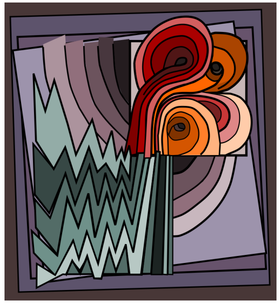
What is stochastic quantisation?



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[made with TeXmacs]

Euclidean quantum fields

a particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$ introduced in the 70s-80s as a tool to construct models of (bosonic) quantum field theories

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + V(\varphi(x))\mathrm{d}x$$

for some non-linear function $V: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$

ill-defined representation

- **large scale problems:** the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- **small scale problems:** sample paths are not expected to be functions, but only distributions, the quantity $V(\varphi(x))$ does not make sense.

some history

▷ Construct rigorously QM models which are compatible with special relativity, (finite speed of signals and Poincaré covariance of Minkowski space $\mathbb{R}^{n+1}$).

▷ Quantum field theory (QM with ∞ many degrees of freedom)

▷ Wightman axioms ('60-'70): Hilbert space, representation of the Poincaré group, fields operators (to construct local observables).

▷ Constructive QFT program ('70-'80): hard to find models of such axioms. Examples in $\mathbb{R}^{1+1}$ were found in the $'60$. Glimm, Jaffe, Nelson, Segal, Guerra, Rosen, Simon, and many others…

▷ Euclidean rotation: $t \to it = x_0$ (imaginary time). $\mathbb{R}^{n+1} \to \mathbb{R}^d$ Minkowski → Euclidean

▷ Osterwalder–Schrader theorem : gives precise condition to perform the passage to/from Euclidean space (OS axioms for Euclidean correlation function).

▷ High point of EQFT: construction of Φ_3^4 (Euclidean version of a scalar field in $\mathbb{R}^{2+1}$ Minkowski space). $(\Phi_3^4)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75) $(\Phi_3^4)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)

▷ Other constructions of Φ_3^4. Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush ('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)

Gaussian free field (GFF)

▷ simplest example of EQFT. We take a Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with covariance

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y \in \mathbb{R}^d$$

and zero mean. Reflection positive, Eucl. covariant and regular. This is the GFF with mass $m>0$.

▷ this measure can be used to construct a QFT in Minkowski space but unfortunately this theory is free, i.e. there is no interaction.

▷ note that $G(0)=+\infty$ if $d \geqslant 2$, this implies that the GFF is not a function.

▷ in particular GFF is a distribution of regularity $\alpha = (2-d)/2-\kappa$ for any small $\kappa>0$, e.g. locally in the sense of the scale of Besov–Holder spaces $(B^{\alpha}_{\infty,\infty})_{\alpha\in\mathbb{R}}$

non-Gaussian Euclieand fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{Z}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side L.

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi)e^{\overbrace{-\frac{1}{2}\sum_{x\in\mathbb{Z}^d_{\varepsilon,L}}|\nabla_\varepsilon\varphi(x)|^2+m^2\varphi(x)^2+V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}}\mathrm{d}\varphi$$

ε is an UV regularisation and L the IR regularisation.

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below. $d = 2,3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon\xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure

❸ study the possible limit points (uniqueness? non-uniqueness? correlations? description?)

stochastic quantisation

Parisi–Wu ('81) introduced a stationary stochastic evolution associated with the EQF

$$\partial_t \Phi(t,x) = -\frac{\delta S(\Phi(t,x))}{\delta \Phi} + 2^{1/2}\eta(t,x), \qquad t \geqslant 0, x \in \mathbb{R}^d,$$

with η space-time white noise

$$\langle \Phi(t,x_1)\cdots\Phi(t,x_n)\rangle = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(t,x_1)\cdots\varphi(t,x_n)e^{-S(\varphi)}\mathrm{d}\varphi, \qquad t \in \mathbb{R}$$

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transport interpretation: the map

$$\eta \mapsto \Phi(t,\cdot)$$

sends the Gaussian measure of the space-time white noise to the EQF measure

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an history of stochastic quantisation (personal & partial)

