Paracontrolled distributions

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Controlled paths/distributions

Controlled paths are paths which "looks like" a *given* path which often is random (but not necessarily).

This proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

Successful approach which mixes functional analytic arguments and probabilistic ones.

Basic analogies

- Itô processes

$$\mathrm{d}X_t = f_t \mathrm{d}M_t + g_t \mathrm{d}t$$

- Amplitude modulation

$$f(t) = g(t)\sin(\omega t)$$

with $|\mathrm{supp}\,\hat{g}| \ll \omega$.

[Joint work with R. Catellier, K. Chouk, P. Imkeller, N. Perkowski]

Some interesting problems (I)

Define and solve the following kind of stochastic partial differential equations.

- Stochastic differential equations (1+0): $u \in [0,T] \to \mathbb{R}^n$

$$\partial_t u = f(u)\xi$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.

- Burgers equations (1+1): $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u = \Delta u + f(u)Du + \xi$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

- Parabolic Anderson model (1+2): $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

$$\partial_t u = \Delta u + f(u)\xi$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.

Recall that

$$\xi \in C^{-d/2-}$$

Some interesting problems (II)

Define and solve the following kind of stochastic partial differential equations.

- Kardar-Parisi-Zhang equation (1+1)

$$\partial_t h = \Delta h + \text{"}(Du)^2 - \infty\text{"} + \xi$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

- Stochastic quantization equation (1+3)

$$\partial_t u = \Delta u + \text{"}u^3\text{"} + \xi$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.

- But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u = \Delta u + f(u)\xi$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

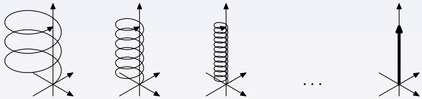
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2}$$



$$I(x^{n,1}, x^{n,2})(t) \not\to I(0,0)(t) = 0$$

The definite integral $I(\cdot,\cdot)(t)$ is not a continuous map $C^\gamma \times C^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Concept of solution

Consider the simple controlled ODE (η smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^{m} f_i(u(t)\eta^i(t)$$

$f_i : \mathbb{R}^d \to \mathbb{R}^d$. $u, \xi : \mathbb{R} \to \mathbb{R}^d$. Solution map: $u = \Psi(\eta)$ is generally **not** continuous for $\eta \in C^{\gamma-1}$ with $\gamma < 1/2$.

▹ We will develop techniques to show that for $\gamma > 1/3$:

$$u = \Phi(\eta, \theta \circ \eta)$$

$$\Phi : C^{\gamma-1} \times C^{2\gamma-1} \to C^{\gamma}$$

where $\partial_t \theta = \eta$ and $\theta \circ \eta = X^2(\eta)$ will be describe later.

▹ Probabilistic step: prove that there exists reasonable definitions of $X^2(\xi)$ when ξ is a white noise. In general X^2 is not a continuous map $C^{\gamma-1} \to C^{2\gamma-1}$.

$$\eta \to (\eta, \theta \circ \eta) \to_{\Phi} u$$

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $C^\gamma = B^\gamma_{\infty,\infty}$.

$f \in C^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \lesssim 2^{-i\gamma}, \qquad i \geqslant 0$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho(2^{-i}|\xi|)\hat{f}(\xi)$$

where $\rho : \mathbb{R} \to \mathbb{R}_+$ is a smooth function with support in $[1/2, 5/2]$ and such that $\rho(x) = 1$ if $x \in [1, 2]$ and there exists $\theta : \mathbb{R} \to \mathbb{R}_+$ smooth and with support $[0, 1]$ such that $\theta(|x|) + \sum_{i\geqslant 0} \rho(2^{-i}|x|) = 1$ for all $x \in \mathbb{R}$.

$$\mathcal{F}(\Delta_{-1} f)(\xi) = \theta(|\xi|)\hat{f}(\xi).$$

$$f = \sum_{i\geqslant -1} \Delta_i f$$

Paraproducts

Deconstruction of a product: $f \in C^\rho, g \in C^\gamma$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = \pi_<(f,g) + \pi_\circ(f,g) + \pi_>(f,g)$$

