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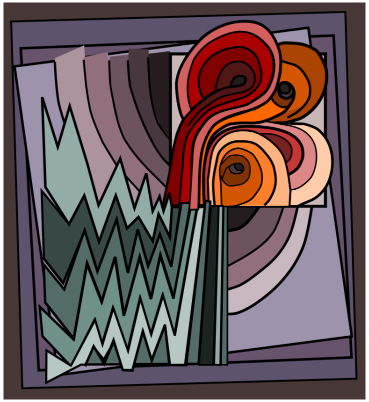
Stochastic analysis of Euclidean quantum fields



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[made with TeXmacs]

Euclidean quantum fields (EQFs)

a particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$:

EQF = regularity + Euclidean invariance + reflection positivity

introduced in the '70-'80 as a tool to constructs models of (bosonic) quantum field theories in the sense of Wightman via the reconstruction theorem of Osterwalder–Schrader.

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + V(\varphi(x))\mathrm{d}x$$

for some non-linear function $V:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$, e.g. a polynomial bounded below, exponentials, trig funcs. ill-defined representation:

- **large scale (IR) problems:** the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- **small scale (UV) problems:** sample paths are not expected to be function, but only distributions, the quantity $V(\varphi(x))$ does not make sense.

EQFs – history

▷ Construct rigorously QM models which are compatible with special relativity, (finite speed of signals and Poincaré covariance of Minkowski space $\mathbb{R}^{n+1}$).

▷ Quantum field theory (QM with ∞ many degrees of freedom)

▷ Wightman axioms ('60-'70): Hilbert space, representation of the Poincaré group, fields operators (to construct local observables).

▷ Constructive QFT program ('70-'80): hard to find models of such axioms. Examples in $\mathbb{R}^{1+1}$ were found in the $'60$. Glimm, Jaffe, Nelson, Segal, Guerra, Rosen, Simon, and many others…

▷ Euclidean rotation: $t \to it = x_0$ (imaginary time). $\mathbb{R}^{n+1} \to \mathbb{R}^d$ Minkowski → Euclidean

▷ Osterwalder–Schrader theorem : gives precise condition to perform the passage to/from Euclidean space (OS axioms for Euclidean correlation function).

▷ High point of EQFT: construction of Φ_3^4 (Euclidean version of a scalar field in $\mathbb{R}^{2+1}$ Minkowski space). $(\Phi_3^4)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75) $(\Phi_3^4)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)

▷ Other constructions of Φ_3^4. Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)

▷ simplest example of EQFT. We take a Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with covariance

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2 + |k|^2} \frac{\mathrm{d}k}{(2\pi)^d} = (m^2 - \Delta)^{-1}(x-y), \qquad x, y \in \mathbb{R}^d$$

and zero mean. Reflection positive, Eucl. covariant and regular. This is the GFF with mass $m > 0$.

▷ this measure can be used to construct a QFT in Minkowski space but unfortunately this theory is free, i.e. there is no interaction.

▷ note that $G(0) = +\infty$ if $d \geqslant 2$, this implies that the GFF is not a function.

▷ in particular GFF is a distribution of regularity $\alpha = (2-d)/2 - \kappa$ for any small $\kappa > 0$, e.g. locally in the sense of the scale of Besov–Holder spaces $(B^{\alpha}_{\infty,\infty})_{\alpha \in \mathbb{R}}$.

▷ heuristically we want

$$\nu(\mathrm{d}\varphi) = \frac{e^{\int_\Lambda V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi).$$

❶ go on a lattice: $\mathbb{R}^d \to \mathbb{Z}^d_\varepsilon = (\varepsilon\mathbb{Z})^d$ with spacing $\varepsilon > 0$ and make it periodic $\mathbb{Z}^d_\varepsilon \to \mathbb{Z}^d_{\varepsilon,L} = (\mathbb{Z}_\varepsilon / 2\pi L\mathbb{N})^d$.

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi) e^{\overbrace{-\frac{1}{2}\sum_{x\in\mathbb{Z}^d_{\varepsilon,L}} |\nabla_\varepsilon\varphi(x)|^2 + m^2\varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}}\mathrm{d}\varphi$$

ε is an UV regularisation and L the IR one.

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below (otherwise integrab. problems). $d = 2, 3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure.

❸ study the possible limit points (uniqueness? non-uniqueness? correlations? description?)

▷ for $d=2$ other choices are possible:

$$V_\varepsilon(\xi)=\lambda\xi^{2l}+\sum_{k=0}^{2l-1}a_{k,\varepsilon}\xi^k,\quad V_\varepsilon(\xi)=a_\varepsilon\cos(\beta\xi)$$

$$V_\varepsilon(\xi)=a_\varepsilon\cosh(\beta\xi),\qquad V_\varepsilon(\xi)=a_\varepsilon\exp(\beta\xi)$$

▷ for $d=3$ "only" 4th order (6th order is critical).

▷ for $d=4$ all the possible limits are Gaussian (see recent work of Aizenmann-Duminil Copin, arXiv:1912.07973)

We are interested in limits of quantities like

$$\lim_{\varepsilon\to 0, L\to\infty} \int \varphi(f_1)\cdots\varphi(f_n)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \int \varphi(f_1)\cdots\varphi(f_n)\nu(\mathrm{d}\varphi)$$

for arbitrary test functions $f_1,\ldots,f_n \in \mathscr{S}(\mathbb{R}^d)$. For $d=2,3$ problem solved in $'70 -' 80$ by Glimm, Jaffe, …

Parisi-Wu, Nelson ('84): introduce a stochastic differential equation (SDE) which has ν as invariant measure.

For clarity we work with $\nu^{\varepsilon,L}$. In Parisi-Wu's approach the SDE is a Langevin equation of the form

$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -\nabla_\varphi S_\varepsilon(\Phi(t,x)) + 2^{1/2}\xi(t,x), \qquad x\in\Lambda_{\varepsilon,L} = \mathbb{Z}^d_{\varepsilon,L}, t\geqslant 0$$

Here $\xi(t,x)$ is a space-time white noise.