- 1984 – Parisi/Wu – SQ (for gauge theories)
- 1985 – Jona-Lasinio/Mitter – “On the stochastic quantization of field theory” (rigorous SQ for Φ_2^4 on bounded domain)
- 1988 – Damgaard/Hüffel – review book on SQ (theoretical physics)
- 1990 – Funaki – Control of correlations via SQ (smooth reversible dynamics)
- 1990–1994 – Kirillov – “Infinite-dimensional analysis and quantum theory as semimartingale calculus”, “On the reconstruction of measures from their logarithmic derivatives”, “Two mathematical problems of canonical quantization.”
- 1993 – Ignatyuk/Malyshev/Sidoravichius – “Convergence of the Stochastic Quantization Method I,II” [Grassmann variables + cluster expansion]
- 2000 – Albeverio/Kondratiev/Röckner/Tsikalenko – “A Priori Estimates for Symmetrizing Measures…” [Gibbs measures via IbP formulas]
- 2003 – Da Prato/Debussche – “Strong solutions to the stochastic quantization equations”
- 2014 – Hairer – Regularity structures, local dynamics of Φ_3^4
- 2017 – Mourrat/Weber – coming down from infinity for Φ_3^4
- 2018 – Albeverio/Kusuoka – “The invariant measure and the flow associated to Φ_3^4…”
- 2021 – Hofmanova/G. – Global space-time solutions for Φ_3^4 and verification of axioms
- 2020-2021 – Chandra/Chevyrev/Hairer/Shen – SQ for Yang–Mills 2d/3d.

an existence result for Φ^4_3

in Parisi–Wu's approach the SDE is a Langevin equation of the form

$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -\nabla_\varphi S_\varepsilon(\Phi(t,x)) + 2^{1/2}\xi(t,x), \qquad x \in \Lambda_{\varepsilon,L} = \mathbb{Z}^d_{\varepsilon,L}, \quad t \geqslant 0$$

here $\xi(t,x)$ is a space-time white noise · if $\mathrm{Law}(\Phi(t=0)) = \nu^{\varepsilon,L}$ then $\mathrm{Law}(\Phi(t)) = \nu^{\varepsilon,L}$ for all $t \geqslant 0$ · the dynamics give a map $\hat{G}_{\varepsilon,L}$ which transform a Gaussian measure into $\nu^{\varepsilon,L}$ · this map passes to the limit as $\varepsilon \to 0$ and $L \to \infty$ and is associated to an SPDE in the limit

$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -(m^2 - \Delta)\Phi(t,x) - {''}\, V'(\Phi(t,x)){''} + 2^{1/2}\xi(t,x)$$

Theorem. $d = 3$ *provided* $(a_\varepsilon)_\varepsilon$ *is chosen approp. there exist a stationary in space and time solution to the limit SPDE. the law of the solution at any given time is a non-Gaussian EQFT* ν *(without rotation invariance) with IbP formula:*

$$\int \nabla_\varphi F(\varphi)\nu(\mathrm{d}\varphi) = \int F(\varphi)(-(m^2 - \Delta)\varphi - \lambda[\![\varphi^3]\!])\nu(\mathrm{d}\varphi).$$

[details in Gubinelli–Hofmanova CMP 2021, "A PDE construction…"]

features of stochastic quantisation

the interacting field ϕ is expressed as a function of the Gaussian free field X:

$$\phi(t) = F(X), \qquad \nu = \mathrm{Law}(\phi(t)) = F_* \mathrm{Law}(X) = F_* \mathrm{GFF}$$

- estimates on ϕ obtained via two ingredients:
  - pathwise PDE estimates for the map F (in weighted Besov spaces)
  - probabilistic estimates for the GFF X
- coupling (ϕ, X)

  $$\phi = X + \psi$$

  where ψ is a random field which is more regular (i.e. smaller at small scale) than X (link with asymptotic freedom/perturbation theory)

  note that

  $$\nu = \mathrm{Law}(\varphi) \not\ll \mathrm{Law}(X(t)) = \mathrm{GFF}$$

estimates

▷ decomposition: $\phi = X + \psi$

$$\partial_t \psi = \frac{1}{2}[(\Delta_x - m^2)\psi - V'(X+\psi)]$$

▷ PDE estimates:

$$\|\psi(t)\| \leqslant H(\|X\|)$$

▷ tightness:

$$\int \|\varphi\|^p \nu(\mathrm{d}\varphi) \lesssim \mathbb{E}\|X\|^p + \mathbb{E}\|\psi(t)\|^p \leqslant \mathbb{E}\|X\|^p + \mathbb{E}[H(\|X\|)^p] < \infty$$

