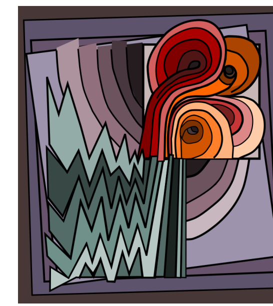
“Quantum many body system and interacting particles” · Münster · June 2022

Towards stochastic quantisation of Euclidean Fermions



Massimiliano Gubinelli · University of Bonn (IAM & HCM)

[made with TeXmacs]

Euclidean Fermions

Fermions: quantum particles satisfying Fermi–Dirac statistics (i.e. living in the anti-symmetric tensor of one-particle states).

EQFT: Wick rotation of QFT. $t \to \tau = it$, $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ Euclidean space. Wightman functions → Schwinger functions.

$$\Psi, \Psi^* \to \psi, \bar{\psi}.$$

☞ K. Osterwalder and R. Schrader. Euclidean Fermi fields and a Feynman-Kac formula for Boson-Fermions models. *Helvetica Physica Acta*, 46:277–302, 1973.

Euclidean fermion fields $\psi, \bar{\psi}$ form a Grassmann algebra

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \quad (\psi_\alpha^2 = 0).$$

Schwinger functions

- Schwinger functions are given by a Berezin integral on $\Lambda = \mathrm{GA}(\psi, \bar{\psi})$

$$\langle O(\psi,\bar{\psi})\rangle = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-S_E(\psi,\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi} e^{-S_E(\psi,\bar{\psi})}} = \frac{\left\langle O(\psi,\bar{\psi}) e^{-V(\psi,\bar{\psi})}\right\rangle_C}{\left\langle e^{-V(\psi,\bar{\psi})}\right\rangle_C}$$

$$S_E(\psi,\bar{\psi}) = \frac{1}{2}(\psi, C\bar{\psi}) + V(\psi,\bar{\psi}) \qquad \langle O(\psi,\bar{\psi})\rangle_C = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-\frac{1}{2}(\psi, C\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{-\frac{1}{2}(\psi, C\bar{\psi})}}$$

- Under $\langle\cdot\rangle_C$ the variables $\psi, \bar{\psi}$ are “Gaussian” (Wicks' rule):

$$\langle \psi(x_1)\cdots\psi(x_{2n})\rangle_C = \sum_\sigma (-1)^\sigma \langle \psi(x_{\sigma(1)})\psi(x_{\sigma(2)})\rangle_C \cdots \langle \psi(x_{\sigma(2n-1)})\psi(x_{\sigma(2n-1)})\rangle_C$$

probability, algebraically

▷ a non-commutative probability space $(\mathscr{A},\omega)$ is given by a C^*-algebra $\mathscr{A}$ and a *state* ω, a linear normalized positive functional on $\mathscr{A}$ (i.e. $\omega(aa^*)\geqslant 0$).

▷ a random variable is in algebra homomorphism into $\mathscr{A}$

☞ L. Accardi, A. Frigerio, and J. T. Lewis. Quantum stochastic processes. *Kyoto University. Research Institute for Mathematical Sciences. Publications*, 18(1):97–133, 1982. 10.2977/prims/1195184017

example. (classical) random variable with values on a manifold $\mathscr{M}$?

$$\Omega \xrightarrow{X} \mathscr{M} \xrightarrow{f} \mathbb{R}$$

$$f\in L^\infty(\mathscr{M};\mathbb{C})\to X(f)\in\mathscr{A}=L^\infty(\Omega;\mathbb{C}), \qquad X(fg)=X(f)X(g), \quad X(f^*)=X(f)^*.$$

algebraic data: $\mathscr{A}=L^\infty(\Omega;\mathbb{C})$, $\omega(a)=\int_\Omega a(\omega)\mathbb{P}(\mathrm{d}\omega)$, $X\in\mathrm{Hom}_*(L^\infty(\mathscr{M}),\mathscr{A})$.

Grassmann probability

▷ random variables with values in a Grassmann algebra Λ are algebra *homomorphisms*

$$\mathscr{G}(V) = \mathrm{Hom}(\Lambda, \mathscr{A})$$

The embedding of ΛV into $\mathscr{A}$ allows to use the topology of $\mathscr{A}$ to do analysis on Grassmann algebras.

$$d_{\mathscr{G}(V)}(X, Y) := \|X - Y\|_{\mathscr{G}(V)} = \sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathscr{A}},$$

analogy. Gaussian processes in Hilbert space. Abstract Wiener space. "a convenient place where to hang our (analytic) hat on".

