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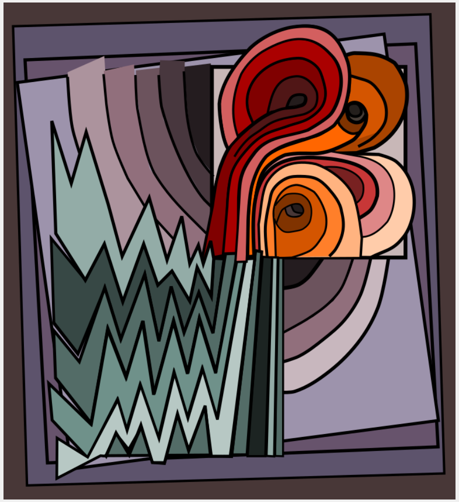
Variational approaches for Euclidean quantum field theory

(part III)



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[ ]

Consider for a generic nice $f: \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ $d = 2,3$

$$e^{-\mathscr{W}_{1/\varepsilon}(f)} := \int e^{f(\varphi)} e^{-\int_{\mathbb{T}^d} \varphi_\varepsilon^4 - a_\varepsilon \varphi_\varepsilon^2 - b_\varepsilon} \mu(\mathrm{d}\varphi) = \mathbb{E}\left[e^{f(\varphi) - \int_{\mathbb{T}^3} \varphi_\varepsilon^4 - a_\varepsilon \varphi_\varepsilon^2 - b_\varepsilon}\right]$$

$$\int e^{f(\varphi)} \nu^\varepsilon(\mathrm{d}\varphi) = \frac{e^{-\mathscr{W}_{1/\varepsilon}(f)}}{e^{-\mathscr{W}_{1/\varepsilon}(0)}} \to \frac{e^{-\mathscr{W}_\infty(f)}}{e^{-\mathscr{W}_\infty(0)}} = \int e^{f(\varphi)} \nu(\mathrm{d}\varphi)$$

Theorem. *(Boué–Dupuis) Let $(B_t)_{t \geqslant 0}$ be a Brownian motion on $\mathbb{R}^n$, then for any bounded $F: C(\mathbb{R}_+; \mathbb{R}^n) \to \mathbb{R}$ we have*

$$\log \mathbb{E}[e^{F(B_\bullet)}] = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[F(B_\bullet + I(u)_\bullet) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

with $u: \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ adapted to B and with $I(u)_t := \int_0^t u_s \mathrm{d}s$.

$$\frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s \approx H(\mathrm{Law}(B_\bullet + I(u)_\bullet) | \mathrm{Law}(B_\bullet)).$$

B cyl BM on $L^2(\mathbb{T}^d)$, $W_T = \int_0^T J_s \mathrm{d}B_s$, $J_s = (1-\Delta)^{-1/2}\rho(|-\Delta|^{1/2}/s)$

$$\mathscr{W}_T(f) := \log \mathbb{E}\left[e^{f(W_\infty) - \int_{\mathbb{T}^3} W_T^4 - a_\varepsilon W_T^2 - b_\varepsilon}\right] = \log \mathbb{E}[e^{F(B_\bullet)}]$$

$$\underset{\text{BD formula}}{=\!=\!=} \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + Z_\infty) - \int_{\mathbb{T}^3} (W_T + Z_T)^4 - a_\varepsilon (W_T + Z_T)^2 - b_\varepsilon - \frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2(\mathbb{T}^d)} \mathrm{d}s\right]$$

with , the GFF acquires a drift Z. $u: \Omega \times \mathbb{R}_+ \to L^2(\mathbb{T}^d)$

$$Z_t = \int_0^t J_s u_s \mathrm{d}s \qquad \sup_{t \geqslant 0} \|Z_t\|^2_{H^1(\mathbb{T}^d)} \lesssim \int_0^\infty \|u_s\|^2_{L^2(\mathbb{T}^d)} \mathrm{d}s$$

$$W_\infty \in \mathscr{C}^{\frac{(2-d)}{2} - \kappa}(\mathbb{T}^d), \quad \sup_{t \geqslant 0} \|W_t\|_{\mathscr{C}^{\frac{(2-d)}{2} - \kappa}} < \infty \ (a.s.), \qquad 0 < \kappa \ll 1.$$

