Stochastic observables and the forward–backward flow of a Euclidean field

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What do we actually observe?

We observe a field at finite resolution. We can improve the resolution, but we never reach the point.

$$\text{coarse information} \longrightarrow \text{finer information}: \qquad (\mathcal{F}_t)_{t\geq 0}, \quad \mathcal{F}_\infty = \sigma\Big(\bigcup_t \mathcal{F}_t\Big).$$

For a singular field φ_∞ the meaningful data are the observations $(\varphi_t)_{t\geq 0}$ at every finite resolution.

- **Framework.** Consistent families of observations: martingales, possibly without a terminal value.
- **Examples.** Brownian deltas, the GFF, Wick powers, OPEs.
- **Construction.** Dynamics in resolution can produce the interacting field; in the Gibbs case an FBSDE, whose backward half is Polchinski and Wetterich.

Wilson–Kadanoff: physics is organised scale by scale

Stochastic observables

A stochastic observable

Fix a filtration $(\mathcal{F}_t)_{t\in I}$, $I = [0,\infty)$ or $[0,T)$, $\mathbb{E}_t = \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$.

Definition (Bailleul–Chevyrev–G.)

An observable is an integrable adapted family $O = (O_t)_{t\in I}$ with

$$O_s = \mathbb{E}_s O_t, \qquad 0 \le s \le t,\ s,t \in I.$$

Its expectation is $\langle O\rangle = \mathbb{E}O_t$.

A true martingale, but no terminal value is required.

- $Y \in L^1(\mathcal{F}_\infty)$ gives a closed observable $O_t = \mathbb{E}_t Y$: these are the uniformly integrable martingales.
- Each O_t is an ordinary random variable; the whole compatible family is the new object.

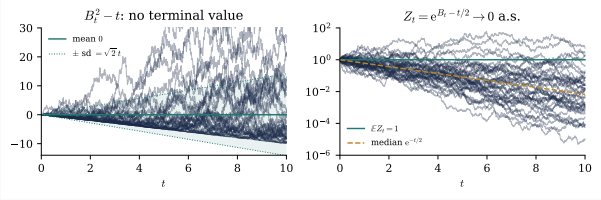
Brownian observables

Standard Brownian motion, $t \in [0, \infty)$, H_n the Hermite polynomials: the martingales

$$B_t, \qquad B_t^2 - t, \qquad B_t^3 - 3tB_t, \qquad \dots, \qquad H_n(B_t; t)$$

are observables without terminal representative, not even bounded in L^1:
$\mathbb{E}|B_t^2 - t| = t\,\mathbb{E}|Z^2 - 1| \to \infty$, $Z \sim N(0,1)$. Prototypes of Wick powers.

$$Z_t = \exp\big(\lambda B_t - \tfrac{\lambda^2}{2} t\big) = \sum_{n \geq 0} \frac{\lambda^n}{n!} H_n(B_t; t), \qquad Z_t \to 0 \text{ a.s. } (\lambda \neq 0), \qquad \mathbb{E} Z_t = 1.$$



The limit is zero, the observable is not: the mass escapes to ever rarer paths, and the a.s. limit forgets all finite-resolution expectations.

Equivalence classes and repair by the future drift

For adapted integrable N, N': $\quad N \sim N' \iff \|N_t - N'_t\|_{L^1} \to 0$ as $t \uparrow \sup I$.

Asymptotic uniqueness

A class contains at most one martingale, and it is recovered by looking back:
$O_s = \lim_{t \uparrow \sup I} \mathbb{E}_s N_t$ in L^1.

Proof. $\mathbb{E}_s$ is an L^1 contraction: $\|O_s - \mathbb{E}_s N_t\| \le \|O_t - N_t\| \to 0$. □

Repair. If $\mathrm{d}N_t = \beta_t \,\mathrm{d}t + \mathrm{d}M_t$, M a true martingale, $\mathbb{E}\int_0^\infty |\beta_s| \mathrm{d}s < \infty$, the observable in its class is

$$O_t = N_t + \mathbb{E}_t \int_t^\infty \beta_s \,\mathrm{d}s : \quad \text{the expected drift still to come.}$$

E.g. $N_t = B_t + \mathrm{e}^{-t} \sin B_t$: $\beta_t = -\frac{3}{2}\mathrm{e}^{-t} \sin B_t$, $\mathbb{E}_t \int_t^\infty \beta = -\mathrm{e}^{-t} \sin B_t$, so $O_t = B_t$.