If $\mathrm{Law}(\Phi(t=0)) = \nu^{\varepsilon,L}$ then $\mathrm{Law}(\Phi(t)) = \nu^{\varepsilon,L}$ for all $t\geqslant 0$

an history of stochastic quantisation (personal & partial)

- 1984 – Parisi/Wu – SQ (for gauge theories)
- 1985 – Jona-Lasinio/Mitter – "On the stochastic quantization of field theory" (rigorous SQ for Φ^4_2 on bounded domain)
- 1988 – Damgaard/Hüffel – review book on SQ (theoretical physics)
- 1990 – Funaki – Control of correlations via SQ (smooth reversible dynamics)
- 1990–1994 – Kirillov – "Infinite-dimensional analysis and quantum theory as semimartingale calculus", "On the reconstruction of measures from their logarithmic derivatives", "Two mathematical problems of canonical quantization."
- 1993 – Ignatyuk/Malyshev/Sidoravichius – "Convergence of the Stochastic Quantization Method I,II" [Grassmann variables + cluster expansion]
- 2000 – Albeverio/Kondratiev/Röckner/Tsikalenko – "A Priori Estimates for Symmetrizing Measures…" [Gibbs measures via IbP formulas]
- 2003 – Da Prato/Debussche – "Strong solutions to the stochastic quantization equations"
- 2014 – Hairer – Regularity structures, local dynamics of Φ^4_3
- 2017 – Mourrat/Weber – coming down from infinity for Φ^4_3
- 2018 – Albeverio/Kusuoka – "The invariant measure and the flow associated to $\Phi^4_3\ldots$"
- 2021 – Hofmanova/G. – Global space-time solutions for Φ^4_3 and verification of axioms
- 2020-2021 – Chandra/Chevyrev/Hairer/Shen – SQ for Yang–Mills 2d/3d.

the dynamics give a map $\hat{G}_{\varepsilon,L}$ which transform a Gaussian measure into $\nu^{\varepsilon,L}$.

this map passes to the limit as $\varepsilon \to 0$ and $L \to \infty$ and is associated to an SPDE in the limit

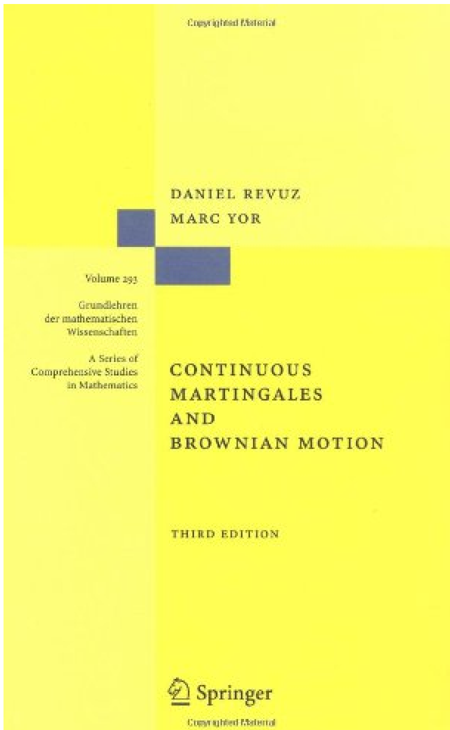
$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -(m^2 - \Delta)\Phi(t,x) - {}'' V'(\Phi(t,x))'' + 2^{1/2}\xi(t,x).$$

Theorem. $d = 3$ *provided* $(a_\varepsilon)_\varepsilon$ *is chosen approp. there exist a stationary in space and time solution to the limit SPDE and moreover the law of the solution at any given time in a non-Gaussian EQFT* ν *(without rotation invariance). It satisfies an IBP formula:*

$$\int \nabla_\varphi F(\varphi)\nu(\mathrm{d}\varphi) = \int F(\varphi)(-(m^2 - \Delta)\varphi - [\![\varphi^3]\!])\nu(\mathrm{d}\varphi).$$

[details in Gubinelli-Hofmanova CMP 2021, "A PDE construction…"]

stochastic analysis of EQFs?



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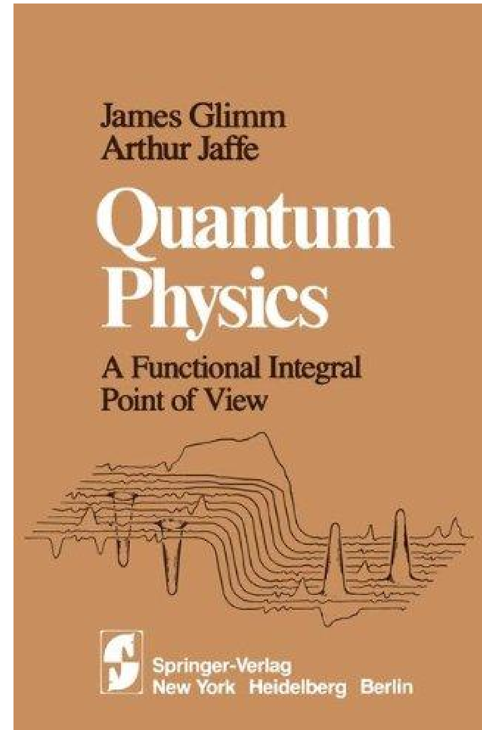

▷ Ito & Dœblin introduced a variety of analysis adapted to the sample paths of a stochastic process.

▷ consider a family of kernels $(P_t)_{t\geqslant 0}$ on $\mathbb{R}^d$ satisfying Chapman–Kolmogorov equation

$$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$$

which defines a probability $\mathbb{P}$ on $C(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$: the law of a continuous Markov process.

▷ sample paths have a *"tangent" process*. Ito identified it as a particular Lévy process: the Brownian motion $(W_t)_t$.

▷ stochastic calculus: from the local picture to the global structure via *stochastic differential equation* (SDE)

$$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$$

▷ these are the basic building blocks of **stochastic analysis**

▷ like in analysis, the fact that we can consider infinitesimal changes simplify the analysis and make appear universal underlying objects:

- polynomials → calculus, Taylor expansion
- Brownian motion and its functionals → Ito calculus, stochastic Taylor expansion

to have an analysis we need:

- a **change parameter** along which consider "change" (*time* for diffusions)
- a suitable **building block** for the infinitesimal changes (*Brownian motion* for diffusion)

▷ other examples: rough paths, regularity structures, SLE,...