Back to QFT: IR & UV problems

QFT requires to consider the formula

$$\langle O(\psi,\bar\psi)\rangle_{C,V} = \frac{\langle O(\psi,\bar\psi)e^{-V(\psi,\bar\psi)}\rangle_C}{\langle e^{-V(\psi,\bar\psi)}\rangle_C}$$

with local interaction

$$V(\psi,\bar\psi) = \int_{\mathbb{R}^d} P(\psi(x),\bar\psi(x))\mathrm{d}x$$

and singular covariance kernel (due to reflection positivity)

$$\langle\bar\psi(x)\psi(y)\rangle \propto |x-y|^{-\alpha}$$

this gives an ill-defined representation

- **large scale (IR) problems**
- **small scale (UV) problems**

well understood in the constructive QFT literature (Gawedzki, Kupiainen, Lesniewski, Rivasseau, Seneor, Magnen, Feldman, Salmhofer, Mastropietro, Giuliani,...)

stochastic quantisation

Parisi–Wu ('81) introduced a stationary stochastic evolution associated with the EQF

$$\partial_t \Phi(t,x) = -\frac{\delta S(\Phi(t,x))}{\delta \Phi} + 2^{1/2}\eta(t,x), \qquad t \geqslant 0, x \in \mathbb{R}^d,$$

with η space-time white noise

$$\langle \Phi(t,x_1)\cdots\Phi(t,x_n)\rangle = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(t,x_1)\cdots\varphi(t,x_n) e^{-S(\varphi)} \mathrm{d}\varphi, \qquad t \in \mathbb{R}.$$

---

transport interpretation: the map

$$\eta \mapsto \Phi(t,\cdot)$$

sends the Gaussian measure of the space-time white noise to the EQF measure

---

▷ many recent progresses for Bosonic theories starting with the work of Hairer on Φ^4_3 | many kinds of stochastic quantisations: parabolic, hyperbolic, elliptic, variational

What about stochastic quantisation for Grassmann measures?

☞ Ignatyuk/Malyshev/Sidoravichius | "Convergence of the Stochastic Quantization Method I,II", 1993. [Grassmann variables + cluster expansion]

weak topology + solution of equations in law + infinite volume limit but no removal of the UV cutoff

*

☞ "Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions" | joint work with Sergio Albeverio, Luigi Borasi, Francesco C. De Vecchi. ArXiv:2004.09637 (PTRF)

algebraic probability viewpoint + strong solutions via Picard interation + infinite volume limit but no removal of the UV cutoff

☞ "Stochastic Quantization of Subcritical Grassmann Measures with forward-backward SDEs" | joint work with Francesco C. De Vecchi and Luca Fresta. (work in progress)

alg. prob. + forward-backward SDE + infinite volume limit & removal of IR cutoff in the whole subcritical regime

Grassmann stochastic analysis

- ▷ filtration $(\mathcal{A}_t)_{t\geqslant 0}$, conditional expectation $\omega_t\colon \mathcal{A}\to\mathcal{A}_t$,

$$\omega_t(ABC) = A\omega_t(B)C, \qquad A, C \in \mathcal{A}_t.$$

- ▷ Brownian motion $(B_t)_{t\geqslant 0}$ with $B_t \in \mathcal{G}(V)$

$$\omega(B_t(v)B_s(w)) = \langle v, Cw\rangle (t\wedge s), \qquad t, s \geqslant 0, v, w \in V.$$

$$\|B_t - B_s\| \lesssim |t-s|^{1/2}.$$

- ▷ Ito formula

$$\Psi_t = \Psi_0 + \int_0^t B_u(\Psi_u)\mathrm{d}u + X_t, \qquad \omega(X_t \otimes X_s) = C_{t\wedge s}$$

$$\omega_s(F_t(\Psi_t)) = \omega_s(F_s(\Psi_s)) + \int_s^t \omega_s[\partial_u F_u(\Psi_u) + \mathcal{L}F_u(\Psi_u)]\mathrm{d}u,$$

$$\mathcal{L}_u F_u = \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle B_u, \mathrm{D}F_u\rangle$$

the forward-backward SDE

let Ψ be a solution of

$$\mathrm{d}\Psi_s = \dot{C}_s\, \omega_s(\mathrm{D}V(\Psi_T))\mathrm{d}s + \mathrm{d}X_s, \qquad s \in [0,T], \quad \Psi_0 = 0.$$

where $(X_t)_t$ is Gaussian martingale with covariance $\omega(X_t \otimes X_s) = C_{t\wedge s}$. Then

$$\omega(\mathrm{e}^{V(X_T)})\omega(\mathrm{e}^{-V(\Psi_T)}) = 1$$

and

$$\omega(O(\Psi_T)) = \frac{\omega(O(X_T)\mathrm{e}^{V(X_T)})}{\omega(\mathrm{e}^{V(X_T)})} = \frac{\langle O(\psi)e^{V(\psi)}\rangle_{C_T}}{\langle e^{V(\psi)}\rangle_{C_T}}$$

for any O.