This correction is the germ of the forward–backward equations.

A delta at a Brownian endpoint

Fix $T > 0$, $y \in \mathbb{R}$, $p_s(z) = (2\pi s)^{-1/2} \mathrm{e}^{-z^2/2s}$, and

$$\Pi_t^y = p_{T-t}(B_t - y), \qquad t < T.$$

- Heat semigroup: $\mathbb{E}_s \Pi_t^y = \Pi_s^y$; and $\int g(y) \Pi_t^y \, \mathrm{d}y = \mathbb{E}_t \, g(B_T)$ for bounded g.
- At fixed y: $\Pi_t^y \to 0$ a.s., but $\mathbb{E}\Pi_t^y = p_T(y) > 0$.

A singular observable

$\delta(B_T - y)$ is not a random variable, but it is an observable: the family of its conditional densities.

Products. For $y \neq z$, exactly,

$$p_\varepsilon(B - y) \, p_\varepsilon(B - z) = c_\varepsilon \, \mathrm{e}^{-(y-z)^2/4\varepsilon} \, p_{\varepsilon/2}\big(B - \tfrac{y+z}{2}\big), \qquad c_\varepsilon = (4\pi\varepsilon)^{-1/2},$$

so $\Pi^y \star \Pi^z = 0$. At $y = z$ the suppression is lost: renormalisation.

Brownian OPE: resolve the delta square

Put $\varepsilon = T - u$ and fix $t < u$. Since $p_\varepsilon(z)^2 = c_\varepsilon\, p_{\varepsilon/2}(z)$, conditioning on the coarse information is exact:

$$\mathbb{E}_t\big[(\Pi^y_{T-\varepsilon})^2\big] = c_\varepsilon\, p_{T-t-\varepsilon/2}(B_t - y).$$

Then $\partial_s p_s = \frac{1}{2}\partial_y^2 p_s$ gives, at fixed t,

$$\mathbb{E}_t\big[(\Pi^y_{T-\varepsilon})^2\big] = c_\varepsilon\, \Pi^y_t - \frac{\varepsilon c_\varepsilon}{4}\,\partial_y^2 \Pi^y_t + \frac{\varepsilon^2 c_\varepsilon}{32}\,\partial_y^4 \Pi^y_t + O_{L^1}\big(\varepsilon^{5/2}\big).$$

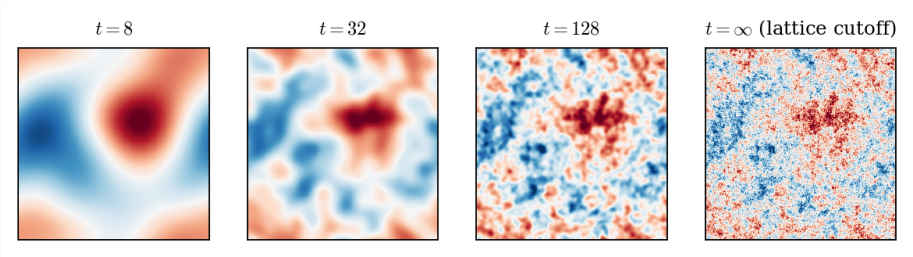
- ε-dependent coefficients (c_ε singular, $\varepsilon c_\varepsilon, \varepsilon^2 c_\varepsilon \to 0$) times observables $\Pi^y, \partial_y^2 \Pi^y, \ldots$: an operator product expansion.
- The order of limits matters: fixed coarse resolution t, then $\varepsilon \downarrow 0$.

The GFF as a martingale of smooth fields

Covariance profile: Q_t symmetric, $Q_0 = 0$, $\dot{Q}_t \geq 0$, $Q_t \to G = (m^2 - \Delta)^{-1}$. Heat profile $Q_t = G\,\mathrm{e}^{-L/t^2}$, $L = m^2 - \Delta$: t is a momentum scale. With W cylindrical Brownian motion,

$$X_t = \int_0^t \dot{Q}_s^{1/2}\,\mathrm{d}W_s, \qquad \mathrm{Law}(X_t) = \mu^{Q_t}, \qquad c_t = Q_t(x,x) \to \infty \ (d \geq 2).$$

$X_t(x)$ is a martingale at every point, although X_∞ is only a distribution.



