



# **Regularisation by noise in PDEs**



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Addition of noise has positive effects on the theory of the equation (in a pathwise sense)

→ ODEs:

$$X_t = x + \int_0^t b(X_s) ds + W_t$$

Many results: Veretennikov, Davie, Krylov–Röckner, Flandoli, Attanasio, Fedrizzi, Proske, Aryasova–Pilipenko, ... Essentially bounded  $b$ . More precisely, Ladyzhenskaya–Prodi–Serrin (LPS) condition :

$$b \in L_t^q L_x^p \quad \frac{d}{p} + \frac{2}{q} < 1.$$

→ Transport (or continuity) equation (Stratonovich integral):

$$d_t u(t, x) + b(x) \cdot \nabla u(t, x) dt = \nabla u(t, x) \circ dW_t$$

good theory for  $L^\infty$  solutions and preservation of regularity. Flandoli–G.–Priola, Flandoli–Attanasio, Flandoli–Maurelli, Neves–Olivera.

→ Flandoli–Beck–G.–Maurelli: full LPS condition ( $\leq 1$ ), new promising method of proof.

[Beck’s talk on monday]

→ Stochastic vector advection equation (Flandoli–Maurelli–Neklyudov):

$$d_t \mathbf{B} + \operatorname{curl}(\mathbf{v} \times \mathbf{B}) dt + \sigma \sum_{k=1}^d \operatorname{curl}(\mathbf{e}_k \times \mathbf{B}) \circ dW_t^k = 0.$$

Noise avoid blow-up of  $\|\mathbf{B}(t, \cdot)\|_{L_x^\infty}$  for  $\mathbf{v} \in C^\alpha$  with  $\alpha \in (0, 1)$ .

→ Non-linear PDEs with transport structure. Point vortices in 2d (Flandoli–G.–Priola), Vlasov–Poisson (Delarue–Flandoli–Vincenzi).

$$du(t, x) + u(t, x) \cdot \nabla u(t, x) = \sum_{k=1}^N \sigma_k(x) \cdot \nabla u(t, x) \circ dW_t^k.$$

(Hypoelliptic) Noise helps to avoid collapse due to peculiar configurations.

→ Modulated non-linear Schrödinger equation in  $d = 1$ . De Bouard–Debussche, Debussche–Tsutsumi.

$$d_t \varphi(t, x) = i \Delta \varphi(t, x) \circ dW_t + i |\varphi(t, x)|^{p-2} \varphi(t, x) dt$$

Motivated by homogenisation in optical wave–guides with dispersion management.

→ Averaging lemmas for kinetic equations. (Fedrizzi–Flandoli–Priola–Vovelle, Lions–Perthame–Souganidis, Gess–Souganidis)

**Goal:** provide a deterministic framework to discuss regularization by “perturbations/modulation” for the following model PDEs:

- **Transport equation:**  $x \in \mathbb{R}^d$ ,  $t \geq 0$ ,  $w: \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) + b(x) \cdot \nabla u(t, x) = 0, \quad u(0, \cdot) = u_0.$$

- **Non-linear Schrödinger equation:**  $x \in \mathbb{T}, \mathbb{R}$ ,  $t \geq 0$ ,  $w: \mathbb{R} \rightarrow \mathbb{R}$

$$\partial_t \varphi(t, x) = i \Delta \varphi(t, x) \dot{w}_t + i |\varphi(t, x)|^{p-2} \varphi(t, x).$$

- **Korteweg–de Vries equation:**  $x \in \mathbb{T}, \mathbb{R}$ ,  $t \geq 0$ ,  $w: \mathbb{R} \rightarrow \mathbb{R}$

$$\partial_t u(t, x) = \partial_x^3 u(t, x) \dot{w}_t + \partial_x (u(t, x))^2.$$

▷ By defining a suitable notion of "irregular"  $w$  we are able to show, in a quantitative way, that the more  $w$  is irregular the more some properties of these equations improves.

▷ The sample paths of Brownian motion or fractional Brownian motion and similar processes have almost surely this kind of irregularity.

