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# *the generator of some singular SPDEs*

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**Goal:** probabilistic well–posedness for (almost) stationary solutions to

$$\partial_t u(t,x) = \Delta u(t,x) + \partial_x(u(t,x)^2) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, \mathbb{R}, \quad t \geqslant 0$$

$u(0) \sim \mu$ and $\mu$ white noise on $\mathbb{T}$ with zero mean. $\xi$ space–time white noise.

Singular equations, related to KPZ ($h = \partial_x u$), well-posedness via regularity structures or paracontrolled distributions.

- **Martingale problem.** (Stroock–Vadadhan) Characterisation of the diffusion $u$ by requiring that for a "large" class of functions $\varphi$

$$\varphi(t, u(t)) = \varphi(0, u(0)) + \int_0^t (\partial_s + \mathscr{L}) \varphi(s, u(s)) \mathrm{d}s + M^{\varphi}(t)$$

with $M^{\varphi}$ a martingale. $\mathscr{L}$ is called the generator, usually unbounded $(\mathscr{L}, D(\mathscr{L}))$.

- In our case, formally,

$$\mathscr{L}\varphi(u) = \underbrace{\int \partial_x^2 u(x) \mathrm{D}_x \varphi(u) \mathrm{d}x + \frac{1}{2} \mathrm{Tr}[\partial_x \otimes \partial_x \mathrm{D}^2 \varphi(u)]}_{\mathscr{L}_0} + \underbrace{\int (\partial_x u(x)^2) \mathrm{D}_x \varphi(u) \mathrm{d}x}_{\mathscr{G}}$$

- $\mathscr{L}_0$ generator of the linear part. $\mathscr{G}$ non-linear drift.

$$\varphi(u) = \Phi(u(f_1), \ldots, u(f_n)) \qquad \Rightarrow \qquad \mathrm{D}_x \varphi(u) = \sum_{k=1}^{n} \partial_k \Phi(u(f_1), \ldots, u(f_n)) f_k(x)$$

$$\mathscr{L}_0 \varphi(u) = \sum_{k=1}^{n} \partial_k \Phi(u(f_1), \ldots, u(f_n)) u(\Delta f_k) + \frac{1}{2} \sum_{k,\ell=1}^{n} \partial_k \partial_\ell \Phi(u(f_1), \ldots, u(f_n)) \langle \partial_x f_k, \partial_x f_\ell \rangle$$

$$\mathscr{G}\varphi(u) = -\sum_{k=1}^{n} \partial_k \Phi(u(f_1), \ldots, u(f_n)) \int u(x)^2 \partial_x f_k(x) \mathrm{d}x$$

- Problem: $u^2(\partial_x f)$ is not a well-defined random variable – not even tested with $\partial_x f$.

$$\mathbb{E}[u^2(f) u^2(f)] \text{ "=" } \int \delta(x-y)^2 f(x) f(y) \mathrm{d}x \mathrm{d}y \quad ?????$$

Indeed, it is a "distribution" on $L^2(\mu)$

diffusion with singular drift & regularisation by noise

[Assing ('03) (pre-generator), Flandoli-Russo-Wolf ('03), Delarue-Diel ('16), Allez-Chouk, Cannizzaro-Chouk]

Gaussian space = symmetric Fock space

$$L^2(\mu) \approx \Gamma_s H = \otimes_{n\geqslant 0} H^{\otimes_s n}, \qquad H = L_0^2(\mathbb{T}) \approx \ell^2(\mathbb{N}_{>0}), \qquad \mathbb{E}|\varphi(u)|^2 = \sum_{n\geqslant 0} n!\|\varphi_n\|^2_{H^{\otimes n}}$$

$$\varphi(u) = \sum_{n\geqslant 0} \underbrace{W_n(\varphi_n)}_{n-\text{th chaos}}, \qquad W_n(\varphi_n) = \sum_{k_1,\ldots,k_n} \varphi_n(k_1,\ldots,k_n)\underbrace{[\![\hat{u}(k_1)\cdots\hat{u}(k_n)]\!]}_{\text{Wick's product}}$$