$\mathbb{T}^2$, heat profile, $m = 4$: $t = 8, 32, 128$ and the lattice cutoff, same white noise.

Wick powers are observables in every dimension

$H_n(a;c)$ Hermite polynomials: $\partial_c H_n = -\frac{1}{2}\partial_a^2 H_n$, $\sum_n \frac{z^n}{n!} H_n(a;c) = \mathrm{e}^{za - z^2 c/2}$.

Proposition

For every d, n, x: $:X_t^n:(x) := H_n(X_t(x); c_t)$ is a martingale and

$$\mathbb{E}\big[:X_t^n:(x)\,:X_t^k:(y)\big] = \delta_{nk}\, n!\, Q_t(x,y)^n \;\longrightarrow\; \delta_{nk}\, n!\, G(x-y)^n \quad (x \neq y).$$

Proof. Itô: the drift is $(\frac{1}{2}\partial_a^2 + \partial_c) H_n\, \dot{c}_t = 0$. □

The counterterms are not arbitrary: they enforce the martingale property.

- $G(x-y) \asymp |x-y|^{2-d}$ ($d \geq 3$; logarithmic in $d = 2$): $:X_t^n:(f)$ converges in L^2 iff $n(d-2) < d$. Wick square: obstruction at $d = 4$; Wick cube: at $d = 3$.
- Beyond that, for $0 \leq f \not\equiv 0$, $:X_t^n:(f)$ is not tight: no terminal value, yet the observable is perfectly well defined: more observables than random distributions.

Multiplication is a partial operation

$O_t P_t$ need not be a martingale. Define $O \star P$ as the observable in the class of $(O_t P_t)_t$, when it exists.

Proposition

If O, P are continuous, square integrable on bounded intervals, and $\mathbb{E}\int_0^\infty |\mathrm{d}\langle O, P\rangle_s| < \infty$, then

$$(O \star P)_t = O_t P_t + \mathbb{E}_t \int_t^\infty \mathrm{d}\langle O, P\rangle_s.$$

GFF at $x \neq y$. $\langle X(x), X(y)\rangle_t = Q_t(x-y)$, so

$$(X(x) \star X(y))_t = X_t(x)X_t(y) + G(x-y) - Q_t(x-y), \qquad \langle X(x) \star X(y)\rangle = G(x-y),$$

the correct two-point function, without ever discussing the regularity of X_∞. At $x = y$ the bracket tail diverges.

The GFF OPE identifies the Wick square

$x_\pm = x \pm \rho e/2$, $|e| = 1$, $\partial = e \cdot \nabla$, $\mathcal{W}_t(U, V) = U_t V_t - \mathbb{E}[U_t V_t]$. At fixed resolution t, as $\rho \to 0$,

$$(X(x_-) \star X(x_+))_t = G(x_- - x_+)\,\mathbb{1} + {:}X_t^2{:}(x) + \rho^2\, C_{2,t}(x) + O_{L^1,t}(\rho^4),$$

$$C_{2,t} = \tfrac{1}{4}\{\mathcal{W}_t(X, \partial^2 X) - \mathcal{W}_t(\partial X, \partial X)\}.$$

Proof. Subtracting G leaves the bilocal Wick product $X_t(x_-)X_t(x_+) - Q_t(x_-, x_+)$, smooth in ρ; symmetry kills odd orders. □

More generally, for $x \neq y$,

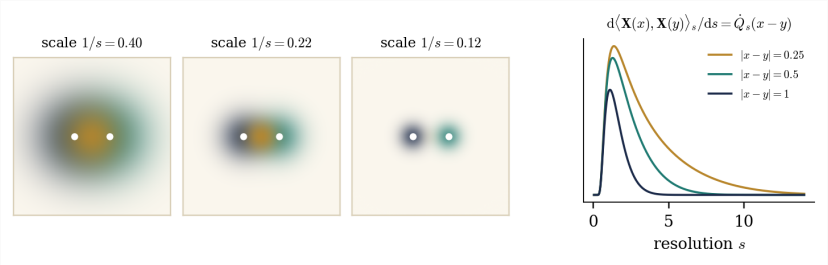
$${:}X^n{:}(x) \star {:}X^k{:}(y) = \sum_{j \le n \wedge k} j!\binom{n}{j}\binom{k}{j}\, G(x-y)^j\, {:}X^{n-j}(x) X^{k-j}(y){:}.$$

The coefficients carry the singularities; the observables are regular as $y \to x$.