▷ tail-estimates:

$$\int e^{c\|\varphi\|^\alpha} \nu(\mathrm{d}\varphi) < \infty$$

[Moinat/Weber, Hofmanova/G., Hairer/Steele]

stochastic analysis

In the '40s Ito introduced an *analysis* adapted to stochastic processes of diffusion:

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y)\in\mathcal{O}\subseteq\mathbb{R}^2$	global description	$P_t(x,\mathrm{d}y)$
$\alpha(x-x_0)^2+\beta(y-y_0)^2=\gamma$	·	$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t)\approx x(t)+a\delta t+o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at+bt^2+\cdots$	building block	$(W_t)_t$
$(\ddot{x}(t),\ddot{y}(t))=F(x(t),y(t))$	local/global link	$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$

$\triangleright$ other examples: rough paths, regularity structures, SLE,…

stochastic quantisation as a stochastic analysis?

	stoch. quantisation
object	EQF
global description	$\nu \in \mathrm{Prob}(\mathscr{S}'(\mathbb{R}^d))$
.	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi$
change parameter	t
local description	$\phi(t+\delta t) \approx \alpha\phi(t) + \beta\delta X(t) + \cdots$
building block	$(X(t))_t$ $\partial_t X = \frac{1}{2}[(\Delta_x - m^2)X] + \xi$
local/global link	$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - V'(\phi)] + \xi$

stochastic analysis of EQFs

- **parabolic stochastic quantisation**

$$\partial_t \phi(t) = \frac{1}{2}[(\Delta_x - m^2)\phi(t) - V'(\phi(t))] + \xi(t)$$

[**MG**, M. Hofmanová · Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space · Comm. Math. Phys. 2019 | **MG**, M. Hofmanová · A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory · Comm. Math. Phys. 2021]

- **canonical stochastic quantisation** · singular stochastic wave equation

$$\partial_t^2 \phi(t) + \partial_t \phi(t) = \frac{1}{2}[(\Delta_x - m^2)\phi(t) - V'(\phi(t))] + \xi(t)$$

[**MG**, H. Koch, T. Oh · Renormalization of the two-dimensional stochastic non- linear wave equations · Trans. Am. Math. Soc. 2018 | **MG**, H. Koch, and T. Oh · Paracontrolled Approach to the Three-Dimensional Stochastic Nonlinear Wave Equation with Quadratic Nonlinearity · Jour. Europ. Math. Soc. 2022]

- **elliptic stochastic quantisation** · supersymmetric proof (Parisi–Sourlas)

$$-\Delta_z \phi(z) = \frac{1}{2}[(\Delta_x - m^2)\phi(z) - V'(\phi(z))] + \xi(z), \quad z \in \mathbb{R}^2$$

[S. Albeverio, F. De Vecchi, **MG** · Elliptic Stochastic Quantization · Ann. Prob. 2020]

- **variational method** · stochastic control problem · Γ-convergence

$$\log \int e^{f(\varphi) - S(\varphi)} \mathrm{d}\varphi = \inf_u \mathbb{E}\left[f(\Phi_\infty^u) + V(\Phi_\infty^u) + \frac{1}{2}\int_0^\infty |u_s| \mathrm{d}s \right]$$

scale parameter $t \in [0, \infty]$ · $\Phi_t^u = X_t + \int_0^t J_s u_s \mathrm{d}s$

[N. Barashkov, **MG** · A Variational Method for Φ_3^4 · Duke Math. Jour. 2020] an example: the variational method for Φ^4_2 in infinite volume

[N. Barashkov, **MG** · On the variational method for Euclidean quantum fields in infinite volume · arXiv:2112.05562]

Boué–Dupuis formula

Theorem. *Let* $(B_t)_{t\geqslant 0}$ *be a Brownian motion on* $\mathbb{R}^n$*, then for any bounded* $F: C(\mathbb{R}_+; \mathbb{R}^n) \to \mathbb{R}$ *we have*

$$\log \mathbb{E}[e^{F(B_\bullet)}] = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[F(B_\bullet + I(u)_\bullet) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

with $u: \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ *adapted to* B *and with*

$$I(u)_t := \int_0^t u_s \mathrm{d}s.$$

$$\frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s \approx H(\mathrm{Law}(B_\bullet + I(u)_\bullet) | \mathrm{Law}(B_\bullet)).$$

[M. Boué and P. Dupuis, A Variational Representation for Certain Functionals of Brownian Motion, *Ann. Prob.* 26(4), 1641–59]