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad \pi_\circ(f,g) = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$\pi_<(f,g) \in C^{\min(\gamma+\rho,\gamma)}$$

$$\pi_\circ(f,g) \in C^{\gamma+\rho} \qquad \text{if } \gamma + \rho > 0$$

Young integral: $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{\pi_<(f,Dg)}_{C^{\gamma-1}} + \underbrace{\pi_\circ(f,Dg) + \pi_>(f,Dg)}_{C^{\gamma+\rho-1}}$$

Recall

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho})$$

(Para)controlled structure

Idea

Use the paraproduct to *define* a controlled structure. We say $y \in \mathcal{D}_x^{\gamma,\rho}$ if $x \in C^\gamma$

$$y = \pi_<(y^x, x) + y^\sharp$$

with $y^x \in C^\rho$ and $y^\sharp \in C^{\gamma+\rho}$.

Paralinearization. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in C^\gamma, \gamma > 0$. Then

$$\varphi(x) = \pi_<(\varphi'(x), x) + C^{2\gamma}$$

▷ A first commutator: $f, g \in C^\rho, x \in C^\gamma$

$$\pi_<(f, \pi_<(g, h)) = \pi_<(fg, h) + C^{\gamma+\rho}$$

Stability. $(\rho \geqslant \gamma)$

$$\varphi(y) = \pi_<(\varphi'(y) y^x, x) + C^{\gamma+\rho}$$

A key commutator

All the difficulty is concentrated in the resonating term

$$\pi_\circ(f,g) = \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$$

which however is smoother than $\pi_<(f,g)$.

Paraproducts decouple the problem from the source of the problem.

Commutator

The linear form $R(f,g,h) = \pi_\circ(\pi_<(f,g),h) - f\pi_\circ(g,h)$ satisfies

$$\|R(f,g,h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

with $\alpha \in (0,1)$, $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$.

Paradifferential calculus allow algebraic computations to simplify the form of the resonating terms ($\pi_\circ$).

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in C^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\pi_<(\varphi(B), DB)}_{\text{the Bad}} + \underbrace{\pi_\circ(\varphi(B), DB)}_{\text{the Ugly}} + \underbrace{\pi_>(\varphi(B), DB)}_{\text{the Good, } C^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \pi_<(\varphi'(B), B) + C^{2\gamma}$$

Then

$$\pi_\circ(\varphi(B), DB) = \pi_\circ(\pi_<(\varphi'(B), B), DB) + \underbrace{\pi_\circ(C^{2\gamma}, DB)}_{\text{OK}}$$

$$= \pi_<(\varphi'(B), \pi_\circ(B, DB)) + C^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \pi_<(\varphi(B), DB) + \pi_<(\varphi'(B), \underbrace{\pi_\circ(B, DB)}_{\text{"Besov area"}}) + \pi_>(\varphi(B), DB) + C^{3\gamma-1}$$

The Besov area

The Besov area $\pi_\circ(B, DB)$ can be defined and studied efficiently using Gaussian arguments:

$$\pi_\circ(B^\varepsilon, DB^\varepsilon) \to \pi_\circ(B, DB)$$

almost surely in $C^{2\gamma-1}$ as $\varepsilon \to 0$.

Remark. If $d = 1$

$$\pi_\circ(B, DB) = \frac{1}{2}(\pi_\circ(B, DB) + \pi_\circ(DB, B)) = \frac{1}{2} D\pi_\circ(B, B)$$

which is well defined.