$$\mathbb{E}(\overline{[\![\hat{u}(k_1)\cdots\hat{u}(k_n)]\!]}[\![\hat{u}(k_1')\cdots\hat{u}(k_n')]\!]) = \sum_{\sigma\in S_n} \mathbb{1}_{k_1=k'_{\sigma(1)},\ldots,k_n=k'_{\sigma(n)}}$$

$$\underbrace{\mathrm{D}_k W_n(\varphi_n) = n W_{n-1}(\varphi_n(k,\cdots))}_{\textit{destruction}} \qquad \underbrace{\mathrm{D}_k^* W_n(\varphi_n) = W_{n+1}(S(\mathbb{1}_k \otimes \varphi_n))}_{\textit{creation}}$$

$$[\![\hat{u}(k_1)\cdots\hat{u}(k_n)]\!] = \mathrm{D}^*_{k_1}\cdots\mathrm{D}^*_{k_n}1$$

$$u_k = D_k + D_k^*, \qquad \overline{u_k} = u_{-k}, \qquad D_k D_\ell^* = D_\ell^* D_k + \mathbb{1}_{\ell=k}$$

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$$\mathcal{N} = \sum_k D_k^* D_k, \qquad -\mathcal{L}_0 = \sum_k k^2 D_k^* D_k \qquad \mathcal{G} = \sum_{k+k_1+k_2=0} \imath k (D_{k_1} + D_{k_1}^*)(D_{k_2} + D_{k_2}^*) D_k$$

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$$\mathcal{G} = \underbrace{\sum_{k+k_1+k_2=0} \imath k D_{k_1} D_{k_2} D_k}_{=0} + \sum_{k+k_1+k_2=0} \imath k \underbrace{D_{k_1}^* D_{k_2}^* D_k}_{1 \text{ particle} \to 2 \text{ particles}} + 2 \sum_{k+k_1+k_2=0} \imath k \underbrace{D_{k_1}^* D_{k_2} D_k}_{2 \text{ particles} \to 1 \text{ particle}}$$

$$\sum_{k+k_1+k_2=0} \imath k D_{k_1} D_{k_2} D_k = \sum_{k+k_1+k_2=0} \imath \frac{k_1 + k_2 + k_3}{3} D_{k_1} D_{k_2} D_k = 0$$

$$u_k = \mathrm{D}_k + \mathrm{D}_k^*, \qquad \overline{u_k} = u_{-k}, \qquad \mathrm{D}_k \mathrm{D}_\ell^* = \mathrm{D}_\ell^* \mathrm{D}_k + \mathbb{1}_{\ell = k}$$

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$$\mathcal{N} = \sum_k \mathrm{D}_k^* \mathrm{D}_k, \qquad -\mathscr{L}_0 = \sum_k k^2 \mathrm{D}_k^* \mathrm{D}_k \qquad \mathscr{G} = \sum_{k+k_1+k_2=0} \imath k (\mathrm{D}_{k_1} + \mathrm{D}_{k_1}^*)(\mathrm{D}_{k_2} + \mathrm{D}_{k_2}^*) \mathrm{D}_k$$

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$$\mathscr{G} = \underbrace{\sum_{k+k_1+k_2=0} \imath k \mathrm{D}_{k_1} \mathrm{D}_{k_2} \mathrm{D}_k}_{=0} + \sum_{k+k_1+k_2=0} \imath k \underbrace{\mathrm{D}_{k_1}^* \mathrm{D}_{k_2}^* \mathrm{D}_k}_{1 \text{ particle} \to 2 \text{ particles}} + 2 \sum_{k+k_1+k_2=0} \imath k \underbrace{\mathrm{D}_{k_1}^* \mathrm{D}_{k_2} \mathrm{D}_k}_{2 \text{ particles} \to 1 \text{ particle}}$$

$$\mathscr{G} = \mathscr{G}^+ - \mathscr{G}^-, \qquad (\mathscr{G}^\pm)^* = \mathscr{G}^\mp$$