Boué–Dupuis for the $d=2$ GFF

$$\mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)(m^2-\Delta)^{-1}(x-y), \qquad t,s\in[0,1].$$

The BD formula gives

$$-\log\int e^{-F(\phi)}\mu(\mathrm{d}\phi) = -\log\mathbb{E}[e^{-F(W_1)}] = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[F(W_1+Z_1)+\frac{1}{2}\int_0^1\|u_s\|_{L^2}^2\mathrm{d}s\right],$$

where

$$Z_t = (m^2-\Delta)^{-1/2}\int_0^t u_s\mathrm{d}s, \qquad u_t = (m^2-\Delta)^{1/2}\dot{Z}_t$$

$$-\log\mathbb{E}[e^{-F(W_1)}] = \inf_{Z\in H^a}\mathbb{E}[F(W_1+Z_1)+\mathscr{E}(Z_\bullet)],$$

with

$$\mathscr{E}(Z_\bullet) := \frac{1}{2}\int_0^1\|(m^2-\Delta)^{1/2}\dot{Z}_s\|_{L^2}^2\mathrm{d}s = \frac{1}{2}\int_0^1(\|\nabla\dot{Z}_s\|_{L^2}^2+m^2\|\dot{Z}_s\|_{L^2}^2)\mathrm{d}s$$

Φ^4_2 in a bounded domain Λ

Fix a compact region $\Lambda \Subset \mathbb{R}^2$ and consider the Φ^4_2 measure θ_Λ on $\mathscr{S}'(\mathbb{R}^2)$ with interaction in Λ and given by

$$\theta_\Lambda(\mathrm{d}\phi) := \frac{e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}{\int e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)} \qquad \phi \in \mathscr{S}'(\mathbb{R}^2) \tag{1}$$

with interaction potential $V_\Lambda(\phi) := \int_\Lambda \phi^4 - c\int_\Lambda \phi^2$. For any $f: \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let

$$e^{-\mathscr{W}_\Lambda(f)} := \int e^{-f(\phi)}\theta_\Lambda(\mathrm{d}\phi).$$

We have the variational representation, $Z = Z_1$, $Z_\bullet = (Z_t)_{t\in[0,1]}$:

$$\mathscr{W}_\Lambda(f) = \inf_{Z\in H^a} F^{f,\Lambda}(Z_\bullet) - \inf_{Z\in H^a} F^{0,\Lambda}(Z_\bullet)$$

where

$$F^{f,\Lambda}(Z_\bullet) := \mathbb{E}[f(W+Z) + \lambda V_\Lambda(W+Z) + \mathscr{E}(Z_\bullet)].$$

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renormalized potential

$$V_\Lambda(W+Z) = \int_\Lambda \Big\{ \underbrace{W^4 - cW^2}_{\mathbb{W}^4} + 4\underbrace{\Big[W^3 - \frac{c}{4}W\Big]}_{\mathbb{W}^3} Z + 6\underbrace{\Big[W^2 - \frac{c}{6}\Big]}_{\mathbb{W}^2} Z^2 + 4WZ^3 + Z^4 \Big\}$$

take $c = 12\mathbb{E}[W^2(x)] = +\infty$

$$V_\Lambda(W+Z) = \int_\Lambda \{4\mathbb{W}^3 Z + 6\mathbb{W}^2 Z^2 + 4WZ^3 + Z^4\} + \cdots$$

$$\mathbb{W}^n \in \mathscr{C}^{-n\kappa}(\Lambda) = B^{-n\kappa}_{\infty,\infty}(\Lambda)$$

Here $B^{-\kappa}_{\infty,\infty}(\Lambda)$ is an Hölder–Besov space. A distribution $f \in \mathscr{S}'(\mathbb{T}^d)$ belongs to $B^{\alpha}_{\infty,\infty}(\Lambda)$ iff for any $n \geqslant 0$

$$\|\Delta_n f\|_{L^\infty} \leqslant (2^n)^{-\alpha} \|f\|_{B^{\alpha}_{\infty,\infty}(\Lambda)}$$

where $\Delta_n f = \mathscr{F}^{-1}(\varphi_n(\cdot)\mathscr{F} f)$ and φ_n is a function supported on an annulus of size $\approx 2^n$. We have $f = \sum_{n \geqslant 0} \Delta_n f$. If $\alpha > 0$ $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d)$ is a space of functions otherwise they are only distributions.

