallel, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$$

which even at the linear level behaves differently: ($\alpha \neq 1 \Rightarrow$ power-law correlations)

$$\mathbb{E}[\varphi(t,x)\varphi(t,y)] = \frac{1}{2(m^2 - \Delta)} \left[\frac{\Delta_\parallel + \Delta_\perp}{(\Delta_\parallel + \alpha\Delta_\perp)^2} \right](x-y)$$

$\alpha \neq 1 \Rightarrow$ no explicit invariant measures (non-equilibrium steady states)

$\triangleright$ what about renormalization? uniqueness/non-uniqueness of stationary solutions?

integration by parts formulas

Q: how to define a EQFT?

one possible approach: IbP formulas

$$\int_{\mathscr{S}'(\mathbb{R}^d)} \left[\frac{\delta}{\delta\varphi} - \frac{\delta S}{\delta\varphi}(\varphi)\right] F(\varphi)\nu(\mathrm{d}\varphi) = 0, \qquad \forall F \text{ in some nice class}$$

$$\left[\frac{\delta}{\delta\varphi} - \frac{\delta S}{\delta\varphi}(\varphi)\right] \approx e^{S(\varphi)} \frac{\delta}{\delta\varphi} e^{-S(\varphi)}$$

▷ existence of solutions, uniqueness?

up to now only for the Høegh–Krohn model (see recent work with Turra & de Vecchi)

☞ relation with cohomological integration, Batalin–Vilkovisky formalism

SPDE for non-trivial fixpoints?

▷ can we write an SPDE for the Wilson–Fisher fixpoint?

$$\partial_t \varphi^\lambda + m^2 \varphi^\lambda - \Delta \varphi^\lambda + \lambda [\![(\varphi^\lambda)^3]\!] = \xi, \qquad \lambda \to \infty.$$

▷ can we write an SPDE for the KPZ fixpoint? inviscid Burgers

$$\partial_t u + u \nabla u = 0$$

but not in viscosity sense and has to leave invariant the white noise

▷ what about SPDEs in kinetic formulation which have noise in the kinetic measure?

...and beyond

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Quantum interaction $:\phi_4^4:$, the construction of quantum field defined as a bilinear form

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We construct the solution $\phi(t,\mathbf{x})$ of the quantum wave equation $\Box\phi+m^2\phi+\lambda:\phi^3:=0$ as a bilinear form which can be expanded over Wick polynomials of the free *in*-field, and where $:\phi^3(t,\mathbf{x}):$ is defined as the normal ordered product with respect to the free *in*-field. The constructed solution is correctly defined as a bilinear form on $D_\theta\times D_\theta$, where D_θ is a dense linear subspace in the Fock space of the free *in*-field. On $D_\theta\times D_\theta$ the diagonal of the Wick symbol of this bilinear form satisfies the nonlinear classical wave equation. *© 2000 American Institute of Physics*. [S0022-2488(00)01001-X]

configuration space or on the space of trajectories is closely connected with dynamical equations of motion and quantum mechanics. However, here we consider a possible description of dynamics and leave a possible description of the vacuum for the future.

In the present paper we consider a self-interacting scalar quantum field in four-dimensional Minkowski space–time satisfying the following relativistic wave equation,

$$\Box\phi(t,\mathbf{x})+m^2\phi(t,\mathbf{x})+\lambda:\phi^3(t,\mathbf{x}):=0, \tag{1.1}$$

or in the form of integral equation

$$\phi(t,\mathbf{x})=\phi_{in}(t,\mathbf{x})-\lambda\int_{-\infty}^{t}\int R(t-\tau,\mathbf{x}-\mathbf{y}):\phi^3(\tau,\mathbf{y}):d\tau d^3y. \tag{1.2}$$

Equations (1.1) and (1.2) contain the relativistic and quantum constants c, $\hbar$ and we put $\hbar$ $=$ *Planck's constant* $=1$ and $c=$ *the light velocity* $=1$.

A principal barrier of this way appears as difficulties associated with the definition of a

a formal analogy

$$\mathscr{L}\varphi+\lambda\varphi^3=\xi$$

$$\mathbb{E}[\xi(t,x)\xi(t,y)]=\delta(x-y)$$

vs.

$$\Box\Phi+\lambda\Phi^3=0$$

$$[\Phi(x),\dot\Phi(y)]=\delta(x-y)$$

thanks

(no human has been harmed with TEX/LATEX to produce this presentation)