- Creation and destruction are unbounded operators

$$\|D_k\varphi\|^2 = \sum_{n\geqslant 0} n!\|(n+1)\varphi_{n+1}(k,\cdot)\|^2 = \sum_{n\geqslant 0} (n+1)!\|(n+1)^{1/2}\varphi_{n+1}(k,\cdot)\|^2 \lesssim \|(\mathcal{N}+1)^{1/2}\varphi\|^2$$

$$\mathrm{D}_k, \mathrm{D}_k^*, u_k \approx \mathcal{N}^{1/2}$$

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- $\mathcal{G}^-$ is nice:

$$(\mathcal{G}^-\varphi)_n = \sum_{k+k_1+k_2=0} \iota k_1 (\mathrm{D}_{k_1}^*\mathrm{D}_{k_2}\mathrm{D}_k\varphi)_n = -\sum_{k+k_1+k_2=0} \iota(k+k_2)\, S(\mathbb{1}_{k_1}\otimes\varphi_{n+1}(k,k_2,\cdot))$$

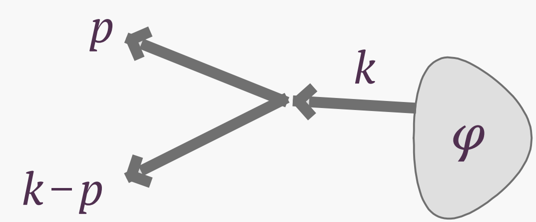
So if $k, k_2$ are bounded, $k_1$ is also bounded and this function is in $(\ell_0^2)^{\otimes n}$.

▸ $\mathcal{G}^+$ is **not**:

$$(\mathcal{G}^+\varphi)_n = \sum_{k+k_1+k_2=0} \iota k_1 \left(\mathrm{D}^*_{k_1}\mathrm{D}^*_{k_2}\mathrm{D}_k\varphi\right)_n = -\sum_{k+k_1+k_2=0} \iota\frac{k}{2} S(\mathbb{1}_{k_1}\otimes\mathbb{1}_{k_2}\otimes\varphi_{n-1}(k,\cdot))$$

No chance that

$$\sum_{k+k_1+k_2=0} \mathbb{1}_{k_1}\otimes\mathbb{1}_{k_2} = \sum_{p\in\mathbb{Z}_0 : p\neq 0,k} \mathbb{1}_{k-p}\otimes\mathbb{1}_p$$




is in $\ell^2\otimes\ell^2$. Too many different possibilities for the created particles, irrespective of the test function $\varphi$.

$\mathcal{G}^+$ is not a well-defined operator in $\Gamma H$.

- $\mathscr{G}\varphi$ is only a (Hida) distribution

$$\|(-\mathscr{L}_0)^{-1/4}\mathscr{G}\varphi\| \lesssim \|(\mathscr{N}+1)(-\mathscr{L}_0)^{1/2}\varphi\|$$

- It looses both $1$ degree of regularity in $\mathscr{N}$ and $3/4$ in $(-\mathscr{L}_0)$. However we “gain” one from $\mathscr{L}_0$. so it remains $(-\mathscr{L}_0)^{1/4}$ to spare.

- Useful in $\mathbb{1}_{|\mathscr{L}_0|\geqslant\mathscr{N}^\alpha}\Gamma H$:

$$\|\mathscr{N}\mathbb{1}_{|\mathscr{L}_0|\geqslant L\mathscr{N}^\alpha}\varphi\| \lesssim L^{-1/\alpha}\|(-\mathscr{L}_0)^{1/\alpha}\mathbb{1}_{|\mathscr{L}_0|\geqslant\mathscr{N}^\alpha}\varphi\| \lesssim L^{-1/\alpha}\|(-\mathscr{L}_0)^{1/\alpha}\varphi\|$$

$$\|(-\mathscr{L}_0)^{-1/2}\mathbb{1}_{|\mathscr{L}_0|\geqslant L\mathscr{N}^\alpha}\mathscr{G}\varphi\| \lesssim \delta\|(-\mathscr{L}_0)^{1/2}\varphi\|$$