Euler–Lagrange equation for minimizers

Lemma. *There exists a minimizer $Z = Z^{f,\Lambda}$ of $F^{f,\Lambda}$. Any minimizer satisfies the Euler–Lagrange equations*

$$\begin{aligned}
&\mathbb{E}\Big(4\lambda\int_\Lambda Z^3 K + \int_0^1 \int_\Lambda (\dot{Z}_s(m^2-\Delta)\,\dot{K}_s)\mathrm{d}s\Big)\\
= \;&\mathbb{E}\Big(\int_\Lambda f'(W+Z)\,K + \lambda\int_\Lambda (\mathbb{W}^3+\mathbb{W}^2 Z + 12\,WZ^2)\,K\Big)
\end{aligned}$$

for any K adapted to the Brownian filtration and such that $K \in L^2(\mu, H)$.

▷ technically one really needs a relaxation to discuss minimizers, we ignore this all along this talk. the actualy object of study is the law of the pair $(\mathbb{W}, Z)$ and not the process Z. (similar as what happens in the Φ^4_3 paper)

apriori estimates

we use polynomial weights $\rho(x) = (1 + \ell|x|)^{-n}$ for large $n > 0$ and small $\ell > 0$.

Theorem. *There exists a constant* C *independent of* $|\Lambda|$ *such that, for any minimizer* Z *of* $F^{f,\Lambda}(\mu)$ *and any spatial weight* $\rho : \Lambda \to [0,1]$ *with* $|\nabla\rho| \leqslant \varepsilon\,\rho$ *for some* $\varepsilon > 0$ *small enough, we have*

$$\mathbb{E}\left[4\lambda\int_\Lambda \rho Z_1^4 + \int_0^1 \int_{\mathbb{R}^2} ((m^2 - \Delta)^{1/2}\rho^{1/2}\dot{Z}_s)^2 \mathrm{d}s\right] \leqslant C.$$

Proof. test the Euler–Lagrange equations with $K = \rho Z$ and then estimate the bad terms with the good terms and objects only depending on $\mathbb{W}$, e.g.

$$\left|\int_\Lambda \rho\, \mathbb{W}^3 Z\right| \leqslant C_\delta \|\mathbb{W}^3\|^2_{H^{-1}(\rho^{1/2})} + \delta\|Z\|^2_{H^1(\rho^{1/2})},$$

$$\left|\int_\Lambda \rho\, \mathbb{W}^2 Z^2\right| \leqslant C_\delta \|\rho^{1/8}\,\mathbb{W}^2\|^4_{C^{-\varepsilon}} + \delta(\|\rho^{1/4}\bar{Z}\|^4_{L^4} + \|\rho^{1/2}\bar{Z}\|^2_{H^{2\varepsilon}}), \cdots$$

tightness and bounds

$$\mathscr{W}_\Lambda(f) = \inf_Z F^{f,\Lambda}(Z) - \inf_Z F^{0,\Lambda}(Z) = F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

Therefore

$$F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{f,\Lambda}) \leqslant \mathscr{W}_\Lambda(f) \leqslant F^{f,\Lambda}(Z^{0,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

and since, for any g,

$$F^{f,\Lambda}(Z^{g,\Lambda}) - F^{0,\Lambda}(Z^{g,\Lambda}) = \mathbb{E}[f(W+Z^{g,\Lambda}) + \lambda V_\Lambda(W+Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})]$$

$$-\mathbb{E}[\lambda V_\Lambda(W+Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})] = \mathbb{E}[f(W+Z^{g,\Lambda})]$$

$$\mathbb{E}[f(W+Z^{f,\Lambda})] \leqslant \mathscr{W}_\Lambda(f) \leqslant \mathbb{E}[f(W+Z^{0,\Lambda})]$$

Consequence: tightness of $(\theta_\Lambda)_\Lambda$ in $\mathscr{S}'(\mathbb{R}^2)$ and optimal exponential bounds (cfr. Hairer/Steele)

$$\sup_\Lambda \int \exp(\delta \|\phi\|^4_{W^{-\kappa,4}(\rho)}) \theta_\Lambda(d\phi) < \infty.$$

Euler–Lagrange equation in infinite volume

moreover

$$\int f(\phi)\,\theta_\Lambda(\mathrm{d}\phi) = \mathbb{E}[f(X + Z^{0,\Lambda})]$$

the family $(Z^{f,\Lambda})_\Lambda$ is converging (provided we look at the relaxed problem) and any limit point $Z = Z^f$ satisfies a EL equation:

$$\mathbb{E}\left\{\int_{\mathbb{R}^2} f'(W+Z)\,K + 4\lambda \int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,K + \int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s (m^2 - \Delta)\,\dot{K}_s \mathrm{d}s\right\} = 0$$

for any test process K (adapted to $\mathbb{W}$ and to Z).