[ ]

Goal: uniform bounds as $T\to\infty$

$$\mathcal{W}_T(f)=\sup_{u\in\mathbb{H}_a}\mathbb{E}\left[f(W_\infty+Z_\infty)-\int_{\mathbb{T}^3}[(W_T+Z_T)^4-a_T(W_T+Z_T)^2-b_T]-\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2(\mathbb{T}^d)}\mathrm{d}s\right]$$

Take $f=0$ and $\lambda=1$ for simplicity. Take $u=0$ then $Z=0$ and

$$W_T(0)\geqslant\mathbb{E}\left[-\int_{\mathbb{T}^3}W_T^4-a_TW_T^2-b_T\right]>-\infty$$

to have finite result we need to take $a_T=3\mathbb{E}[W_T^2]$ and b_T suitably. Let's tackle the other bound, with the choice of a_T one has

$$(W_T+Z_T)^4-a_T(W_T+Z_T)^2-b_T=[\![W_T^4]\!]+4[\![W_T^3]\!]Z_T+12[\![W_T^2]\!]Z_T^2+4W_TZ_T^3+Z_T^4$$

Wick products: $[\![W_T^4]\!]=\frac{1}{4}\int_0^T[\![W_t^3]\!]\mathrm{d}W_t$, $[\![W_T^3]\!]=\frac{1}{3}\int_0^T[\![W_t^2]\!]\mathrm{d}W_t$, $[\![W_T^2]\!]=\frac{1}{2}\int_0^TW_t\mathrm{d}W_t$ they are martingales wrt. T.

Upper bound (note that $\mathbb{E}[\int_{\mathbb{T}^3}[\![W_T^4]\!]=0)$

$$\mathscr{W}_T(0)=\sup_{u\in\mathbb{H}_a}\mathbb{E}\Big[-\underbrace{\int_{\mathbb{T}^3}\Big\{4[\![W_T^3]\!]Z_T+12[\![W_T^2]\!]Z_T^2+4W_TZ_T^3\Big\}}_{\text{bad terms}}-\underbrace{\Big\{\int_{\mathbb{T}^3}Z_T^4+\frac{1}{2}\int_0^\infty\|u_s\|_{L^2(\mathbb{T}^d)}^2\mathrm{d}s\Big\}}_{\text{good terms}}\Big]$$

recall that $\mathbb{H}_a$ are all the "adapted" processes with values in $L^2(\mathbb{R}_+;L^2(\mathbb{T}^d))$. At this point we "forget" about probability. The good coercive terms contol the L^4 and the H^1 norm of $(Z_t)_t$. The bad terms can be controlled as (in $d=2$: we have uniform in T control in $\mathscr{C}^{-\kappa}$ for $[\![W_T^3]\!],[\![W_T^2]\!],W_T$). Pathwise estimates

$$\Big|\int_{\mathbb{T}^d}\underbrace{[\![W_T^3]\!]}_{\mathscr{C}^{-\kappa}}Z_T\Big|\lesssim\|[\![W_T^3]\!]\|_{\mathscr{C}^{-\kappa}}\|Z_T\|_{B_{1,1}^{\kappa}}\lesssim\|[\![W_T^3]\!]\|_{\mathscr{C}^{-\kappa}}\|Z_T\|_{H^1}\lesssim C_\delta\|[\![W_T^3]\!]\|_{\mathscr{C}^{-\kappa}}^2+\delta\|Z_T\|_{H^1}^2$$

for small $\delta>0$. We are in good shape since $\sup_T\mathbb{E}[\|[\![W_T^3]\!]\|_{\mathscr{C}^{-\kappa}}^2]<\infty$ and the second term is controlled by the good terms because we can choose $\delta>0$ small.

$$\Big|\int_{\mathbb{T}^d}[\![W_T^2]\!]Z_T^2\Big|\lesssim\|[\![W_T^2]\!]\|_{\mathscr{C}^{-\kappa}}\|Z_T^2\|_{B_{1,1}^{\kappa}}\lesssim C_\delta\|[\![W_T^2]\!]\|_{\mathscr{C}^{-\kappa}}^M+\delta\Big[\int_{\mathbb{T}^3}Z_T^4+\frac{1}{2}\int_0^\infty\|u_s\|_{L^2(\mathbb{T}^d)}^2\mathrm{d}s\Big]$$

same for the remaining term. Some care in needed in these estimates.