# controlled functions

- To use $\mathscr{L}_0\varphi$ to compensate for $\mathscr{G}^+\varphi$: we look for "controlled" $\varphi$ such that

$$\mathscr{L}_0\varphi \approx -\mathscr{G}^+\varphi$$

We don't need to be greedy.

$$\mathscr{G}^{\succ} := \mathbb{1}_{|\mathscr{L}_0| \geqslant L\mathscr{N}^{\alpha}}\mathscr{G}, \qquad \mathscr{G}^{\prec} = \mathscr{G} - \mathscr{G}^{\succ}$$

$\mathscr{G}^{\succ}$ models the large momentum behaviour of $\mathscr{G}$. $L$ is a cutoff to be chosen later.

$$\varphi = -\mathscr{L}_0^{-1}\mathscr{G}^{\succ}\varphi + \varphi^{\#}, \qquad \varphi = \mathscr{K}\varphi^{\#}$$

$$\mathscr{L}\varphi = \mathscr{L}_0\varphi + \mathscr{G}\varphi = \mathscr{L}_0\varphi^{\#} + \mathscr{G}^{\prec}\varphi$$

# estimates & domain

- For $\gamma \in (1/4, 1/2]$

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-1}\mathcal{G}^{\succ}\varphi\| \lesssim \varepsilon|w|\|(-\mathcal{L}_0)^{\gamma}w(\mathcal{N})\varphi\|$$

$$\|(-\mathcal{L}_0)^{\gamma}w(\mathcal{N})\mathcal{K}\varphi^{\#}\| + (|w|\varepsilon)^{-1}\|(-\mathcal{L}_0)^{\gamma}w(\mathcal{N})(\mathcal{K}\varphi^{\#}-\varphi^{\#})\| \lesssim \|(-\mathcal{L}_0)^{\gamma}w(\mathcal{N})\varphi^{\#}\|$$

- For all $\gamma \geqslant 0$ $\delta > 0$

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma}\mathcal{G}^{\prec}\varphi\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{9/2+7\gamma}(-\mathcal{L}_0)^{1/4+\delta}\varphi^{\#}\|$$

so $\mathcal{L}\varphi = \mathcal{L}_0\varphi^{\#} + \mathcal{G}^{\prec}\varphi$ is well defined for controlled functions.

$$\mathcal{D}_w(\mathcal{L}) = \{\varphi = \mathcal{K}\varphi^{\#} : \|w(\mathcal{N})(-\mathcal{L}_0)\varphi^{\#}\| + \|w(\mathcal{N})(1+\mathcal{N})^{9/2}(-\mathcal{L}_0)^{1/2}\varphi^{\#}\|\}$$

is *dense* in $w(\mathcal{N})^{-1}\Gamma H$ and $\mathcal{D}(\mathcal{L}) = \mathcal{D}_1(\mathcal{L})$.

- Densely defined operator

$$(\mathscr{L}, \mathscr{D}(\mathscr{L}))$$

- For $\mathscr{L}^{(\lambda)} = \mathscr{L}_0 + \lambda \mathscr{G}$ with $\lambda \in \mathbb{R}$ similar construction: $\mathscr{D}(\mathscr{L}^{(\lambda)}) \cap \mathscr{D}(\mathscr{L}^{(\lambda')}) = \{\text{constants}\} \dots$

- $\mathscr{L}$ is dissipative

$$\langle \varphi, \mathscr{L}\varphi \rangle = -\|(-\mathscr{L}_0)^{1/2}\varphi\|^2 \leqslant 0, \qquad \varphi \in \mathscr{D}(\mathscr{L})$$

$$\langle \psi, \mathscr{L}\varphi \rangle = \langle \mathscr{L}^{(-1)}\psi, \varphi \rangle, \qquad \varphi \in \mathscr{D}(\mathscr{L}), \psi \in \mathscr{D}(\mathscr{L}^{(-1)})$$