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y)\in\mathcal{O}\subseteq\mathbb{R}^2$	global description	$P_t(x,\mathrm{d}y)$
$\alpha(x-x_0)^2+\beta(y-y_0)^2=\gamma$	·	$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t)\approx x(t)+a\delta t+o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at+bt^2+\cdots$	building block	$(W_t)_t$
$(\ddot{x}(t),\ddot{y}(t))=F(x(t),y(t))$	local/global link	$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x, \mathrm{d}y)$	global description	$\nu \in \mathrm{Prob}(\mathscr{S}'(\mathbb{R}^d))$
$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$		$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-S(\varphi)} \mathrm{d}\varphi$
t	change parameter	t
$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}},$	local description	$\phi(t+\delta t) \approx \alpha\phi(t) + \beta\delta X(t) + \cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X = \frac{1}{2}[(\Delta_x - m^2)X] + 2^{1/2}\xi$
$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$	local/global link	$\partial_t\phi = \frac{1}{2}[(\Delta_x - m^2)\phi - V'(\phi)] + 2^{1/2}\xi$

- **parabolic stochastic quantisation.** the parameter is an additional "fictious" coordinate $t \in \mathbb{R}$, playing the röle of a simulation time. The EQF is viewed as the invariant measure of a Markov process (SDE). Building block is a space-time white noise. [Parisi/Wu, Nelson, Jona-Lasinio/Mitter, Kirillov, Funaki, Albeverio/Röckner, Da Prato/Debussche, Hairer, Catellier/Chouk, Mourrat/Weber, G./Hofmanova, Albeverio/Kusuoka, Chandra/Moinat/Weber, Shen, Garban, many others…]

$$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **canonical stochastic quantisation.** same as for parabolic, but the evolution takes place in "phase space" and the SDE is second order in time, giving rise to a stochastic wave equation. [G./Koch/Oh, Tolomeo, Oh/Robert/Wang]

$$\partial_t^2 \phi + \partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **elliptic stochastic quantisation.** the parameter is a coordinate $z \in \mathbb{R}^2$. Building block is a white noise in $\mathbb{R}^{d+2}$. An elliptic stochastic partial differential equation describes the EQF as a function of the white noise. Link with supersymmetry.

  [Parisi/Sourlas, Klein/Landau/Perez, Albeverio/De Vecchi/G., Barashkov/De Vecchi]

$$-\Delta_z \phi(z,x) = \frac{1}{2}[(\Delta_x - m^2)\phi(z,x) - V'(\phi(z,x))] + 2^{1/2}\xi(z,x)$$

- **variational method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The EQF is described as the solution of a stochastic optimal control problem. [Barashkov/G.]
- **rg method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The effective action of the EQF satisfies an Hamilton–Jacobi–Bellmann equation. [Wilson, Wegner, Polchinski, Salmhofer, Brydges/Kennedy, Mitter, Gawedzki/Kupiainen, Brydges/Bauerschmidt/Slade, Bauerschmidt/Bodineau, Bauerschmidt/Hofstetter, also many others…]

the interacting field ϕ is expressed as a function of the Gaussian free field X:

$$\phi(t) = F(X), \qquad \nu = \mathrm{Law}(\phi(t)) = F_* \mathrm{Law}(X) = F_* \mathrm{GFF}$$

- estimates on ϕ obtained via two ingredients:
  - pathwise PDE estimates for the map F (in weighted Besov spaces)
  - probabilistic estimates for the GFF X
- coupling (φ, X)

  $$\phi = X + \psi$$

  where ψ is a random field which is more regular (i.e. smaller at small scale) than X (link with asymptotic freedom/perturbation theory)

  note that

  $$\nu = \mathrm{Law}(\varphi) \not\ll \mathrm{Law}(X(t)) = \mathrm{GFF}$$

estimates

- ▷ decomposition: $\phi = X + \psi$

$$\partial_t \psi = \frac{1}{2}[(\Delta_x - m^2)\psi - V'(X + \psi)]$$

- ▷ PDE estimates:

$$\| \psi(t) \| \leqslant H(\| X \|)$$

- ▷ tightness:

$$\int \| \varphi \|^p \nu(\mathrm{d}\varphi) \lesssim \mathbb{E} \| X \|^p + \mathbb{E} \| \psi(t) \|^p \leqslant \mathbb{E} \| X \|^p + \mathbb{E}[H(\| X \|)^p] < \infty$$

- ▷ tail-estimates:

$$\int e^{c \| \varphi \|^\alpha} \nu(\mathrm{d}\varphi) < \infty$$

[Moinat/Weber, Hofmanova/G., Hairer/Steele]

properties of the stochastically quantized EQF

Φ^4_3 measure. $p(\varphi) = \lambda\varphi^4 - c\varphi^2$, $d = 3$. [Hofmanova/G. - CMP 2021]

▷ non-gaussianity:

$$\langle\varphi\varphi\varphi\varphi\rangle_c = \langle XXXX\rangle_c + 4\langle XXX\psi\rangle_c + 12\langle XX\psi\psi\rangle_c + 4\langle X\psi\psi\psi\rangle_c + \langle\psi\psi\psi\psi\rangle_c$$

$$= 4\langle XXX\psi\rangle_c + \cdots \neq 0$$

▷ renormalized cube:

$$[\![\varphi^3]\!] = \lim_{\varepsilon\to 0} [(\rho_\varepsilon * \varphi)^3 - c_\varepsilon(\rho_\varepsilon * \varphi)] = [\![X^3]\!] + \{([\![X^2]\!] *_r \psi) + X\psi^2 + \psi^3\}$$

result: $[\![\varphi^3]\!]$ is not a random variable but a distribution on $\mathrm{Cyl}(\mathscr{S}'(\mathbb{R}^3))$.

▷ Dyson–Schwinger equation (IBP formula for ν):

$$\int \mathrm{D}_\varphi F(\varphi)\nu(\mathrm{d}\varphi) = \int F(\varphi)\left\{(\Delta - m^2)\varphi - \lambda[\![\varphi^3]\!]\right\}\nu(\mathrm{d}\varphi)$$

outlook

goal: develop a stochastic analysis of EQFs
(at least for superrenormalizable models)

- identify "building blocks" and describe EQFs (non-perturbatively) in terms of these simpler objects.
- small scales behaviour/renormalization: well understood in most models in some of the approaches (see e.g. recent results of Hairer et al. on Yang-Mills fields).
- coercivity (large fields problem) plays a key role for global control and infinite volume limit. So far, difficult for YM (or even Higgs).
- uniqueness (high or low temp)? still open (in sq) in most models, especially $\Phi^4_{2,3}$.

some recent literature on SQ for EQF

[I list here some results which apply to the $\varepsilon \to 0$ and $L \to \infty$. More results are available on a finite box]