*[Joint work with Remi Catellier and Khalil Chouk]*

Consider the linear transport PDE

$$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) = f(x), \quad u(0, \cdot) = 0.$$

Solutions are given explicitly by

$$u(t, x) = \int_0^t f(x + w_s - w_t) ds = T_t^w f(x - w_t)$$

where for any continuous function  $w: [0, 1] \rightarrow \mathbb{R}^d$  we define the **averaging operator**

$$T_t^w f(x) = \int_0^t f(x + w_s) ds, \quad T_{t,s}^w f = T_t^w f - T_s^w f$$

acting on functions (or distributions)  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ .

**Question:** What is the relation between  $w$ , the regularity of  $f$  and that of  $u(t, \cdot)$ ?

If  $w$  is smooth we do not expect anything special to happen and  $u$  to have the same regularity of  $f$ .

▷  $d=1$ ,  $w_t=t$ . Then if  $F'(x) = f(x)$  we have  $T_t^w f(x) = \int_0^t F'(x+s)ds = F(x+t) - F(x)$  and  $T^w: L^\infty \rightarrow \text{Lip}$ :

$$|T_t^w f(x) - T_t^w f(y)| \leq \|f\|_\infty |x - y|, \quad |T_{t,s}^w f(x)| \leq \|f\|_\infty |t - s|$$

▷ Tao–Wright: if  $w$  “wiggles enough” then  $T_t^w$  maps  $L^q$  into  $L^{q'}$  with  $q' > q$ .

▷ Davie: if  $w$  is a sample of BM then a.s. (the exceptional set depends on  $f$ )

$$|T_{t,s}^w f(x) - T_{t,s}^w f(y)| \leq C_w \|f\|_\infty |x - y|^{1-|t-s|^{1/2}}$$

**Problem:** study the mapping properties of  $T^w$  with  $w$  sample path of a stochastic process.

Consider

$$Y_t^w(\xi) = \int_0^t e^{i\langle \xi, w_s \rangle} ds$$

then  $T_t^w f = \mathcal{F}^{-1}(Y_t^w \mathcal{F}(f))$ . Mapping properties of  $T^w$  in  $(H^s)_{s \in \mathbb{R}}$  spaces can be discussed in terms of  $Y^w$ :

$$\|T_{t,s}^w f\|_{H^s} = \left\| (1 + \xi^2)^{s/2} Y_{t,s}^w(\xi) \mathcal{F}f(\xi) \right\|_{H_\xi^s}.$$

In our setting more convenient to look at the scale  $(\mathcal{FL}^\alpha)_\alpha$  :

$$\|f\|_{\mathcal{FL}^\alpha} = \int |f(\xi)| (1 + \xi^2)^{\alpha/2} d\xi$$

since  $\mathcal{FL}^\alpha \subseteq C^\alpha$ .

**Definition 1** (Catellier–G.) *We say that  $w$  is  $(\rho, \gamma)$ -irregular if there exists a constant  $K$  such that for all  $\xi \in \mathbb{R}^d$  and  $0 \leq s \leq t \leq 1$ :*

$$|Y_{t,s}^w(\xi)| \leq K(1 + |\xi|)^{-\rho} |t - s|^\gamma.$$

- ▷ In  $d = 1$  smooth functions are  $(\rho, \gamma)$  irregular for  $\rho + \gamma = 1$ . In particular if we insist on  $\gamma > 1/2$  we have  $\rho < 1/2$ .
- ▷ Not easy to say if a function is irregular.

**Theorem 2** *The fBM of Hurst index  $H$  is  $\rho$ -irregular for any  $\rho < 1/2H$ .*

⇒ there exists functions of arbitrarily high irregularity and arbitrarily  $L^\infty$ -near any given continuous function.

**Lemma 3** *An irregular function cannot be too regular.*

**Proof.** If  $w \in C^\theta$  with  $\alpha\theta + \gamma > 1$  and  $\alpha \in [0, 1]$ , using the Young integral, we find

$$|t - s| = |e^{ia}(t - s)| = \left| \int_s^t \underbrace{e^{ia - iaw_r}}_{C^{\alpha\theta}} d_r \underbrace{Y_r^w(a)}_{C^\gamma} \right|$$

$$\leq C K_w (|t - s|^\gamma + |t - s|^{\alpha\theta + \gamma} |a|^\alpha) \|w\|_\theta (1 + |a|)^{-\rho} \rightarrow 0$$

if  $t > s$  and  $\alpha < \rho$ . This implies that is not possible that  $\theta > (1 - \gamma) / \rho$ .