Paulovics: systematic study of the OPE in this framework

Local observables and locality

A germ of O: functionals $\mathring{O}_t$ of smooth fields with $\|O_t - \mathring{O}_t(X_t)\|_{L^1} \to 0$; it determines O. $O(x)$ is local if it has germs $\mathring{\mathcal{O}}_t\big(X_t(x), \nabla X_t(x), \dots\big)$. Germs are not unique: this freedom is the choice of counterterms.



Kernels of $\dot{Q}_s^{1/2}$ at x, y: their overlap is $\dot{Q}_s(x-y) = \mathrm{d}\langle X(x), X(y)\rangle_s/\mathrm{d}s$, negligible for $s \gg 1/|x-y|$ (Gaussian tails; exactly zero for finite-range profiles). Separated supports multiply asymptotically freely; colliding supports call for OPEs and renormalised local operators.

Constructing interactions: dynamics in resolution

From a language to a construction programme

Specify an interacting field through its local observables:

$$\text{local germs} \longrightarrow \text{compatible martingales} \longrightarrow \text{correlations.}$$

Definition (Wilson–Itô diffusion; Bailleul–Chevyrev–G.)

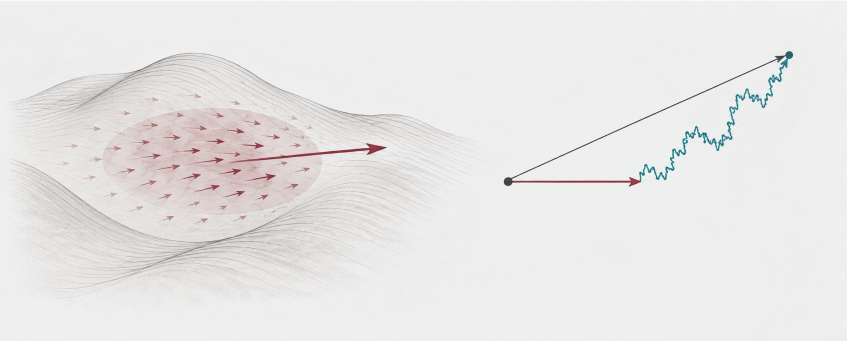
$(\varphi_a)_{a \geq a_0}$, a the resolution, $a_0 > 0$, solves

$$\mathrm{d}\varphi_a = A_a(f_a)\,\frac{\mathrm{d}a}{a} + A_a^{1/2}\sigma_a\,\frac{\mathrm{d}W_a}{\sqrt{a}}.$$

- **Itô:** the field is revealed by a diffusion in resolution.
- **Wilson:** A_a symmetric positive block averaging on scale a^{-1}, $A_a \to \mathrm{Id}$; force f and diffusivity σ^2 are local observable fields.

Trivial example: $f = 0$, $\sigma = 1$: fresh white innovations at every scale and unbounded variance; formally the flat "infinite-dimensional Lebesgue measure".

Local averaging, drift and innovation



$$\mathrm{d}\varphi_a = A_a(f_a)\,\frac{\mathrm{d}a}{a} + A_a^{1/2}\sigma_a\,\frac{\mathrm{d}W_a}{\sqrt{a}}$$

Predictable drift: local forces averaged over a neighbourhood of size a^{-1}.

Innovation: fresh noise, zero conditional mean.

Predictable means known from current information, not deterministic. Conceptual illustration.

The Gibbs sector: effective potentials and Polchinski

Gaussian reference with profile Q_t, a different parametrisation ($\dot{Q}_t$ is a covariance derivative, not the block average A_a). Finite volume, UV cutoff $T < \infty$, V_T bounded below (e.g. $\int \lambda{:}\varphi^4{:}_T + \gamma_T{:}\varphi^2{:}_T$), $\nu_T \propto \mathrm{e}^{-V_T}\mu^{Q_T}$, and with $Q_{T,t} = Q_T - Q_t$:

$$\mathrm{e}^{-V_t(\varphi)} = \int \mathrm{e}^{-V_T(\varphi+\zeta)}\, \mu^{Q_{T,t}}(\mathrm{d}\zeta).$$

Theorem (Polchinski 1984)

(finite dim.)

$$\partial_t V_t = \tfrac{1}{2}\langle DV_t, \dot{Q}_t DV_t\rangle - \tfrac{1}{2}\operatorname{Tr}\big(\dot{Q}_t D^2 V_t\big).$$

Nonlinear term: trees. Trace term: loops. $u = \mathrm{e}^{-V}$ solves a backward heat equation (Cole–Hopf).