- construction of Φ_3^4 by G./Hofmanova (CMP 2021) and IbP formula
- construction of the $(\exp(\beta\varphi))_2$ model via elliptic SQ (arXiv:1906.11187)
- construction of Sinh-Gordon $d=2$ (all axioms) by Barashkov/de Vecchi via elliptic SQ (arXiv:2108.12664)
- optimal bounds by Hairer/Steele (arXiv:2102.11685)
- some results on phase transition by Chandra/Gunaratnam/Weber (arXiv:2006.15933)
- ongoing work on control of correlations by G./Hofmanova/Rana
- recent paper on perturbation theory for Φ_2^4 by Shen/Zhu/Zhu (arXiv:2108.11312)
- work on the $N \to \infty$ limit of the $O(N)$ model by Shen/Zhu/Zhu

open problems

- how to apply these ideas to gauge theories/geometric models? Higgs model, Yang-Mills? [Hairer/Zambotti/Chandra/Chevyrev/Shen/...] coercivity not well understood.
- Grassmann fields? [partial progress in Albeverio/Borasi/De Vecchi/G., no renorm yet]
- small coupling regime? (proof of Borel-summability?)
- decay of correlations at high temperature? [some results Rana/Hofmanova/G.]
- Dyson-Schwinger eq. / IbP formulas determines the measure?
- weak-universality and triviality of models above the critical dimension?
- how to apply these ideas directly in Minkowski space? (i.e. develop a non-commutative stochastic analysis for fields)
- ...

thanks.

some references

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some details on the construction of Φ^4_2

coupling to the GFF

- we work on $\Lambda_{\varepsilon,L} = \mathbb{Z}^d_{\varepsilon,L}$. The solution $X\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{\Lambda_{\varepsilon,L}}$ to

$$\mathrm{d}X_t(x) = -(AX_t)(x)\mathrm{d}t - \frac{1}{2}V_\varepsilon'(X_t(x))\mathrm{d}t + 2^{1/2}\mathrm{d}B_t(x) \qquad x \in \Lambda_{\varepsilon,L}$$

with $A = m^2 - \Delta$ (discrete Laplacian) leaves the measure

$$\nu^{\varepsilon,L}(\mathrm{d}\varphi) = Z^{-1} e^{-\sum_{x\in\Lambda_{\varepsilon,L}} V_\varepsilon(\varphi(x))} \mu^{\varepsilon,L}(\mathrm{d}\varphi), \qquad V_\varepsilon(\xi) = \lambda\xi^4 - \beta_\varepsilon\xi^2$$

invariant. here $(B_t(x))_{t\geqslant 0, x\in\Lambda_{\varepsilon,L}}$ are iid BM and $\mu^{\varepsilon,L}$ is the GFF (i.e. $\mathscr{N}(0, A^{-1})$).

- let Y be the solution of the linear equation (dynamic GFF):

$$\mathrm{d}Y_t = -AY_t\mathrm{d}t + 2^{1/2}\mathrm{d}B_t,$$

with invariant measure $\mu^{\varepsilon,L}$. define $Z = X - Y$ which solves a RDE:

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V_\varepsilon'(Y_t + Z_t).$$

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V_\varepsilon'(Y_t + Z_t)$$

$$V_\varepsilon'(\varphi) = \lambda\varphi^3 - \beta\varphi$$

- introduce a polynomial weight $\rho\colon \Lambda = (\varepsilon\mathbb{Z})^d \to \mathbb{R}$

$$\rho(x) = (1 + \ell\,|x|)^{-\sigma}, \qquad \sigma > 0, \ell > 0,$$

- test the equation for Z with $\rho^2 Z$ summing over the full lattice Λ

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Lambda_\varepsilon} |\rho(x)Z_t(x)|^2 + G(Z_t) \leqslant -\lambda \sum_{x\in\Lambda_\varepsilon} \rho(x)(Y_t(x)^3 Z_t(x) + 3Y_t(x)^2 Z_t(x)^2 + 3Y_t(x)Z_t(x)^3)$$

$$+\beta \sum_{x\in\Lambda_\varepsilon} \rho(x)(Z_t(x)Y_t(x) + Z_t(x)^2) + C_{\rho,\ell} \sum_{x\in\Lambda_\varepsilon} \rho(x)Z_t(x)^2$$

$$G(\varphi) = \|\rho\nabla\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + m^2\|\rho\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + \lambda\|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4.$$

▷ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leqslant C_\delta \|\rho^{1/2} Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + \delta G(Z_t)$$

indeed the interaction terms can be estimated as

$$\lambda \left| \sum_{x\in\Lambda_\varepsilon} \rho(x) Y_t(x)^3 Z_t(x) \right| \leqslant \lambda \left| \sum_{x\in\Lambda_\varepsilon} (\rho(x)^{3/2} Y_t(x)^3)(\rho(x)^{1/2} Z_t(x)) \right|$$

$$\leqslant \lambda \frac{C}{\delta} \|\rho^{1/2} Y_t\|_{L^4}^4 + \delta\lambda \|\rho^{1/2} Z_t\|_{L^4}^4 \leqslant \lambda \frac{C}{\delta} \|\rho^{1/2} Y_t\|_{L^4}^4 + \delta G(Z_t)$$

for any small $\delta > 0$.

$$\boxed{\|\rho Z_t\|_{L^2(\Lambda)}^2 + \frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s \leqslant \|\rho Z_0\|_{L^2(\Lambda)}^2 + C\int_0^t \|\rho^{1/2} Y_s\|_{L^4(\Lambda)}^4 \mathrm{d}s}$$

tightness

▷ use a stationary coupling of (Y,Z):

$$\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2=\mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2$$

so

$$\mathbb{E}G(Z_0)=\frac{1}{t}\int_0^t\mathbb{E}G(Z_s)\mathrm{d}s\leqslant\frac{2C}{t}\int_0^t\mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s=\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4$$

$$\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4=\mathbb{E}\sum_{x\in\Lambda_\varepsilon}\rho(x)^2|Y_0(x)|^4=\sum_{x\in\Lambda_\varepsilon}\rho(x)^2\mathbb{E}|Y_0(x)|^4=C\sum_{x\in\Lambda_\varepsilon}\rho(x)^2<\infty$$

uniformly in L. from this estimate one can deduce that

$$\sup_L\int\|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4\nu^{\varepsilon,L}(\mathrm{d}\varphi)<\infty$$

this is a key estimate to take the infinite volume limit since it allows to use tightness on the family $(\nu^{\varepsilon,L})_L$ in the topology of local convergence.

it gives also a stationary infinite volume limit coupling to the GFF.

bounds

▷ the local (or weighted) $L^p(\Lambda_\varepsilon)$ norms of $\varphi: \mathbb{R}^{\Lambda_\varepsilon} \to \mathbb{R}$ under the measure $\nu^{\varepsilon,M}$ have finite moments:

$$\sup_L \int \|\rho\varphi\|_{L^p}^p \nu^{\varepsilon,L}(\mathrm{d}\varphi) < \infty$$

for any $p > 1$.