▷ For  $d > 1$  smooth functions are not irregular: if  $|t - s| \ll 1$

$$\int_s^t e^{i\langle a, w_r \rangle} dr \simeq \int_s^t e^{i\langle a, w'_s \rangle (t-s)} dr \simeq (1 + |\langle a, w'_s \rangle|)^{-1} \not\lesssim (1 + |a|)^{-\rho}.$$

▷ If  $w$  is  $\rho$ -irregular and  $\varphi$  is a  $C^1$  perturbation then  $w + \varphi$  is at least  $\rho - (1 - \gamma)$  irregular since:

$$Y_{t,s}^{w+\varphi}(\xi) = \int_s^t e^{i\langle \xi, w_r + \varphi_r \rangle} dr = \int_s^t e^{i\langle \xi, \varphi_r \rangle} dr Y_{s,r}^w(\xi)$$

and we can use Young integral estimates.

▷ If  $W$  is a fBM and  $\Phi$  an adapted smooth perturbation then  $W + \Phi$  is as irregular as  $W$  (via Girsanov theorem).

▷ Other results (see Catellier thesis): relation with intersection local times, irregularity for  $\alpha$ -stable Levy processes, relation with local non-determinism.

**Theorem 4** *If  $w$  is  $\rho$ -irregular then*

$$T^w: H^s \rightarrow H^{s+\rho}$$

and

$$T^w: \mathcal{FL}^\alpha \rightarrow \mathcal{FL}^{\alpha+\rho}.$$

**Proof.** Indeed

$$\begin{aligned} \|T_{t,s}^w f\|_{\mathcal{FL}^{\alpha+\rho}} &= \int d\xi (1 + |\xi|)^{\alpha+\rho} |Y_{t,s}^w(\xi)(\mathcal{F}f)(\xi)| \\ &\leq K_w |t - s|^\gamma \int d\xi (1 + |\xi|)^\alpha |(\mathcal{F}f)(\xi)| = K_w |t - s|^\gamma \|f\|_{\mathcal{FL}^\alpha}. \end{aligned}$$

**Remark 5** More difficult to understand the mapping properties in other spaces, for example Hölder spaces  $C^\alpha$ . Only partial results available. Wide open problem.

- ▷ Consider the transport equation with a perturbation:

$$\partial_t u(t, x) + \dot{w}_t \cdot \nabla u(t, x) + b(x) \cdot \nabla u(t, x) = 0, \quad u(0, \cdot) = u_0.$$

- ▷ In the Lipschitz case there is only one solution  $u$  given by the method of characteristics:

$$u(t, x) = u_0(\phi_t^{-1}(x))$$

where  $\phi_t(x) = x_t$  is the flow of the ODE

$$\begin{cases} \dot{x}_t = b(x_t) + \dot{w}_t \\ x_0 = x \end{cases}$$

- ▷ Uniqueness of solutions is related to the uniqueness (and smoothness) theory of the flow.

In order to exploit the averaging properties of  $w$  in the study of the ODE

$$x_t = x_0 + \int_0^t b(x_s) ds + w_t$$

we rewrite it in order to make the action of the averaging operator explicit: let  $\theta_t = x_t - w_t$ :

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s) ds = \theta_0 + \int_0^t (d_s G_s)(\theta_s)$$

where  $G_s(x) = T_s^w b(x)$  so that  $d_s G_s(x) = f(w_s + x)$ .

If we assume that  $G$  is  $C^\gamma$  in time ( $\gamma > 1/2$ ) with values in a space of regular enough functions we can study this equation as a Young type equation for  $\theta \in C^\gamma$ .

▷ **Non-linear Young integral:**

$$\int_0^t (d_s G_s)(\theta_s) = \lim_{\Pi} \sum_i G_{t_{i+1}, t_i}(\theta_{t_i})$$

This limit exists if  $\theta \in C_t^\gamma$  and  $G \in C_t^\gamma C_x^\nu$  with  $\gamma(1 + \nu) > 1$ . The integral is in  $C_t^\gamma$ .

**Theorem 6** *The integral equation*

$$\theta_t = \theta_0 + \int_0^t (d_s G_s)(\theta_s)$$

is well defined for  $\theta \in C^\gamma$  and  $G \in C_t^\gamma C_{x,\text{loc}}^\nu$  with  $(1 + \nu)\gamma > 1$ .