Tools: Besov embeddings $L^p(\Omega; C^\theta) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Au delá des paraproduits

$u : \mathbb{R} \to \mathbb{R}^d$, $\xi \in C^{-1/2-}$ is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz $(\partial_t X = \xi,\ X \in C^{1/2-})$

$$u = f(u) \prec X + u^\sharp \quad \Rightarrow \quad \partial_t u = \partial_t f(u) \prec X + f(u) \prec \xi + \partial_t u^\sharp$$

so

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f(u) \circ \xi + f(u) \succ \xi \in C^{0-}$$

▷ Paralinearization: $f(u) = f'(u) \prec u + R(f,u)$

$$f(u) = (f'(u)f(u)) \prec X + R(f,u,X)$$

▷ Commutator lemma:

$$f(u) \circ \xi = ((f'(u)f(u)) \prec X) \circ \xi + R(f,u,X) \circ \xi$$

$$= (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f,u,X) \circ \xi$$

SDEs

The SDE

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X =& \xi \\
\partial_t u^\sharp =& (f'(u)f(u))(X \circ \xi) - \partial_t f(u) \prec X \\
&+ f(u) \succ \xi + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi \\
u =& f(u) \prec X + u^\sharp
\end{aligned}$$

▷ We can check that indeed

$$X \in C^{1/2-}, \qquad (X \circ \xi) \in C^{0-}$$

▷ The system can be solved by fixed point.

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = f(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times C^{\gamma-1} \times C^{2\gamma-1} \to C^\gamma$$

is continuous for any $\gamma > 1/3$.
▷ So if $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $C^{\gamma-1} \times C^{2\gamma-1}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then

$$u^n \to u$$

where $u = \Phi(u_0, \xi, \eta)$.
▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

Relaxed form of the equation

▹ Take ξ^n, ξ smooth but $\xi^n \to \xi$ only in $C^{\gamma-1}$ then in general we could have

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in C^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

So this limit procedure generates correction terms to the equation. The original equation **relaxes** to another form.

Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - \mathrm{D}^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in C^{-1}$ space white noise.

$$\mathcal{L} u = f(u)\xi$$

▷ Paracontrolled ansatz $\qquad \mathcal{L}X = \xi$ so $X \in C^{1-}$

$$u = f(u) \prec X + u^\sharp \quad \Rightarrow \quad \mathcal{L}u = \mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + f(u) \prec \xi + \mathcal{L}u^\sharp$$

▷ Paralinearization: $\qquad f(u) = (f'(u)f(u)) \prec X + R(f, u, X)$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

Problem

$$X \circ \xi = X \circ \mathcal{L}X = c + C^{0-}$$

with $c = +\infty$.

Renormalization

To cure the problem we add a suitable counterterm to the equation.

$$\mathcal{L}u = f(u)\xi - c(f'(u)f(u))$$

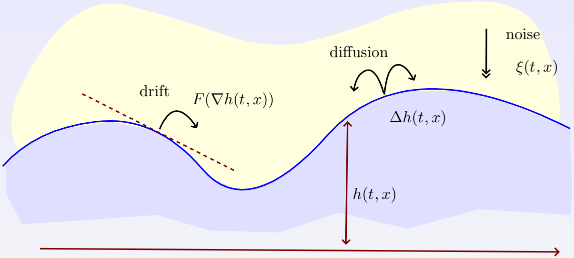
$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The gPAM is equivalent to the equation

$$\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c) \\ + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

$$X \in C^{1-}, \qquad (X \circ \xi - c) \in C^{0-}, \quad u^\sharp \in C^{2-}$$

The Kardar–Parisi–Zhang equation



Large scale dynamics of the height $h : [0,T] \times \mathbb{T} \to \mathbb{R}$ of an interface

$$\partial_t h \simeq \Delta h + F(Dh) + \xi$$

The universal limit should coincide with the large scale fluctuations of the KPZ equation