▷ by working a bit harder one can prove uniform integrability of functions like $\exp(\|\rho\varphi\|_{L^2})$. (see Gubinelli-Hofmanova CMP 2021)

▷ another approach is to use the “coming down from infinity” to remove dependence on the initial condition (see Mourrat-Weber CMP 2017, Gubinelli-Hofmanova CMP 2020, Moinat-Weber CPAM 2020)

optimal bounds: Hairer/Steele argument

▷ we want to bound $Z_H = \int e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi)$ for some nice function $H(\varphi) \geqslant 0$.

▷ the idea of Hairer/Steele (slightly revisited here) is to consider the new measure

$$\rho^H(\mathrm{d}\varphi) = \frac{e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi)}{Z_H} = \frac{e^{H(\varphi) - V_\varepsilon(\varphi)}}{Z_H Z_{\varepsilon,L}} \mu^{\varepsilon,L}(d\varphi)$$

and observe that by Jensen's:

$$1 = \int e^{-H(\varphi)} e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi) = Z_H \int e^{-H(\varphi)} \rho^H(\mathrm{d}\varphi) \geqslant Z_H \exp\left(-\int H(\varphi) \rho^H(\mathrm{d}\varphi)\right)$$

so

$$\log Z_H \leqslant \int H(\varphi) \rho^H(\mathrm{d}\varphi).$$

▷ the SQ of ρ^H can be used as before to obtain bounds which depends only on the GFF provided (e.g.) H is controlled by G (with natural hypothesis):

$$\left| \sum_\Lambda \rho^2 \varphi H'(\psi + \varphi) \right| \leqslant Q(\psi) + \delta G(\varphi), \qquad |H(\psi + \varphi)| \leqslant Q(\psi) + G(\varphi)$$

▷ shifted SQ equation

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V_\varepsilon'(Y_t + Z_t) + H'(Y_t + Z_t)$$

▷ bounds + stationary coupling

$$\mathbb{E}G(Z_0) = \frac{1}{t}\int_0^t \mathbb{E}G(Z_s)\mathrm{d}s \leqslant \frac{2C}{t}\int_0^t \mathbb{E}\{\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4 + Q(Y_s)\}\mathrm{d}s = \mathbb{E}\{\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 + Q(Y_0)\}$$

therefore

$$\int H(\varphi)\rho^H(\mathrm{d}\varphi) = \mathbb{E}[H(X_0)] = \mathbb{E}[H(Y_0 + Z_0)] \leqslant C\,[\mathbb{E}\{\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 + Q(Y_0)\}] < \infty.$$

example: $H(\varphi) = \eta\|\rho\varphi\|_{L^4}^4$ for $\eta > 0$ small gives the optimal bound

$$\sup_L \int e^{\eta\|\rho\varphi\|_{L^4}^4}\nu^{\varepsilon,L}(\mathrm{d}\varphi) < \infty.$$

coupling of two solutions

▷ let (Z^1, Y^1) and (Z^2, Y^2) be two solutions of the shifted SQ equation. then $H = Z^1 - Z^2$ solves

$$\partial_t H - AH = Q := -[V'(X^1) - V'(X^1 + H + K)] = -\underbrace{\int_0^1 \mathrm{d}\tau V''(X^1 + \tau(H+K))}_{=:G \geqslant -\chi}(H+K)$$

with $K := Y^1 - Y^2$ and $X^1 = Y^1 + Z^1$. assume that $V''(\varphi) \geqslant -\chi$ for some $\chi > 0$.

▷ estimates with $\rho(x) = e^{-\theta|x|}$ e.g. when $K = Y^1 - Y^2$ is stationary:

$$\mathbb{E}\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct}\mathbb{E}\|\rho H_0\|_{L^2}^2 + C\sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}\, K_0^4(x))^{1/2}$$

- by coupling two different invariant measures via a common dynamics ($K = 0$) one can show that the two measures are equal. this gives uniqueness.
- one can use noises which coincide in a bounded region Ω to drive two different dynamics. in this case $K = 0$ in Ω and this shows that the two solutions X^1 and X^2 are near inside $\Omega' \subseteq \Omega$.
- one can modify this setup to obtain decay of correlations in SQ (work in progress with Hofmanova and Rana, already used by Funaki in more regular setting).

the small scale limit $\varepsilon \to 0$ for L fixed

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2)\, Z_t - V'(Y_t + Z_t), \qquad \text{with } V'(\varphi) = \lambda \varphi^3 + \beta\, \varphi.$$

▷ renormalized drift term:

$$V'(Y + Z) - \lambda Z^3 = \lambda\, \mathbb{Y}^3 + 3\lambda\, \mathbb{Y}^2 Z + 3\lambda\, \mathbb{Y}^1 Z^2$$

▷ pde estimates

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1 - \delta) \int_{\mathbb{T}_\varepsilon^d} \left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2} |Z_t|^4 \right] \leqslant Q(\mathbb{Y}_t^\varepsilon),$$

where

$$Q(\mathbb{Y}_t^\varepsilon) := 1 + C \sum_{k=1,2,3} \| \mathbb{Y}_t^k \|_{\mathscr{C}^{k\alpha}}^K,$$

▷ probabilistic estimates

$$\sup_\varepsilon \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

putting all together

▷ improving the renormalized apriori estimates with a spatial weight ρ (+ some results on weighted Besov spaces) one can prove the same estimates in weighted spaces:

$$\sup_{\varepsilon,L} \int (\|\rho\psi\|_{\mathscr{C}^\alpha}^2 + \|\rho\nabla\zeta\|_{L^2}^2 + m^2\|\rho\zeta\|_{L^2}^2 + \lambda\|\rho^{1/2}\zeta\|_{L^4}^4)\gamma^{\varepsilon,L}(\mathrm{d}\psi\times\mathrm{d}\zeta)$$

$$\leqslant 1 + C\sup_{\varepsilon,L} \sum_{k=1,2,3} \mathbb{E}\|\rho^\sigma \mathbb{Y}_0^{\varepsilon,L,k}\|_{\mathscr{C}^{k\alpha}}^K < +\infty$$

(some $\sigma > 0$)

▷ using Hairer/Steele kind of arguments also have uniform exponential bounds of the form

$$\sup_{\varepsilon,L} \int e^{\eta\|\rho(\psi+\zeta)\|_{B_{4,4}^\alpha}^4} \gamma^{\varepsilon,L}(\mathrm{d}\psi\times\mathrm{d}\zeta) < \infty$$

for some small η

Theorem. *Provided $d=2$ and we take $\beta=-3\lambda c_\varepsilon+\beta'$ for some constant $\beta'\in\mathbb{R}$ and $c_\varepsilon=\mathbb{E}[Y_t^\varepsilon(x)^2]$ then the family $(\nu^{\varepsilon,L})_{\varepsilon,L}$ is tight in $\mathscr{S}'(\mathbb{R}^2)$.*

Any accumulation point ν is regular, RP and translation invariant and satisfies

$$\int e^{\eta\|\rho\varphi\|_{B_{4,4}^\alpha}^4}\nu(\mathrm{d}\varphi)<\infty \tag{1}$$

for small $\eta>0$. (no rotation invariance due to lack of uniqueness)

▷ any limiting measure ν is non-Gaussian due to (1) (cfr. Hairer/Steele for $d=3$).