- Existence of global solutions if  $G$  of linear growth.
- Uniqueness if  $G \in C_t^\gamma C_{x,\text{loc}}^{\nu+1}$  and differentiable flow.
- Smooth flow if  $G \in C_t^\gamma C_x^{\nu+k}$ .

**Theorem 7** *The equation*

$$x_t = x_0 + \int_0^t b(x_s) ds + w_t$$

has a unique solution for  $w$   $\rho$ -irregular and  $b \in \mathcal{FL}^\alpha$  for  $\alpha > 1 - \rho$ . In this case we can take  $\theta \in C^1$  above and the condition for uniqueness (and Lipschitz flow) is  $G \in C_t^\gamma C_x^{3/2}$ .

▷ Say that  $x$  is controlled by  $w$  if  $\theta = x - w \in C^\gamma$ . In this case we have

$$I_x(b) = \int_0^t b(x_s) ds = \int_0^t (d_s T_s^w b)(\theta_s)$$

and the r.h.s. is well defined as soon as  $T^w b \in C_t^\gamma C_x^\nu$ .

▷ If  $w$  is  $\rho$  irregular and  $b \in \mathcal{FL}^\alpha$  then  $T^w b \in C_t^\gamma \mathcal{FL}_x^{\alpha+\rho}$  so if  $\alpha + \rho \geq \nu$  we have  $T^w b \in C_t^\gamma C_x^\nu$ .

In this case  $I_x(b)$  can be extended by continuity to all  $b \in \mathcal{FL}^\alpha$  and in particular we have given a meaning to

$$\int_0^t b(x_s) ds$$

when  $b$  is a distribution *provided*  $x$  is controlled by a  $\rho$ -irregular path.

▷ For controlled paths the ODE

$$x_t = x_0 + \int_0^t b(x_s) ds + w_t$$

make sense even for certain distributions  $b$  as a Young equation for  $\theta = x - w$ .

(work of R. Catellier)

We want to give a meaning and study the uniqueness problem for the transport equation

$$(\partial_t + b(x) \cdot \nabla + \dot{w}_t \cdot \nabla)u(t, x) = 0$$

for  $u \in L^\infty$  and  $w \in C^\sigma$  with  $\sigma > 1/3$  such that  $(w, \mathbb{W})$  is a geometric  $\sigma$ -Hölder rough path such that  $w$  is  $\rho$ -irregular. For the moment only in the case  $\operatorname{div} b = 0$ .

▷ **Weak formulation:** We consider  $u$  as a distribution:  $u_t(\varphi) = \int dx \varphi(x) u(t, x)$  for all  $\varphi \in L^1(\mathbb{R}^d)$ . The integral formulation of the equation is

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi)) dr + \int_s^t u_r(\nabla \varphi) d_r w_r$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  and  $0 \leq s \leq t$ .

We need to give a meaning to such an integral equation in order to discuss the regularization by noise phenomenon. (No way out!)

▷ Possible via the theory of **controlled rough paths** (G. 2004).

Let  $(X, \mathbb{X})$  be a  $\sigma$ -Hölder rough path with  $\sigma > 1/3$ :

$$\mathbb{X}_{t,s} = \mathbb{X}_{t,u} + \mathbb{X}_{u,s} + (X_t - X_u) \otimes (X_u - X_s), \quad |X_t - X_s| + |\mathbb{X}_{s,t}|^{1/2} = O(|t - s|^\sigma)$$

▷ We say that  $y \in C_t^\sigma$  is **controlled by  $X$**  if there exists  $y^X \in C_t^\sigma$  such that

$$y_t - y_s - y_s^X (X_t - X_s) =: y_{s,t}^\# = O(|t - s|^{2\sigma}).$$

▷ For a controlled path  $y$  we can define the integral against  $X$  by compensated Riemman sums:

$$I_t = \int_0^t y_s dX_s := \lim_{\Pi} \sum_i y_{t_i} (X_{t_{i+1}} - X_{t_i}) + y_{t_i}^X \mathbb{X}_{t_{i+1}, t_i}$$

▷ This integral is the only function (up to constants) which has the following property

$$I_t - I_s = y_s (X_t - X_s) + y_s^X \mathbb{X}_{t,s} + O(|t - s|^{3\sigma}).$$

In particular, the integral is itself controlled by  $X$  and  $I^X = y$ .