- $\mathscr{L}^m$ Galerkin approximation for $\mathscr{L}$, $(T_t^m)_t$ Markov semigroup

$$\partial_t \varphi^m(t) = \mathscr{L}^m \varphi^m(t)$$

- To pass to the limit we need to control the growth of solutions in weighted spaces

$$\frac{1}{2}\partial_t \|w(\mathscr{N})\varphi^m(t)\|^2 + \|w(\mathscr{N})(-\mathscr{L}_0)^{1/2}\varphi^m(t)\|^2 = \langle \varphi^m(t), w(\mathscr{N})^2 \mathscr{G}^m \varphi^m(t)\rangle$$

- We have for $\gamma > 1/4$ and uniformly in $m$

$$\|w(\mathscr{N})(-\mathscr{L}_0)^{-\gamma}\mathscr{G}_+^m \psi\| \lesssim \|w(\mathscr{N})\mathscr{N}(-\mathscr{L}_0)^{3/4-\gamma}\psi\| \qquad \textit{(roughly)}$$

$$\langle \varphi^m(t), w(\mathcal{N})^2 \mathcal{G}^m \varphi^m(t)\rangle = \langle \varphi^m(t), w(\mathcal{N})^2 (\mathcal{G}_+^m + \mathcal{G}_-^m)\varphi^m(t)\rangle$$

$$= \langle \varphi^m(t), w(\mathcal{N})^2 \mathcal{G}_+^m \varphi^m(t)\rangle + \langle \varphi^m(t), \mathcal{G}_-^m w(\mathcal{N}+1)^2 \varphi^m(t)\rangle$$

$$= \langle \varphi^m(t), [w(\mathcal{N})^2 - w(\mathcal{N}+1)^2] \mathcal{G}_+^m \varphi^m(t)\rangle \approx \left\langle \varphi^m(t), w(\mathcal{N}) \underbrace{w'(\mathcal{N})}_{\approx w(\mathcal{N})\mathcal{N}^{-1}} \mathcal{G}_+^m \varphi^m(t) \right\rangle$$

$$\lesssim \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t)\|^2 + c_\delta \| w(\mathcal{N})(-\mathcal{L}_0)^{-1/2} \mathcal{N}^{-1} \mathcal{G}_+^m \varphi^m(t)\|^2$$

$$\lesssim \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t)\|^2 + c_\delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/4} \varphi^m(t)\|^2$$

$$\frac{1}{2}\partial_t \| w(\mathcal{N}) \varphi^m(t)\|^2 + \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t)\|^2 \lesssim_\delta \| w(\mathcal{N}) \varphi^m(t)\|^2$$

# controlled regularity

- To pass to the limit in the Kolmogorov equation we need further regularity to put $\lim_m \varphi^m$ in the domain of $\mathscr{L}$. We need control of

$$\varphi^{m,\#}(t) = \varphi^m(t) + \mathscr{L}_0^{-1}\mathscr{G}^{m,\succ}\varphi^m(t)$$

- The equation for $\varphi^{m,\#}$ gives the required apriori estimates

$$\partial_t\varphi^{m,\#}(t) = \mathscr{L}^m\varphi^m(t) + \mathscr{L}_0^{-1}\mathscr{G}^{m,\succ}\partial_t\varphi^m(t) = \mathscr{L}_0\varphi^{m,\#}(t) + \mathscr{G}^{m,\prec}\varphi^m(t) + \mathscr{L}_0^{-1}\mathscr{G}^{m,\succ}\partial_t\varphi^m(t)$$

For $\gamma \in (3/8, 5/8)$, exists $p(\alpha)$ s.t.