▷ we actually construct a stationary coupling (Y,Z) with $Y+Z\sim\nu$ which solves the system

$$\frac{\partial}{\partial t}Z_t=(\Delta-m^2)\,Z_t-\lambda Z_t^3-\lambda\,\mathbb{Y}_t^3+3\lambda\,\mathbb{Y}_t^2 Z_t+3\lambda\,\mathbb{Y}_t^1\,Z_t^2+\beta'\,Y_t+\beta'\,Z_t$$

$$\frac{\partial}{\partial t}Y_t=(\Delta-m^2)\,Y_t+\xi(t,\cdot)$$

[Detailed construction of the $d=3$ case in Hofmanova/G. - CMP 2021]

▷ the SQ proof of uniqueness sketched on the lattice fails for the renomalized equation since we do not have anymore convexity (we subtracted an infinite 2nd order polynomial):

$$H = Z^{(1)} - Z^{(2)}, \qquad \Upsilon^{(1)} = \Upsilon^{(2)} = \Upsilon$$

$$\frac{\partial}{\partial t} H_t = (\Delta - m^2) H_t - \lambda \int_0^1 \mathrm{d}\tau \left\{ 3[Z_t^{(2)} + \tau H_t]^2 + 6\, \mathbb{Y}_t^1 [Z_t^{(2)} + \tau H_t] + 3\, \mathbb{Y}_t^2 \right\} H_t + \beta' H_t$$

OPEN PROBLEM

▷ the "standard" approach to uniqueness of the limit (in certain conditions) is via correlation inequalities or cluster expansion [see Glimm-Jaffe's book].

▷ uniqueness in finite volume via Markovian techniques (irreducibility, see e.g. Hairer-Steele)

renormalized cube

▷ any limit coupling $\gamma(\mathrm{d}X\times\mathrm{d}\psi)$ is supported on

$$\mathscr{C}^{\alpha}(\rho)\times(H^1(\rho)\cap L^4(\rho^{1/2}))$$

more regularity of the second component can be obtained by using parabolic estimates on the equation, essentially one can arrive to $2+\alpha$ spatial regularity.

▷ under the measure $\gamma(\mathrm{d}X\times\mathrm{d}\psi)$ we have $\varphi=X+\psi\sim\nu$ and

$$[(\theta_\varepsilon*\varphi)^3-3c_\varepsilon(\theta_\varepsilon*\varphi)]=\underbrace{[(\theta_\varepsilon*X)^3-3c_\varepsilon(\theta_\varepsilon*X)]}_{\to[\![X^3]\!]\text{ in }\mathscr{C}^{3\alpha}(\rho)}+3\underbrace{[(\theta_\varepsilon*X)^2-c_\varepsilon]}_{\to[\![X^2]\!]\text{ in }\mathscr{C}^{2\alpha}(\rho)}\underbrace{(\theta_\varepsilon*\psi)}_{\to\psi\text{ in }H^{1-\kappa}(\rho)}$$

$$+3\underbrace{(\theta_\varepsilon*X)}_{\to X\text{ in }\mathscr{C}^{\alpha}(\rho)}\underbrace{(\theta_\varepsilon*\psi)^2}_{\to\psi^2\text{ in }B^{1-\kappa}_{1,1}(\rho)}+(\theta_\varepsilon*\psi)^3$$

$$\xrightarrow[\varepsilon\to 0]{}[\![X^3]\!]+\{[\![X^2]\!]\psi+X\psi^2+\psi^3\}=:[\![\varphi^3]\!]$$

the terms in the r.h.s are under control as products of Besov functions

integration by parts formula

▷ at the discrete level we have

$$\int \nabla_{\varphi} F(\varphi) \nu^{\varepsilon,L}(\varphi) = \int F(\varphi) \Big\{ (\Delta_{\varepsilon} - m^2)\varphi - \lambda(\varphi^3 - c_{\varepsilon}\varphi) \Big\} \nu^{\varepsilon,L}(\varphi)$$

▷ estimates and tightness allow to pass to the limit in this equation and obtain an IBP formula for any accumulation point ν

$$\int \nabla_{\varphi(f)} F(\varphi) \nu(\varphi) = \int F(\varphi) \Big\{ \varphi((\Delta - m^2) f) - \lambda [\![\varphi^3]\!](f) \Big\} \nu(\varphi)$$

where appears the renormalized square $[\![\varphi^3]\!]$ which is well defined under ν as

$$[\![\varphi^3]\!](f) = \lim_{\varepsilon \to 0} \left[\int (\theta_{\varepsilon} * \varphi)^3 f - 3 c_{\varepsilon} \int (\theta_{\varepsilon} * \varphi) f \right]$$

▷ Dyson–Schwinger equations for correlation functions by taking $F(\varphi) = \varphi(f_1) \cdots \varphi(f_n)$:

$$\sum_k \int \varphi(f_1) \cdots \varphi(\cancel{f_k}) \cdots \varphi(f_n) \nu(\varphi) = \int \varphi(f_1) \cdots \varphi(f_n) \Big\{ \varphi((\Delta - m^2) f) - \lambda [\![\varphi^3]\!](f) \Big\} \nu(\varphi)$$

the variational method for Φ^4_2

Theorem. *(Let $(B_t)_{t\geqslant 0}$ be a Brownian motion on $\mathbb{R}^n$, then for any bounded F: $C(\mathbb{R}_+;\mathbb{R}^n)\to\mathbb{R}$ we have*

$$\log\mathbb{E}[e^{F(B_\bullet)}]=\sup_{u\in\mathbb{H}_a}\mathbb{E}\left[F(B_\bullet+I(u)_\bullet)-\frac{1}{2}\int_0^\infty|u_s|^2\mathrm{d}s\right]$$

with $u:\Omega\times\mathbb{R}_+\to\mathbb{R}^n$ adapted to B and with

$$I(u)_t:=\int_0^t u_s\mathrm{d}s.$$

$$\frac{1}{2}\int_0^\infty|u_s|^2\mathrm{d}s\approx H(\mathrm{Law}(B_\bullet+I(u)_\bullet)\,|\,\mathrm{Law}(B_\bullet)).$$