**Definition 8** We say that  $u$  is a function controlled by  $w$  if for all  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  we have

$$u_t(\varphi) - u_s(\varphi) = u_s^w(\varphi)(w_t - w_s) + u_{t,s}^\sharp(\varphi)$$

where  $u_s^w(\varphi) \in C^\sigma$  and  $|u_{t,s}^\sharp(\varphi)| \lesssim |t - s|^{2\sigma}$ .

**Definition 9** If  $u$  is controlled we say that it is a  $L^\infty$  solution of the rough transport equation (RTE) if

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))dr + \int_s^t u_r(\nabla \varphi)d_r w_r$$

holds for all  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,  $0 \leq s \leq t$ .

**Remark:** If  $\sigma > 1/2$  we can just assume that  $u_t(\nabla \varphi) \in C_t^\sigma$  so that the rough integral becomes a Young integral.

Equivalently,  $u$  is a solution to the RTE iff

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))dr + u_s(\nabla \varphi)(w_t - w_s) + u_s(\nabla^2 \varphi)\mathbb{W}_{t,s} + O(|t - s|^{3\sigma})$$

**Lemma 10** *If  $b$  is Lipschitz there exists a solution to the RTE given by  $u(t, x) = u_0(\phi_t^{-1}(x))$ .*

**Theorem 11** *Let  $b \in \mathcal{FL}^\alpha$  for  $\alpha > 0$  and  $\alpha + \rho > 3/2$  and let  $w$  be  $\rho$ -irregular. Then there exists a unique solution to the RTE given by the method of characteristics.*

**Proof.** Approximate  $b$  by  $b_\varepsilon$ , then by the previous Lemma there exists a unique solution  $u_\varepsilon$  to the RTE. Analysis of the approximate flow  $\phi_\varepsilon$  shows that this solution converges to a controlled solution  $u$  of the RTE with vectorfield  $b$ . Since  $\phi$  is Lipschitz we can prove again uniqueness.  $\square$

**Remark 12** The above result is path-wise. In particular  $b$  can depend on  $w$ .

**Remark 13** If  $b \in C^\alpha$ ,  $b$  deterministic and  $w$  is a fBm of Hurst index  $H$  then the uniqueness holds almost surely when  $\alpha > 1 - 1/(2H)$  and  $\alpha > 0$ . This recovers the results of Flandoli–Gubinelli–Priola for the Brownian case but extend them well beyond the Brownian context.

(joint work with K. Chouk)

Two simple dispersive models with  $\rho$ -irregular modulation  $w$ :

- **Non-linear Schrödinger equation:**  $x \in \mathbb{T}, \mathbb{R}, \mathbb{R}^2, t \geq 0$

$$\partial_t \varphi(t, x) = i\Delta \varphi(t, x) \partial_t w_t + i|\varphi(t, x)|^{p-2} \varphi(t, x).$$

- **Korteweg–de Vries equation:**  $x \in \mathbb{T}, \mathbb{R}, t \geq 0$

$$\partial_t u(t, x) = \partial_x^3 u(t, x) \partial_t w_t + \partial_x (u(t, x))^2.$$

To be compared to the non-modulated setting where  $\partial_t w_t = 1$  and studied in the scale of  $(H^s)_s$  spaces.

The equations are understood in the mild formulation

$$u(t) = U_t^w u(0) + \int_0^t U_t^w (U_s^w)^{-1} \partial_x (u(s))^2 ds.$$

with  $U_t^w = e^{iw_t \partial_x^3}$ . (similarly for NLS). Here  $w$  can be an arbitrary continuous function.