Marginals. Under $\mathrm{d}\mathbb{P} \propto \mathrm{e}^{-V_T(X_T)}\mathrm{d}\mathbb{P}_0$: $\mathrm{Law}(X_t) \propto \mathrm{e}^{-V_t}\mu^{Q_t}$, and $\mathrm{Law}(X_T|\mathcal{F}_t) = \mu_t^{X_t}$ (the tilt of $\mu^{Q_{T,t}}(\cdot - X_t)$ by e^{-V_T}) goes from ν_T to δ_{X_T}: a stochastic localisation (Eldan) of ν_T.

The interacting field solves an FBSDE

Theorem (Polchinski process)

(finite dim.) Under $\mathbb{P}$, $\Phi_t := X_t$ *solves*

$$\mathrm{d}\Phi_t = \dot{Q}_t F_t \,\mathrm{d}t + \dot{Q}_t^{1/2} \mathrm{d}W_t^{\mathbb{P}}, \qquad F_t = -DV_t(\Phi_t) = \mathbb{E}_t^{\mathbb{P}}\big[-DV_T(\Phi_T)\big].$$

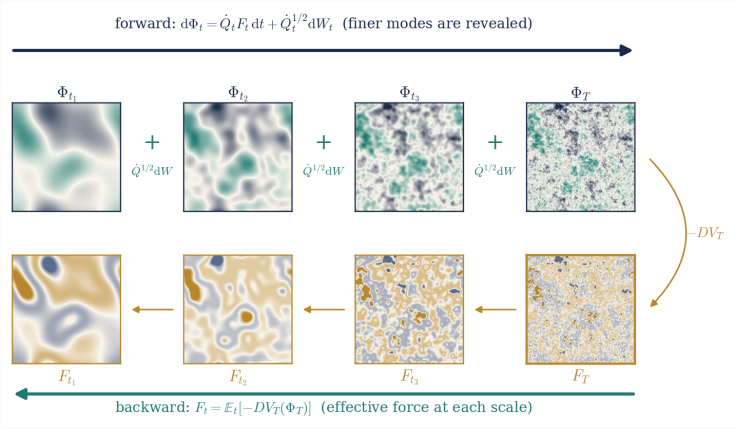
Proof. $M_t \propto \mathrm{e}^{-V_t(X_t)}$ is driftless by Polchinski; Girsanov. □

Read it as an equation for the pair (Φ, F):

- the field is driven forward by a force determined backward from the terminal field;
- the force is an observable; the field coordinate Φ is not (a semimartingale);
- no Polchinski equation needed: $\nu_T = \mathrm{Law}(\Phi_T)$.

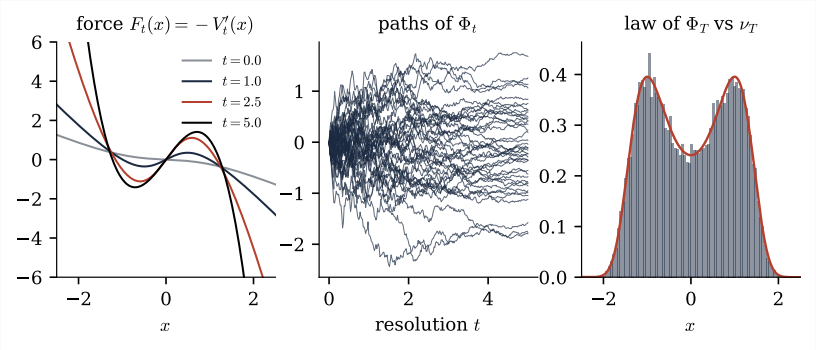
Strategy. Renormalised V_T, estimates uniform in T and in the volume, limits.

Forward field, backward force



$\mathbb{T}^2$, heat profile, same noise. The force is shown to first order in λ: $F_t = -4\lambda H_3(\Phi_t; c_t)$. The backward arrows are conditional expectations, not time reversal.