<u>**a kind of stochastic "elliptic" problem**</u>

the stochastic equation

rewrite the EL equation as

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \left(f'(W_1+Z_1) + 4\lambda\llbracket (W_1+Z_1)^3\rrbracket + \dot{Z}_s(m^2-\Delta)\right)\dot{K}_s \mathrm{d}s\right\} = 0$$

then

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \mathbb{E}\left[f'(W_1+Z_1) + 4\lambda\llbracket (W_1+Z_1)^3\rrbracket + (m^2-\Delta)\dot{Z}_s \middle| \mathscr{F}_s\right]\dot{K}_s \mathrm{d}s\right\} = 0$$

which implies that

$$(m^2-\Delta)\dot{Z}_s = -\mathbb{E}\left[f'(W_1+Z_1) + 4\lambda\llbracket (W_1+Z_1)^3\rrbracket \middle| \mathscr{F}_s\right]$$

Open questions

- Uniqueness??
- Γ-convergence of the variational description of $\mathscr{W}_\Lambda(f)$?

not clear. We lack sufficient knowledge of the dependence on f of the solutions to the EL equations above.

exponential interaction

we can study similarly the model with

$$V^{\xi}(\varphi)=\int_{\mathbb{R}^2}\xi(x)[\![\exp(\beta\varphi(x))]\!]\mathrm{d}x$$

for $\beta^2<8\pi$ and $\xi:\mathbb{R}^2\to[0,1]$ a smooth spatial cutoff function

$$V^{\xi}(W+Z)=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))\underbrace{[\![\exp(\beta W(x))]\!]\mathrm{d}x}_{M^{\beta}(\mathrm{d}x)}$$

$$=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))M^{\beta}(\mathrm{d}x), \qquad \text{[Gaussian multiplicative chaos]}$$

BD formula

$$\begin{aligned}\mathcal{W}^{\xi,\exp}(f) &= -\log\int\exp(-f(\phi))\mathrm{d}\nu^{\xi}\\ &= \inf_{Z\in\mathfrak{H}_a}\mathbb{E}\left[f(W+Z)+\int\xi\exp(\beta Z)\mathrm{d}M^{\beta}+\frac{1}{2}\int_0^1\int((m^2-\Delta)^{1/2}\dot{Z}_t)^2\mathrm{d}t\right]\end{aligned}$$

▷ the function $Z\mapsto V^{\xi}(W+Z)$ is convex!

variational description of the infinite volume limit

▷ thanks to convexity the EL equations have a unique limit Z in the ∞ volume limit

▷ moreover we have the Γ-convergence of the variational description:

$$\mathscr{W}_{\mathbb{R}^2}(f) = \lim_{n\to\infty}\left[-\log\int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp}\right]$$

$$= \lim_{n\to\infty}\,[\mathscr{W}_{\xi_n}(f) - \mathscr{W}_{\xi_n}(0)] = \inf_K G^{f,\infty,\exp}(K)$$

with functional

$$G^{f,\infty,\exp}(K) = \mathbb{E}\Big[f(W+Z+K) + \underbrace{\int \exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^\beta + \mathcal{E}(K)}_{\geqslant 0}\Big]$$

which depends via Z on the infinite volume measure for the exp interaction.

stochastic analysis of Grassmann variables

EQF with Fermion fields involve *anti-commuting* variables (Grassmann algebras)

$$\psi_1\psi_2 = -\psi_2\psi_1$$

there is a notion of Grassmann Gaussian variables (and Brownian motion)

$$\mathbb{E}[\psi_1 \cdots \psi_n] = \sum_{\text{pairs}(i,j)} (-1)^{\#}\, \mathbb{E}[\psi_i\psi_j]$$

stochastic analysis on Grassmann algebras and stochastic quantisation of fermionic EQFs

[S. Albeverio, L. Borasi, F. De Vecchi, **MG** · Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions · preprint 2021]

some papers

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