$$\|(1+\mathscr{N})^\alpha(-\mathscr{L}_0)^{1+\gamma}\varphi^{m,\#}(t)\| + \|(1+\mathscr{N})^\alpha(-\mathscr{L}_0)^\gamma\partial_t\varphi^{m,\#}(t)\| \lesssim \|(1+\mathscr{N})^{p(\alpha)}(-\mathscr{L}_0)^{1+\gamma}\varphi^{m,\#}(0)\|$$

# Kolmogorov backward equation

- Given

$$\|(1+\mathcal{N})^{p(\alpha)}(-\mathscr{L}_0)^{1+\gamma}\varphi(0)\| < \infty$$

with $\alpha > 9/2$ and $\gamma \in (3/8, 5/8)$ then

$$\partial_t \varphi(t) = \mathscr{L}\varphi(t)$$

has a solution

$$\varphi \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{L})) \cap C^1(\mathbb{R}_+, \Gamma H)$$

- Unique by dissipativity (but we cannot define flow $e^{t\mathscr{L}}$)

# martingale problem



- By Galerkin approximation we can construct a stationary process $(u_t^m)_t$ such that

$$\varphi(u^m(t)) = \varphi(u^m(0)) + \int_0^t \mathscr{L}^m \varphi(u^m(s))\mathrm{d}s + M_t^{m,\varphi}$$

- Compactness by energy solution methods [Gonçalves–Jara] [Gubinelli–Jara].
- For all $\varphi \in \mathscr{D}(\mathscr{L}) \subseteq \Gamma H$

$$\varphi(u(t)) = \varphi(u(0)) + \int_0^t \mathscr{L}\varphi(u(s))\mathrm{d}s + M_t^{\varphi}$$

- *Incompressible solutions.* Makes sense only if $\mathrm{Law}(u(t)) \ll \mu$.
- Uniqueness by duality with the backward equation

$$\mathbb{E}[\varphi(u_t)\psi(u_s)] = \mathbb{E}\Big[\Big(\varphi(t-s,u_t) + \int_s^t (\partial_r + \mathscr{L})\varphi(t-r,u_r)\mathrm{d}r\Big)\psi(u_s)\Big] = \mathbb{E}[\varphi(t-s,u_s)\psi(u_s)]$$

# extensions



- Multi-component Burgers eq. [Funaki-Hoshino '17, Kupiainen-Marcozzi '17]

$$\partial_t u^i = \Delta u^i + \sum_{j,k} \Gamma^i_{jk} \partial_x (u^j u^k) + \partial_x \xi^i$$

under "trilinear condition" [Funaki-Hoshino '17]: $\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^j_{ki}$.

- Fractional Burgers eq. [G.-Jara '13]

$$\partial_t u = -(-\Delta)^{\theta} u + \partial_x u^2 + (-\Delta)^{\theta/2} \xi$$

for $\theta > 3/4$; note that $\theta = 3/4$ is critical, ∞ expansion in reg. str.!

- Weak universality for fractional Burgers [Sethuraman '16, Gonçalves-Jara '18] and multi-component Burgers [Bernardin-Funaki-Sethuraman '19+]

- 2d NS with small hyperdissipation and energy invariant measure (G., Turra, in prep.) $\kappa > 0$

$$\partial_t u = -(-\Delta)^{1+\kappa} u + u \cdot \nabla u + (-\Delta)^{(1+\kappa)/2} \xi, \qquad u : \mathbb{T}^2 \to \mathbb{R}^2.$$

# summary



- Probabilistic theory for singular SPDEs $\leftrightarrow$ $\infty$-dim singular operator $\mathscr{L} = \mathscr{L}_0 + \mathscr{G}$.
- Under Gaussian (invariant) measure: use chaos decomposition $\rightarrow$ work on Fock space
- Construct $\mathscr{D}(\mathscr{L})$ via ideas from paracontrolled distributions.
- Existence for martingale problem via Galerkin approximation.
- Existence for backward equation $\partial_t \varphi = \mathscr{L} \varphi$ via energy estimates.
- Duality gives uniqueness for martingale prob. and backward eq.
- (multi-component, fractional) Burgers, down to criticality.
- Need Gaussian measure. beyond: unclear.