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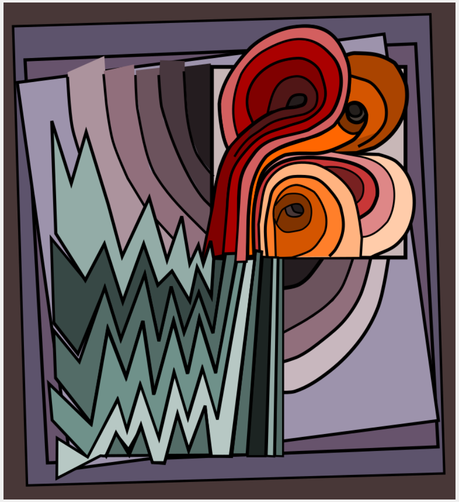
Variational approaches for Euclidean quantum field theory

(part I)



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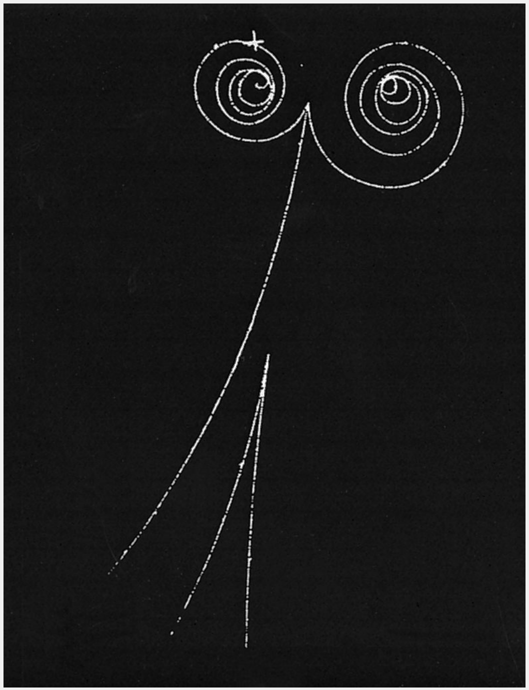

- Euclidean QFT & connection with probability
- Measures on distributions and renormalization
- Gibbs measures which describe interacting fields
- Variational approach (j.w. with Barashkov): a way to describe measures which works also in singular cases and uses a stochastic optimal control problem
- Connections: with optimal control, Gamma-convergence and renormalization group & certain HJB equation in ∞-dimensions.
- Goal: construction of the Φ^4_d theory for $d = 2, 3$.

Euclidean QFT (history)

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Construct rigorously QM models which are compatible with special relativity, (finite speed of signals and Poincaré covariance of Minkowski space $\mathbb{R}^{n+1}$).

Quantum field theory (QM with ∞ many degrees of freedom)

Wightman axioms ('60-'70): Hilbert space, representation of the Poincaré group, fields operators (to construct local observables).

Constructive QFT program: Hard to find models of such axioms. Examples in $\mathbb{R}^{1+1}$ were found in the $'60$.

Euclidean rotation: $t \to it = x_0$ (imaginary time). $\mathbb{R}^{n+1} \to \mathbb{R}^d$ Minkowski → Euclidean

Oserwalder–Schrader theorem : gives precise condition to perform the passage to/from Euclidean space (OS axioms for Euclidean correlation function).

Surprise: in some cases the Euclidean theory is a probability measure on $\mathscr{S}'(\mathbb{R}^d)$.

High point of CQFT: construction of Φ^4_3 (Euclidean version of a scalar field in $\mathbb{R}^{2+1}$ Minkowski space).

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An EQFT is a prob. measure μ on $\mathscr{S}'(\mathbb{R}^d)$ such that the following holds (OS axioms)

1. **Regularity**: $\int_{\mathscr{S}'(\mathbb{R}^d)} e^{\alpha\|\varphi\|_s}\mu(\mathrm{d}\varphi) < \infty$ where $\|\varphi\|_s$ is some norm on $\mathscr{S}'(\mathbb{R}^d)$ and $\alpha > 0$.