M. Boué and P. Dupuis, `A Variational Representation for Certain Functionals of Brownian Motion', *The Annals of Probability* 26, no. 4: 1641–59 https://doi.org/10.1214/aop/1022855876

$$\mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)(m^2-\Delta)^{-1}(x-y), \qquad t,s\in[0,1].$$

The BD formula gives

$$-\log\int e^{-F(\phi)}\mu(\mathrm{d}\phi) = -\log\mathbb{E}[e^{-F(W_1)}] = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[F(W_1+Z_1)+\frac{1}{2}\int_0^1\|u_s\|_{L^2}^2\mathrm{d}s\right],$$

where

$$Z_t = (m^2-\Delta)^{-1/2}\int_0^t u_s\mathrm{d}s, \qquad u_t = (m^2-\Delta)^{1/2}\dot{Z}_t$$

$$-\log\mathbb{E}[e^{-F(W_1)}] = \inf_{Z\in H^a}\mathbb{E}[F(W_1+Z_1)+\mathscr{E}(Z_\bullet)],$$

with

$$\mathscr{E}(Z_\bullet) := \frac{1}{2}\int_0^1\|(m^2-\Delta)^{1/2}\dot{Z}_s\|_{L^2}^2\mathrm{d}s = \frac{1}{2}\int_0^1(\|\nabla\dot{Z}_s\|_{L^2}^2+m^2\|\dot{Z}_s\|_{L^2}^2)\mathrm{d}s$$

Φ^4_2 in a bounded domain Λ

Fix a compact region $\Lambda \Subset \mathbb{R}^2$ and consider the Φ^4_2 measure θ_Λ on $\mathscr{S}'(\mathbb{R}^2)$ with interaction in Λ and given by

$$\theta_\Lambda(\mathrm{d}\phi) := \frac{e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}{\int e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)} \qquad \phi \in \mathscr{S}'(\mathbb{R}^2) \tag{2}$$

with interaction potential $V_\Lambda(\phi) := \int_\Lambda \phi^4 - c\int_\Lambda \phi^2$. For any $f: \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let

$$e^{-\mathscr{W}_\Lambda(f)} := \int e^{-f(\phi)}\theta_\Lambda(\mathrm{d}\phi).$$

We have the variational representation, $Z = Z_1$, $Z_\bullet = (Z_t)_{t\in[0,1]}$:

$$\mathscr{W}_\Lambda(f) = \inf_{Z \in H^a} F^{f,\Lambda}(Z_\bullet) - \inf_{Z \in H^a} F^{0,\Lambda}(Z_\bullet)$$

where

$$F^{f,\Lambda}(Z_\bullet) := \mathbb{E}[f(W+Z) + \lambda V_\Lambda(W+Z) + \mathscr{E}(Z_\bullet)].$$

renormalized potential

$$V_\Lambda(W+Z)=\int_\Lambda\left\{\underbrace{W^4-cW^2}_{\mathbb{W}^4}+4\underbrace{\left[W^3-\frac{c}{4}W\right]}_{\mathbb{W}^3}Z+6\underbrace{\left[W^2-\frac{c}{6}\right]}_{\mathbb{W}^2}Z^2+4WZ^3+Z^4\right\}$$

take $c=12\mathbb{E}[W^2(x)]=+\infty$

$$V_\Lambda(W+Z)=\int_\Lambda\left\{4\mathbb{W}^3Z+6\mathbb{W}^2Z^2+4WZ^3+Z^4\right\}+\cdots$$

$$\mathbb{W}^n\in\mathscr{C}^{-n\kappa}(\Lambda)=B^{-n\kappa}_{\infty,\infty}(\Lambda)$$

Here $B^{-\kappa}_{\infty,\infty}(\Lambda)$ is an Hölder–Besov space. A distribution $f\in\mathscr{S}'(\mathbb{T}^d)$ belongs to $B^{\alpha}_{\infty,\infty}(\Lambda)$ iff for any $n\geqslant 0$

$$\|\Delta_n f\|_{L^\infty}\leqslant(2^n)^{-\alpha}\|f\|_{B^{\alpha}_{\infty,\infty}(\Lambda)}$$

where $\Delta_n f=\mathscr{F}^{-1}(\varphi_n(\cdot)\mathscr{F}f)$ and φ_n is a function supported on an annulus of size $\approx 2^n$. We have $f=\sum_{n\geqslant 0}\Delta_n f$. If $\alpha>0$ $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d)$ is a space of functions otherwise they are only distributions.

Lemma. *There exists a minimizer* $Z = Z^{f,\Lambda}$ *of* $F^{f,\Lambda}$*. Any minimizer satisfies the Euler–Lagrange equations*

$$\begin{aligned}&\mathbb{E}\left(4\lambda\int_{\Lambda} Z^3 K + \int_0^1 \int_{\Lambda} (\dot{Z}_s (m^2 - \Delta) \dot{K}_s) \mathrm{d}s\right)\\ = \ &\mathbb{E}\left(\int_{\Lambda} f'(W+Z) K + \lambda \int_{\Lambda} (\mathbb{W}^3 + \mathbb{W}^2 Z + 12 W Z^2) K\right)\end{aligned}$$

for any K *adapted to the Brownian filtration and such that* $K \in L^2(\mu, H)$*.*

▷ technically one really needs a relaxation to discuss minimizers, we ignore this all along this talk. the actualy object of study is the law of the pair $(\mathbb{W}, Z)$ and not the process Z. (similar as what happens in the Φ_3^4 paper) we use polynomial weights $\rho(x)=(1+\ell|x|)^{-n}$ for large $n>0$ and small $\ell>0$.