Wrap-up: pathwise control of the bad terms:

$$\text{(bad terms)} \leqslant C_\delta \left(\sum_{\ell=1,2,3} \| [\![W_T^\ell]\!] \|_{\mathscr{C}^{-\kappa}} \right)^M + \delta \left[\int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \| u_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right]$$

$$\text{(bad terms)} - \left[\int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \| u_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right] \leqslant C_\delta \left(\sum_{\ell=1,2,3} \| [\![W_T^\ell]\!] \|_{\mathscr{C}^{-\kappa}} \right)^M$$

Take averages to get

$$\mathscr{W}_T(0) = \sup_{u \in \mathbb{H}_a} \mathbb{E} \Bigg[- \underbrace{\int_{\mathbb{T}^3} \left\{ 4 [\![W_T^3]\!] Z_T + 12 [\![W_T^2]\!] Z_T^2 + 4 W_T Z_T^3 \right\}}_{\text{bad terms}} - \underbrace{\left\{ \int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \| u_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right\}}_{\text{good terms}} \Bigg]$$

$$\leqslant C_\delta \sup_T \mathbb{E} \left(\sum_{\ell=1,2,3} \| [\![W_T^\ell]\!] \|_{\mathscr{C}^{-\kappa}} \right)^M < \infty$$

the last step is a probabilistic estimate (very easy using hypercontractivity).

This shows that $\mathscr{W}_T(f)$ is uniformly in T bounded $\Rightarrow$ tightness of ν^ε (in $d = 2$).

[ ]

Provided we choose a_T, b_T as explained above we have that

$$\mathscr{W}_T(f) = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + Z_\infty) - \int_{\mathbb{T}^3} [(W_T + Z_T)^4 - a_T(W_T + Z_T)^2 - b_T] - \frac{1}{2}\int_0^\infty \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right]$$

converge as $T \to \infty$ to the functional

$$\mathscr{W}_\infty(f) = \sup_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + Z_\infty) - \int_{\mathbb{T}^3} \left\{ 4[\![W_\infty^3]\!] Z_\infty + 12[\![W_\infty^2]\!] Z_\infty^2 + 4 W_\infty Z_\infty^3 + Z_\infty^4 \right\} - \frac{1}{2}\int_0^\infty \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right]$$

and the Φ_2^4 measure ν satisfy

$$-\log \int e^{f(\varphi)} \nu(\mathrm{d}\varphi) = -\log \int e^{f(\varphi) - \int [\![\varphi^4]\!]} \mu(\mathrm{d}\varphi) = \mathscr{W}_\infty(f) - \mathscr{W}_\infty(0)$$

for functions $f: \mathscr{C}^{-\kappa}(\mathbb{T}^2) \to \mathbb{R}$ with linear growth.

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In this case things gets more complicated since

$$W_T \to W_\infty \text{ in } \mathscr{C}^{-1/2-\kappa}, \qquad [\![W_T^2]\!] \to [\![W_\infty^2]\!] \text{ in } \mathscr{C}^{-1-\kappa}, \qquad [\![W_T^3]\!] \text{ does not converge } \|[\![W_T^3]\!]\|_{\mathscr{C}^{-3/2-\kappa}} \approx \log T$$

and the problem is still the same: (the good terms are alwys L^4, H^1)

$$\mathscr{W}_T(0) = \sup_{u \in \mathbb{H}_a} \mathbb{E}\Big[-\underbrace{\int_{\mathbb{T}^3} \Big\{ 4[\![W_T^3]\!] Z_T + 12 [\![W_T^2]\!] Z_T^2 + 4 W_T Z_T^3 \Big\}}_{\text{bad terms}} - \underbrace{\Big\{ \int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \Big\}}_{\text{good terms}} \Big]$$

There is no hope of direct control of $\int_{\mathbb{T}^d} [\![W_T^3]\!] Z_T$, $\int_{\mathbb{T}^d} [\![W_T^2]\!] Z_T^2$ using only H^1 regularity for Z_T. We need to kill it using the good terms.