2. **Euclidean covariance**: The Euclidean group G (rotation+translation) acts on $\mathscr{S}'(\mathbb{R}^d)$ and the measure μ is invariant under this action. Example:

$$\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(f(\cdot + h))\mu(\mathrm{d}\varphi) = \int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(f(\cdot))\mu(\mathrm{d}\varphi), \qquad f \in \mathscr{S}(\mathbb{R}^d).$$

3. **Reflection positivity**: Let $\theta(x_1, \ldots x_d) = (-x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then for any bounded measurable $F: \mathscr{S}'(\mathbb{R}_{>0} \times \mathbb{R}^{d-1}) \to \mathbb{C}$ we have

$$\int \overline{F(\theta\varphi)} F(\varphi)\mu(\mathrm{d}\varphi) \geqslant 0.$$

Example: for $x_1 > 0$,

$$\int \varphi(-x_1, x_2, \ldots, x_d)\varphi(x_1, x_2, \ldots x_d)\mu(\mathrm{d}\varphi) \geqslant 0, \qquad \int \varphi(y)\varphi(y')\overline{\varphi(\theta y)}\,\overline{\varphi(\theta y')}\mu(\mathrm{d}\varphi) \geqslant 0.$$

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The simplest example of EQFT. We take a Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with covariance

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean. This measure is reflection positive, Eucl. covariant and regular. This is the GFF with mass $m>0$. This measure can be used to construct a QFT in Minkowski space but unfortunately this theory is free, i.e. there is no interaction.

Other gaussian measures which are reflection positive, Eucl. covariant and regular can be constructed by positive linear combinations of Gaussians, i.e. by taking

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} \lambda(\mathrm{d}r)\int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{r+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d}.$$

These are the only known RP Gaussian measures.

Note that $G(0)=+\infty$ if $d\geqslant 2$, this implies that the GFF is not a function.

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Can we construct a non-Gaussian EQFT?

The heuristic idea is to try to maintain the "Makovianity" of the GFF μ. Heuristically we want something like

$$\nu(\mathrm{d}\varphi) = \frac{e^{\int_\Lambda V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi),$$

with $\Lambda = \Lambda_+ \cup \theta\Lambda_+$ and $V: \mathbb{R} \to \mathbb{R}$ so that

$$\int_\Lambda V(\varphi(x))\mathrm{d}x = \int_{\Lambda_+} V(\varphi(x))\mathrm{d}x + \int_{\Lambda_+} V((\theta\varphi)(x))\mathrm{d}x$$

since it will be RP:

$$\int \overline{F(\theta\varphi)} F(\varphi)\nu(\mathrm{d}\varphi) = \int \frac{\overline{F(\theta\varphi) e^{\int_{\Lambda_+} V(\theta\varphi(x))\mathrm{d}x}} F(\varphi) e^{\int_{\Lambda_+} V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi) \geqslant 0.$$

Unfortunately even if we can make sense of it this measure will not be translation invariant, ideally we would like to have $\Lambda = \mathbb{R}^d$.

Goal

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The aim of the next lectures is to make this picture rigorous when $d=2,3$ and for certain V, in particular we will simplify the problem by taking $\mathbb{T}^d$ instead of $\mathbb{R}^d$ (RP can be generalize to the torus).

We will consider an approximate measure

$$\nu^\varepsilon(\mathrm{d}\varphi)=\frac{e^{-\int_{\mathbb{T}^d}V_\varepsilon((\rho_\varepsilon*\varphi)(x))\mathrm{d}x}}{Z^\varepsilon}\mu(\mathrm{d}\varphi),$$

where ρ_ε is a smoothing function such that $\rho_\varepsilon\to\delta$ as $\varepsilon\to0$. This give a well defined measure ν^ε our main goal is to show that $\nu^\varepsilon\to\nu$ as $\varepsilon\to0$ and describe the limit. When we take

$$V_\varepsilon(\xi)=\xi^4-a_\varepsilon\xi^2-b_\varepsilon$$

for suitable $a_\varepsilon,b_\varepsilon\to\infty$ we say that the limit is the Φ_d^4 on the periodic domain $\mathbb{T}^d$.

It will turn out that for $d=2$ the measure $\nu\ll\mu$ but not in $d=3$.