$$\partial_t h = \Delta h + [(Dh)^2 - \infty] + \xi$$

with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

$$u = u_1 + u_2 + \cdots = u_1 + u_{\geqslant 2}$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \mathrm{D}\xi + \mathrm{D}u_1^2 + 2\mathrm{D}u_1 u_{\geqslant 2} + \mathrm{D}u_{\geqslant 2}^2$$
$$\mathcal{L}u_1 = \mathrm{D}\xi \Rightarrow u_1 \in C^{-1/2-}$$
$$\mathcal{L}u_2 + \mathcal{L}u_{\geqslant 3} = \mathrm{D}u_1^2 + 2\mathrm{D}(u_1 u_2) + 2\mathrm{D}(u_1 u_{\geqslant 3}) + \mathrm{D}u_2^2 + 2\mathrm{D}(u_{\geqslant 3} u_2) + \mathrm{D}u_{\geqslant 3}^2$$
$$\mathcal{L}u_2 = \mathrm{D}u_1^2 \Rightarrow u_2 \in C^{0-}$$
$$\mathcal{L}u_3 + \mathcal{L}u_{\geqslant 4} = 2\mathrm{D}(u_1 u_2) + 2\mathrm{D}(u_1 u_3) + 2\mathrm{D}(u_1 u_{\geqslant 4}) + \mathrm{D}u_2^2 + 2\mathrm{D}u_{\geqslant 3} u_2 + \mathrm{D}u_{\geqslant 3}^2$$
$$\mathcal{L}u_3 = 2\mathrm{D}(u_1 u_2) \Rightarrow u_3 \in C^{1/2-}$$
$$\mathcal{L}u_{\geqslant 4} = 2\mathrm{D}(u_1 u_3) + 2\mathrm{D}(u_1 u_{\geqslant 4}) + \mathrm{D}u_2^2 + 2\mathrm{D}(u_{\geqslant 3} u_2) + \mathrm{D}u_{\geqslant 3}^2$$

Paracontrolled ansatz for SBE

Recall:

$$u_1 \in C^{-1/2-},\ u_2 \in C^{0-},\ u_3 \in C^{1/2-}$$

$$\mathcal{L}u_{\geqslant 4} = 2\mathrm{D}(u_1 u_3) + 2(u_{\geqslant 4} \prec \mathrm{D}u_1) + \mathrm{D}u_2^2 + 2\mathrm{D}(u_1 \circ u_{\geqslant 4}) + 2(\mathrm{D}u_{\geqslant 4} \prec u_1)$$
$$+2\mathrm{D}(u_1 \succ u_{\geqslant 4}) + 2\mathrm{D}u_{\geqslant 3}u_2 + \mathrm{D}u_{\geqslant 3}^2$$

▷ Ansatz: $u_{\geqslant 4} = Q + v \prec X + v^\sharp$

$$\mathcal{L}u_{\geqslant 4} = \mathcal{L}Q + \mathcal{L}v \prec X + v \prec \mathcal{L}X - \mathrm{D}v \prec \mathrm{D}X + \mathcal{L}v^\sharp$$
$$\mathcal{L}Q = 2\mathrm{D}(u_1 u_3), \quad v = 2u_{\geqslant 4}, \qquad \mathcal{L}X = \mathrm{D}u_1$$
$$X \in C^{3/2-}, \qquad Q \in C^{1/2-}$$

▷ The Ugly:

$$u_1 \circ u_{\geqslant 4} = u_1 \circ (Q + v \prec X + v^\sharp) = u_1 \circ Q + u_1 \circ (v \prec X) + u_1 \circ v^\sharp$$
$$= u_1 \circ Q + v(u_1 \circ X) + R(v, u_1, X) + u_1 \circ v^\sharp$$

▷ Final equation:

$$\mathcal{L}v^\sharp = 2\mathrm{D}u_{\geqslant 4} \prec \mathrm{D}X + \mathcal{L}u_{\geqslant 4} \prec X + \mathrm{D}u_2^2 + 2\mathrm{D}(u_1 \circ u_{\geqslant 4})$$
$$+2(\mathrm{D}u_{\geqslant 4} \prec u_1) + 2\mathrm{D}(u_1 \succ u_{\geqslant 4}) + 2\mathrm{D}u_{\geqslant 3}u_2 + \mathrm{D}u_{\geqslant 3}^2$$

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3$: $\xi \in C^{-5/2-}$, $u \in C^{-1/2-}$, $u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 - c_1))$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects: $(u_1^2 - c_1)$, $(u_1^3 - 3c_1 u_1)$, $(3\lambda(X \circ (u_1^2 - c_1)) - c_2)$, $(u_2 u_1)$, $(u_2^2 u_1)$

Thanks