Zero dimensions



$Q_t = 1 - \mathrm{e}^{-t}$, $T = 5$, $V_T(x) = \frac{1}{2}x^4 - \frac{3}{2}x^2$. Left: $F_t = -V_t'$, almost linear at low resolution, the cubic force of the double well as $t \to T$. Centre: FBSDE paths split between the wells. Right: histogram of Φ_T vs. the density of ν_T.

Closed versus local

At finite T the force is closed, $F_t = \mathbb{E}_t F_T$, and local:

$$F_T(x) = -4\lambda H_3(\Phi_T(x); c_T) - 2\gamma_T H_1(\Phi_T(x); c_T).$$

As $T \to \infty$

F should converge as an observable. In $d = 2$ it is closed, $F_\infty = -4\lambda{:}\Phi^3_\infty{:}$. In $d = 3$ a closed limit $F_t = \mathbb{E}_t F_\infty$ would let the DS identity pass to $t \to \infty$, and conditioning on $\sigma(\Phi_\infty)$ would give an integrable logarithmic derivative of Φ^4_3 as a function of the field: not expected (Albeverio–Liang–Zegarlinski 2006: the regularised ones do not converge in L^2).

- Locality of the limiting force is the crux of the construction.
- The general WIDE problem: replace the terminal condition by asymptotic local-germ data (coherent germs; backup).

The backward half: forecasts, Wetterich, variational structure

Removing the predictable drift: the forecast

$S_t = Q_T - Q_t$, the covariance still unresolved. Since F is a martingale, $\mathbb{E}_t \int_t^T \dot{Q}_s F_s \, \mathrm{d}s = S_t F_t$ (repair by the future drift):

$$\psi_t := \mathbb{E}_t[\Phi_T] = \Phi_t + S_t F_t, \qquad \mathrm{d}\psi_t = \big(\mathbb{1} + S_t D F_t(\Phi_t)\big)\dot{Q}_t^{1/2}\mathrm{d}W_t.$$

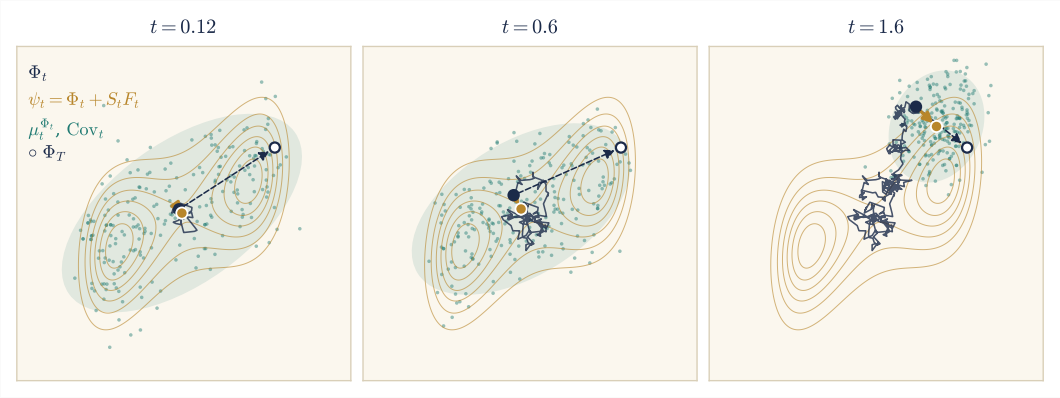
- (Φ, F) is state and costate of a control problem; the noiseless flow is the linear shear $\Sigma_t : (x, p) \mapsto (x + S_t p, p)$, and $(\psi_t, F_t) = \Sigma_t(\Phi_t, F_t)$.
- The forward–backward coupling moves from the drift to the volatility.

Proposition (Conditional moments)

(finite dim.) $F = -DV_t$: $\mathrm{Cov}_t := \mathrm{Cov}(\Phi_T | \mathcal{F}_t) = S_t - S_t D^2 V_t(\Phi_t) S_t > 0$,
$\mathrm{d}\langle \psi, \psi \rangle_t = \mathrm{Cov}_t \, \dot{\mathcal{R}}_t \, \mathrm{Cov}_t \, \mathrm{d}t$, $\mathcal{R}_t := S_t^{-1} - Q_T^{-1}$.