Theorem. *There exists a constant C independent of $|\Lambda|$ such that, for any minimizer Z of $F^{f,\Lambda}(\mu)$ and any spatial weight $\rho:\Lambda\to[0,1]$ with $|\nabla\rho|\leqslant\varepsilon\rho$ for some $\varepsilon>0$ small enough, we have*

$$\mathbb{E}\left(4\lambda\int_\Lambda \rho Z_1^4+\int_0^1\int_{\mathbb{R}^2}((m^2-\Delta)^{1/2}\rho^{1/2}\dot{Z}_s)^2\mathrm{d}s\right)\leqslant C.$$

Proof. test the Euler–Lagrange equations with $K=\rho Z$ and then estimate the bad terms with the good terms and objects only depending on $\mathbb{W}$, e.g.

$$\left|\int_\Lambda \rho\,\mathbb{W}^3 Z\right|\leqslant C_\delta\|\mathbb{W}^3\|_{H^{-1}(\rho^{1/2})}^2+\delta\|Z\|_{H^1(\rho^{1/2})}^2,$$

$$\left|\int_\Lambda \rho\mathbb{W}^2 Z^2\right|\leqslant C_\delta\|\rho^{1/8}\mathbb{W}^2\|_{C^{-\varepsilon}}^4+\delta(\|\rho^{1/4}\bar{Z}\|_{L^4}^4+\|\rho^{1/2}\bar{Z}\|_{H^{2\varepsilon}}^2),\cdots$$

$$\mathscr{W}_\Lambda(f) = \inf_Z F^{f,\Lambda}(Z) - \inf_Z F^{0,\Lambda}(Z) = F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

Therefore

$$F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{f,\Lambda}) \leqslant \mathscr{W}_\Lambda(f) \leqslant F^{f,\Lambda}(Z^{0,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

and since, for any g,

$$F^{f,\Lambda}(Z^{g,\Lambda}) - F^{0,\Lambda}(Z^{g,\Lambda}) = \mathbb{E}[f(W + Z^{g,\Lambda}) + \lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})]$$

$$-\mathbb{E}[\lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})] = \mathbb{E}[f(W + Z^{g,\Lambda})]$$

$$\boxed{\mathbb{E}[f(W + Z^{f,\Lambda})] \leqslant \mathscr{W}_\Lambda(f) \leqslant \mathbb{E}[f(W + Z^{0,\Lambda})]}$$

Consequence: tightness of $(\theta_\Lambda)_\Lambda$ in $\mathscr{S}'(\mathbb{R}^2)$ and optimal exponential bounds (cfr. Hairer/Steele)

$$\sup_\Lambda \int \exp(\delta \|\phi\|^4_{W^{-\kappa,4}(\rho)}) \theta_\Lambda(d\phi) < \infty.$$

Euler–Lagrange equation in infinite volume

The family $(Z^{f,\Lambda})_\Lambda$ is also converging (provided we look at the relaxed problem) and any limit point $Z = Z^f$ satisfies a EL equation:

$$\mathbb{E}\left\{\int_{\mathbb{R}^2} f'(W+Z)K + 4\lambda\int_{\mathbb{R}^2} [\![(W+Z)^3]\!]K + \int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\dot{K}_s \mathrm{d}s\right\} = 0$$

for any test process K (adapted to $\mathbb{W}$ and to Z).

a new kind of stochastic "elliptic" problem

Open questions

- Uniqueness??
- Γ-convergence of the variational description of $\mathscr{W}_\Lambda(f)$?

not clear. We lack sufficient knowledge of the dependence on f of the solutions to the EL equations above.

For any $f\colon \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let $\mathscr{W}_\Lambda^\hbar(f)$ be defined by:

$$e^{-\frac{1}{\hbar}\mathscr{W}_\Lambda^\hbar(f)} := \int e^{-f(\phi)}\,\theta_\Lambda^\hbar(\mathrm{d}\phi).$$

where

$$\mathrm{d}\theta_\Lambda^\hbar(\phi) = \exp\left(-\frac{1}{\hbar}V_\Lambda^\hbar(\phi)\right)\mathrm{d}\mu^\hbar(\phi) = \exp\left(-\frac{\lambda}{\hbar}\int_\Lambda [\![\phi^4]\!]\right)\mathrm{d}\mu^\hbar(\phi)$$

and $\mu^\hbar$, is the Gaussian measure with covariance $\hbar(m^2-\Delta)^{-1}$.

Theorem. *Any accumulation point $\theta^\hbar$ of $\theta_\Lambda^\hbar$ satisfies a Laplace principle with rate function*

$$J(\phi) = \lambda\int_{\mathbb{R}^2}\phi^4\mathrm{d}x + \int_{\mathbb{R}^2}\phi(m^2-\Delta)\phi\,\mathrm{d}x.$$

That is

$$\lim_{\hbar\to 0}\mathscr{W}^\hbar(f) = \inf_\psi\{f(\psi)+J(\psi)\}.$$

exponential interaction

we can study similarly the model with

$$V^{\xi}(\varphi)=\int_{\mathbb{R}^2}\xi(x)[\![\exp(\beta\varphi(x))]\!]\mathrm{d}x$$

for $\beta^2<8\pi$ and $\xi\colon\mathbb{R}^2\to[0,1]$ a smooth spatial cutoff function.

$$V^{\xi}(W+Z)=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))\underbrace{[\![\exp(\beta W(x))]\!]\mathrm{d}x}_{M^{\beta}(\mathrm{d}x)}$$

$$=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))M^{\beta}(\mathrm{d}x),\qquad \text{[Gaussian multiplicative chaos]}$$

BD formula

$$\begin{aligned}\mathcal{W}^{\xi,\exp}(f) &= -\log\int\exp(-f(\phi))\mathrm{d}\nu^{\xi}\\ &= \inf_{Z\in\mathfrak{H}_a}\mathbb{E}\left[f(W+Z)+\int\xi\exp(\beta Z)\mathrm{d}M^{\beta}+\frac{1}{2}\int_0^1\int((m^2-\Delta)^{1/2}\dot{Z}_t)^2\mathrm{d}t\right]\end{aligned}$$

▷ the function $Z\mapsto V^{\xi}(W+Z)$ is convex!

variational description of the infinite volume limit

▷ thanks to convexity the EL equations have a unique limit Z in the ∞ volume limit

▷ moreover we have the Γ-convergence of the variational description:

$$\mathscr{W}_{\mathbb{R}^2}(f) = \lim_{n\to\infty} \left[-\log \int \exp(-f(\varphi)) \mathrm{d}\nu^{\xi_n, \exp} \right]$$

$$= \lim_{n\to\infty} [\mathscr{W}_{\xi_n}(f) - \mathscr{W}_{\xi_n}(0)] = \inf_K G^{f,\infty,\exp}(K)$$

with functional

$$G^{f,\infty,\exp}(K) = \mathbb{E}\left[f(W + Z + K) + \underbrace{\int \exp(\beta Z)(\exp(\beta K) - 1) \mathrm{d}M^{\beta} + \mathcal{E}(K)}_{\geqslant 0} \right]$$

which depends via Z on the infinite volume measure for the exp interaction.