Let's focus on the two terms:

$$\int_{\mathbb{T}^3} 4[\![W_T^3]\!] Z_T + \frac{1}{2} \int_0^\infty \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s$$

Idea: complete the square:

$$\int_{\mathbb{T}^3} 4[\![W_T^3]\!]Z_T + \frac{1}{2}\int_0^T \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s$$

Step one: use Ito formula (we don't care about the martingale parts, recall that $\mathrm{d}Z_t = J_t u_t \mathrm{d}t$)

$$\mathrm{d}_t \int_{\mathbb{T}^3} 4[\![W_t^3]\!]Z_t = \left[\int_{\mathbb{T}^3} 4[\![W_t^3]\!](J_t u_t)\right]\mathrm{d}t + \text{mart.}$$

$$\int_{\mathbb{T}^3} 4[\![W_T^3]\!]Z_T + \frac{1}{2}\int_0^T \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s = \frac{1}{2}\int_0^T \int_{\mathbb{T}^3} \left[8(J_s[\![W_s^3]\!])\, u_s + u_s^2\right]\mathrm{d}s$$

$$= \frac{1}{2}\int_0^T \int_{\mathbb{T}^3} \left[(4(J_s[\![W_s^3]\!]) + u_s)^2 - 16(J_s[\![W_s^3]\!])^2\right]\mathrm{d}s = \frac{1}{2}\int_0^T \|l_s\|^2 \mathrm{d}s - 8\int_0^T \int_{\mathbb{T}^3} (J_s[\![W_s^3]\!])^2 \mathrm{d}s$$

with $l_s := 4(J_s[\![W_s^3]\!]) + u_s$. So we have a new energy a new bad term which I don't care about because do not depends on u anymore can be cancel by choosing b_T

$$\int_{\mathbb{T}^3} 4[\![W_T^3]\!]Z_T + \frac{1}{2}\int_0^T \|u_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s = \frac{1}{2}\int_0^T \|l_s\|^2 \mathrm{d}s - 8\int_0^T \int_{\mathbb{T}^3} (J_s[\![W_s^3]\!])^2 \mathrm{d}s$$

At this point the energy depends on

$$\frac{1}{2}\int_0^T \|l_s\|^2 \mathrm{d}s$$

where the control u_s is now given by

$$u_s = -4(J_s [\![W_s^3]\!]) + l_s$$

where now

$$Z_T = \int_0^T J_s u_s \mathrm{d}s = \underbrace{-4\int_0^T J_s^2 [\![W_s^3]\!] \mathrm{d}s}_{=:\mathbb{W}_T^{[3]}} + K_T, \qquad K_T = \int_0^T J_s l_s \mathrm{d}s.$$

with K_T controlled in H^1 by $\frac{1}{2}\int_0^T \|l_s\|^2 \mathrm{d}s$. The new term $\mathbb{W}_T^{[3]}$ will have regularity $\mathscr{C}^{1/2-\kappa}$ uniformly in T. In particular $Z_T \notin H^1$. New variational problem:

$$\mathscr{W}_T(0) = \text{const} + \sup_{l \in \mathbb{H}_a} \mathbb{E}\left[\underbrace{-\int_{\mathbb{T}^3} \left\{ 12 [\![W_T^2]\!] Z_T^2 + 4 W_T Z_T^3 \right\}}_{\text{bad terms}} - \underbrace{\left\{ \int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2}\int_0^\infty \|l_s\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right\}}_{\text{good terms}} \right]$$

Summary: new variational problem:

$$\mathscr{W}_T(0) = \text{const} + \sup_{l \in \mathbb{H}_a} \mathbb{E}\Big[- \underbrace{\int_{\mathbb{T}^3} \Big\{ 12 [\![W_T^2]\!] Z_T^2 + 4 W_T Z_T^3 \Big\}}_{\text{bad terms}} - \underbrace{\Big\{ \int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \| l_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \Big\}}_{\text{good terms}} \Big]$$

with $Z_T = -4\mathbb{W}_T^{[3]} + K_T$, $\mathbb{W}_T^{[3]}$ uniformly in $\mathscr{C}^{1/2-\kappa}$ and K_T uniformly in H^1. We two new problems the two bad terms are still not controlled.