The forecast coordinates are the first two moments of the stochastic localisation.

Stochastic localisation: a picture



Two-component field, V_T a tilted double well (contours: ν_T). Path Φ_t, forecast $\psi_t = \Phi_t + S_t F_t$, conditional law $\mu_t^{\Phi_t}$ and 2σ ellipse of Cov_t; open circle: Φ_T.

The Wetterich equation is the backward half of the FBSDE

Gradient sector: the shear maps the graph of $F_t = -DV_t$ to the graph of $-D\Gamma_t$ over ψ; V_t is the Moreau transform of Γ_t.

Theorem (Wetterich 1993; Morris 1994)

(finite dim.) For $\Gamma_t^{\mathrm{EAA}} = \Gamma_t + \frac{1}{2}\langle \psi, Q_T^{-1}\psi\rangle$,

$$\partial_t \Gamma_t^{\mathrm{EAA}} = \tfrac{1}{2}\,\mathrm{Tr}\Big[\dot{\mathcal{R}}_t \big(D^2\Gamma_t^{\mathrm{EAA}} + \mathcal{R}_t\big)^{-1}\Big] + \mathit{const}, \qquad \Gamma_T = V_T.$$

Wetterich	FBSDE
field argument ψ	forecast $\mathbb{E}_t[\Phi_T]$
$D\Gamma_t(\psi_t)$	minus the force, $\mathbb{E}_t[DV_T(\Phi_T)]$
propagator $(D^2\Gamma_t^{\mathrm{EAA}} + \mathcal{R}_t)^{-1}$	conditional covariance Cov_t
regulator $\mathcal{R}_t = S_t^{-1} - Q_T^{-1}$	precision of the unresolved shell minus the prior's

In forecast coordinates the trees are integrated exactly; loops come from $\langle \psi, \psi\rangle$ only.

Variational structure

$X^u = X + \int_0^\cdot \dot{Q}_s^{1/2} u_s \, \mathrm{d}s$, u adapted with finite energy.

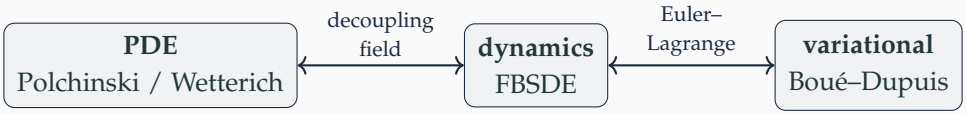
Theorem (Boué–Dupuis 1998; Polchinski = HJB)

(finite dim.)

$$-\log \mathbb{E}\left[\mathrm{e}^{-V_T(X_T)}\right] = \inf_u \mathbb{E}\left[V_T(X_T^u) + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s\right],$$

with optimal feedback $u_s^* = -\dot{Q}_s^{1/2} DV_s(X_s^{u^*})$ *and* $\mathrm{Law}(X_T^{u^*}) = \nu_T$.

Euler–Lagrange. Perturb $u \to u + \varepsilon v$, v adapted: $u_s = -\dot{Q}_s^{1/2}\mathbb{E}_s[DV_T(X_T^u)] = \dot{Q}_s^{1/2} F_s$: the FBSDE.



Φ_3^4: the leading control $u = -4\lambda \dot{Q}^{1/2} {:}X^3{:}$ has cost $\sim T$; paracontrolled changes of variables leave a coercive remainder. Barashkov–Gubinelli, Duke 2020

Dyson–Schwinger equations at every resolution

Finite dimensions: $\mathbb{E}_\nu[D_h\Theta] = \mathbb{E}_\nu\big[\Theta\,\langle K\varphi + DV(\varphi), h\rangle\big]$ for $\nu \propto \mathrm{e}^{-\frac{1}{2}\langle\varphi, K\varphi\rangle - V}$. E.g. 0 dimensions, $K = m^2$, $V = \lambda q^4$: $m^2\mathbb{E}q^{n+1} + 4\lambda\mathbb{E}q^{n+3} = n\mathbb{E}q^{n-1}$: exact, not closed.

Theorem

(finite dim.) $K = Q_T^{-1}$, $J_t = KQ_t$, *all operators functions of* L *(commuting),* $\Phi_0 = 0$. *The noise observable* $\eta_t = K\Phi_t - J_tF_t$ *is a centred martingale, and*

$$\mathbb{E}\big[D_{J_th}\Theta(\Phi_t)\big] = \mathbb{E}\big[\Theta(\Phi_t)\,\langle h, K\Phi_t - J_tF_t\rangle\big].$$

At $t = T$*: the DS equation of* ν_T.