Rewrite the mild formulation as  $(U_t^w = e^{\partial_x^3 w t})$

$$v(t) = (U_t^w)^{-1}u(t) = u(0) + \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds.$$

**Theorem 14** *Let*

$$X_t(\varphi) = X_t(\varphi, \varphi) = \int_0^t (U_s^w)^{-1} \partial_x (U_s^w \varphi)^2 ds$$

*If  $w$  is  $\rho$  irregular then  $X \in C^\gamma \text{Lip}_{\text{loc}}(H^\alpha)$  for  $\alpha > -\rho$  and  $\rho > 3/4$ .*

For  $v \in C^\gamma H^\alpha$  we can give a meaning to the non-linearity as a Young integral

$$\int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds := \int_0^t (d_s X_s)(v(s)) := \lim_{\Pi} \sum_i X_{t_{i+1}}(v(t_i)) - X_{t_i}(v(t_i))$$

The continuity of the Young integral implies that if  $v_n \rightarrow v$  in  $C^\gamma H^\alpha$  then

$$\int_0^t (U_s^w)^{-1} \partial_x (U_s^w v(s))^2 ds = \lim_n \int_0^t (U_s^w)^{-1} \partial_x (U_s^w v_n(s))^2 ds$$

**Theorem 15** *The Young equation for  $v \in C^\gamma H^\alpha$  :*

$$v(t) = u(0) + \int_0^t (d_s X_s)(v(s))$$

*has local solutions for initial conditions in  $H^\alpha$  with locally Lipschitz flow. Uniqueness in  $C^\gamma H^\alpha$ .*

▷ Equivalent “differential” formulation:

$$v(t) - v(s) = X_{t,s}(v(s)) + O(|t - s|^{2\gamma}), \quad v(0) = u_0$$

**Regularization by modulation.** In the non-modulated case it is known that there cannot be a continuous flow for  $\alpha \leq -1/2$  on  $\mathbb{T}$  and  $\alpha \leq -3/4$  on  $\mathbb{R}$ .

- ▷ Global solutions thanks to the  $L^2$  conservation and smoothing for  $\alpha > 0$  or an adaptation of the I-method for  $-3/2 \leq \alpha < 0$  and  $\alpha > -\rho/(3 - 2\gamma)$ .
- ▷ **NLS:** 1d, global solutions for  $\alpha \geq 0$  and  $\rho > 1/2$ . 2d, local solutions for  $\alpha \geq 1/2$ .
- ▷ Global solutions for 1d NLS with  $\alpha > 0$  come from a smoothing effect of the non-linearity which is due to the irregularity of the driving function.

A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same  $\rho$ -irregularity assumption.

**Theorem 16** *Let  $T > 0$ ,  $p \in (2, 5]$ ,  $\rho > \min(\frac{3}{2} - \frac{2}{p}, 1)$  then there exists a finite constant  $C_{w,T} > 0$  and  $\gamma^*(p) > 0$  such that the following inequality holds:*

$$\left\| \int_0^\cdot U^\cdot(U_s^w)^{-1} \psi_s ds \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq C_w T^{\gamma^*(p)} \|\psi\|_{L^1([0,T], L^2(\mathbb{R}))}$$

for all  $\psi \in L^1([0, T], L^2(\mathbb{R}))$ .

▷ In the deterministic case the Strichartz estimate does not have the factor of  $T$  in the critical case  $p=5$ . This is a sign of a *mild* regularization effect of the noise.

**Remark 17** Similar path-wise statements (in  $w$ ) holds true for averaging lemmas in kinetic equations with irregular perturbations (similar to the results of Lions–Perthame–Souganidis in the Brownian case).

As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity  $i e: \mathcal{N}(\phi) = |\phi|^\mu \phi$ : (Debussche–de Bouard, Debussche–Tsutsumi)

**Theorem 18** *Let  $\mu \in (1, 4]$ ,  $p = \mu + 1$ ,  $\rho > \min(1, 3/2 - \frac{2}{p})$  and  $u^0 \in L^2(\mathbb{R})$  then there exists  $T^* > 0$  and a unique  $u \in L^p([0, T], L^{2p}(\mathbb{R}))$  such that the following equality holds:*

$$u_t = U_t^w u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|u_s|^\mu u_s) ds$$

*for all  $t \in [0, T^*]$ . Moreover we have that  $\|u_t\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$  and then we have a global unique solution  $u \in L_{loc}^p([0, +\infty), L^{2p}(\mathbb{R}))$  and  $u \in C([0, +\infty), L^2(\mathbb{R}))$ . If  $u^0 \in H^1(\mathbb{R})$  then  $u \in C([0, \infty), H^1(\mathbb{R}))$ .*

Thanks.