$\triangleright$ this FBSDE provides a stochastic quantisation of the Grassmann Gibbs measure along the interpolation $(X_t)_t$ of its Gaussian component.

the backwards step

let F_t be such that $F_T = \mathrm{D}V$. By Ito formula

$$B_s := \omega_s(\mathrm{D}V(\Psi_T)) = \omega_s(F_T(\Psi_T))$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

letting $R_t = B_t - F_s(\Psi_s)$ we have now the forwards-backwards system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

with

$$Q_u := \partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{DF}_u\rangle$$

solution theory

▷ standard interpolation for $C_\infty = (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}$, $\gamma \leqslant d/2$. $\chi \in C^\infty(\mathbb{R}_+)$, compactly supported around 0:

$$C_t := (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}\chi(2^{-2t}(-\Delta_{\mathbb{R}^d})), \qquad \|\dot{C}\|_{\mathscr{L}(L^\infty,L^\infty)} \lesssim 2^{2\gamma-d}, \|\dot{C}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim 2^{2\gamma}$$

▷ the system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s)+R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

can be solved by standard fixpoint methods for small interaction, uniformly in the volume since X stays bounded as long as $T<\infty$:

$$\|X_t\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{\gamma t}.$$

▷ decay of correlations can be proved by coupling different solutions (Funaki '96).

▷ limit $T\to\infty$ requires renormalization when $\gamma \in [0, d/2]$.

relation with the continuous RG

if we take F such that $Q=0$ we have $R=0$ and then

$$\Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s))\mathrm{d}s + X_t,$$

with

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{D} \mathrm{F}_u \rangle = 0, \qquad F_T = \mathrm{D}V.$$

define the effective potential V_t by the solution of the HJB equation

$$\partial_u V_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} V_u + \langle \mathrm{D}V_u, \dot{C}_u \mathrm{D}V_u \rangle = 0, \qquad V_T = V.$$

then $F_t = \mathrm{D}V_t$ and the FBSDE computes the solution of the RG flow equation along the interacting field.

▷ so far a full control of the Fermionic HJB equation has not been achieved (work by Brydges, Disertori, Rivasseau, Salmhofer,...). Fermionic RG methods rely on a discrete version of the RG iteration.

approximate flow equation

thanks for the FBSDE we are not bound to solve exactly the flow equation and we can proceed to approximate it.

▷ **linear approximation.** take

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u = 0, \qquad F_T = \mathrm{D}V.$$

this corresponds to Wick renormalization of the potential V:

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \, (F_s(\Psi_s) + R_s) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

the key difficulty is to show uniform estimates for

$$\int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u) \rangle] \mathrm{d}u$$

as $T \to \infty$. we cannot expect better than $\|\Psi_t\| \approx \|X_t\| \approx 2^{\gamma t}$.

polynomial truncation

a wiser approximation is to truncate the equation to a (large) finite polynomial degree

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u + \Pi_{\leqslant K} \langle F_u, \dot{C}_u \mathrm{D} F_u \rangle = 0$$

where $\Pi_{\leqslant K}$ denotes projection on Grassmann polynomials of degree $\leqslant K$ and take

$$F_t(\psi) = \sum_{k \leqslant K} F_t^{(k)} \psi^{\otimes k}.$$

With this approximation one can solve the flow equation and get estimates

$$\|F_t^{(k)}\| \leqslant \frac{2^{(\alpha - \beta k)t}}{(k+1)^2}, \qquad t \geqslant 0,$$

with $\alpha = 3\beta$, $\beta = d/2 - \gamma$, provided the initial condition $F_T = \mathrm{D}V$ is appropriately renormalized.

FBSDE in the full subcritical regime

with the truncation Π_K we have

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s \left(F_s(\Psi_s) + R_s \right) \mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{DF}_u \rangle (\Psi_u)] \mathrm{d}u + \int_t^T \omega_t [\langle R_u, \dot{C}_u \mathrm{DF}_u (\Psi_u) \rangle] \mathrm{d}u \end{cases}$$

but now observe that

$$\|\Psi_t\| \approx \|X_t\| \lesssim 2^{\gamma t} \qquad \|F_t^{(k)} \Psi_t^{\otimes k}\| \lesssim 2^{(\gamma k - \beta(k-3))t}$$

which is exponentially small for k large as long as $\gamma \leqslant d/4$ (full subcrititcal regime).

now the term

$$\int_t^T \omega_t [\Pi_{>K} \langle F_u, \dot{C}_u \mathrm{DF}_u \rangle (\Psi_u)] \mathrm{d}u$$

can be controlled uniformly as $T \to \infty$ and also the full FBSDE system. (!)

thanks