$$\int_{\mathbb{T}^3} [\![W_T^2]\!] Z_T^2 = \int_{\mathbb{T}^3} [\![W_T^2]\!] (-4\mathbb{W}_T^{[3]} + K_T)^2 = \int_{\mathbb{T}^3} \underbrace{[\![W_T^2]\!]}_{\mathscr{C}^{-1-\kappa}} \big[(-4\mathbb{W}_T^{[3]})^2 - 8\mathbb{W}_T^{[3]} K_T + K_T^2 \big]$$

In order to solve this problem one has to split in the hope to complete the square again

$$\int_{\mathbb{T}^3} [\![W_T^2]\!] (-8\mathbb{W}_T^{[3]} K_T + K_T^2) = \int_{\mathbb{T}^3} \big[[\![W_T^2]\!] (-8\mathbb{W}_T^{[3]} + K_T) \big] K_T$$

$$= \int_{\mathbb{T}^3} \big[(-8 [\![W_T^2]\!] \mathbb{W}_T^{[3]} + [\![W_T^2]\!] \succ K_T) \big] K_T + \int_{\mathbb{T}^3} \big[[\![W_T^2]\!] \preccurlyeq K_T \big] K_T$$

where $\succ, \preccurlyeq$ are paraproducts such that $f \succ g + f \preccurlyeq g = gf$.

The idea is that

$$[\![W_T^2]\!] \succ K_T \approx [\![W_T^2]\!] \in \mathscr{C}^{-1-\kappa}, \qquad [\![W_T^2]\!] \preccurlyeq K_T \text{ is more regular} \in H^{1/2-\kappa}$$

$$\int_{\mathbb{T}^3} \left[(-8[\![W_T^2]\!]\mathbb{W}_T^{[3]} + [\![W_T^2]\!] \succ K_T)\right] K_T + \underbrace{\int_{\mathbb{T}^3} \left[[\![W_T^2]\!] \preccurlyeq K_T\right] K_T}_{\text{ok}}$$

On the first part we complete the square with $\int \|l\|^2$ as we did for $\int_{\mathbb{T}^d} [\![W^3]\!] Z_T + \int \|u_s\|^2$. At the end we obtain another change of variables and another variational problem

$$\mathscr{W}_T(0) = \text{const} + \sup_{h \in \mathbb{H}_a} \mathbb{E}\Bigg[-\underbrace{\int_{\mathbb{T}^3} \left\{4 W_T Z_T^3 + \cdots\right\}}_{\text{bad terms}} - \underbrace{\left\{\int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2}\int_0^\infty \|h_s\|^2_{L^2(\mathbb{T}^d)} \mathrm{d}s\right\}}_{\text{good terms}}\Bigg]$$

with

$$Z_T = -4\mathbb{W}_T^{[3]} + \int_0^T J_s^2(-8[\![W_s^2]\!]\mathbb{W}_s^{[3]} + [\![W_s^2]\!] \succ K_s)\mathrm{d}s + \int_0^T J_s h_s \mathrm{d}s.$$

At this point a bit work allow to conclude using the $d=2$ strategy and this form pass to the limit as $T \to \infty$ to give $\mathscr{W}_\infty(f)$.

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Provided we choose a_T, b_T as explained above we have that

$$\mathscr{W}_T(f) = \sup_{h \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + Z_\infty) - \int_{\mathbb{T}^3} \left\{ 4 W_T Z_T^3 + \cdots \right\} - \left\{ \int_{\mathbb{T}^3} Z_T^4 + \frac{1}{2} \int_0^\infty \| h_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right\} \right]$$

converge as $T \to \infty$ to the functional

$$\mathscr{W}_\infty(f) = \sup_{h \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + Z_\infty) - \int_{\mathbb{T}^3} \left\{ 4 W_\infty Z_\infty^3 + \cdots \right\} - \left\{ \int_{\mathbb{T}^3} Z_\infty^4 + \frac{1}{2} \int_0^\infty \| h_s \|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s \right\} \right]$$

and the Φ_3^4 measure ν satisfy

$$-\log \int e^{f(\varphi)} \nu(\mathrm{d}\varphi) = \mathscr{W}_\infty(f) - \mathscr{W}_\infty(0)$$

for functions $f: \mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3) \to \mathbb{R}$ with linear growth.

The renormalization of the var. prob. is a signal of singularity of ν wrt. μ. (also $H(\nu|\mu) = +\infty$).