Proof. $\dot J_t = K\dot Q_t$: the drifts cancel; IBP for μ^{Q_t} in the direction J_th ($t > 0$). □

As $T \to \infty$: $L\Phi_t = LQ_t\,F_t + \eta_t$, $L = m^2 - \Delta$: the field equation seen at resolution t. No products of distributions: the nonlinearity enters through the force observable. For Φ^4_3, $t \to \infty$ corresponds to the renormalised DS equation $L_xS_2(x,y) + 4\lambda\langle[\varphi^3]_{\mathrm{ren}}(x)\varphi(y)\rangle = \delta(x-y)$.

Gubinelli–Hofmanová 2021; De Vecchi–Gubinelli–Turra

Constructions and outlook

Rigorous constructions along these lines

Sine-Gordon (Gubinelli–Meyer). A scale FBSDE constructs the massive sine-Gordon field on $\mathbb{R}^2$ for $\beta^2 < 6\pi$, at small coupling: UV and volume cutoffs removed, local correlations, large deviations, and

$$|\mathrm{Cov}(O_{1,x}, O_{2,y})| \leq C_{O_1,O_2}\, \mathrm{e}^{-c|x-y|} \quad \text{(bounded Lipschitz local observables).}$$

Vertex observables $\mathrm{e}^{i\beta X_t + \beta^2 c_t/2}$ are martingales at every scale.

Φ^4_3 (Barashkov–Gubinelli). Variational problem with renormalised stochastic objects, coercive estimates, Γ-convergence; a scale-filtration Girsanov construction: a random shift of the GFF, then an absolutely continuous tilt. The measure is singular w.r.t. the GFF.

Infinite volume. Only intensive objects survive: the force, $F_t(x) = -\mathbb{E}_t[v_T'(\Phi_T(x))]$, $V_T = \int v_T(\varphi)$, its Burgers flow, and differences of free energies. Non-uniqueness of the FBSDE is expected to reflect phase transitions.

arXiv:2401.13648; Duke 2020; EJP 2021; PMP 2023

Dunlap–Gu: a forward–backward limit in the wild

2D nonlinear stochastic heat equation with weak noise,

$$\partial_t u_\varepsilon = \tfrac{1}{2}\Delta u_\varepsilon + \frac{\sigma(u_\varepsilon)}{\sqrt{\log \varepsilon^{-1}}}\,\xi^\varepsilon, \qquad \sigma \geq 0,\ \sigma(0) = 0,\ \mathrm{Lip}(\sigma) < \sqrt{2\pi}.$$

For constant initial data $x_0 \geq 0$, the one-point law at fixed $t > 0$ (heat-kernel mollified noise, fixed x) converges to the law of X_2, where

$$\mathrm{d}X_q = \sqrt{Y_q}\,\mathrm{d}B_q, \qquad Y_q = \frac{1}{4\pi}\,\mathbb{E}\big[\sigma^2(X_2) \,\big|\, \mathcal{G}_q\big], \qquad X_0 = x_0.$$

- The scale parameter $q \in [0, 2]$ is logarithmic; not physical time.
- The effective diffusivity is itself a martingale observable, determined by the terminal value of the diffusion it drives.
- Structural evidence for WIDE (not an identification).

Dunlap–Gu, Ann. Probab. 2022; Dunlap–Graham arXiv 2023, rev. 2026

Beyond scalar fields: transport before averaging

Sigma models $\Phi : \mathbb{R}^d \to M$. Force covectors at nearby points live in different fibres: raise and transport before averaging,

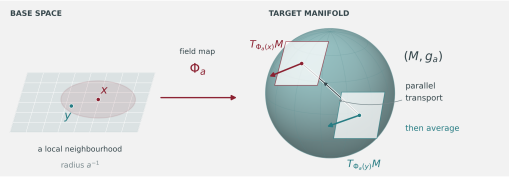
$$(\mathcal{K}_a\omega)(x) = \int K_a(x,y) P^{(a)}_{y\to x} g_a^{-1} \omega(y)\, \mathrm{d}y,$$

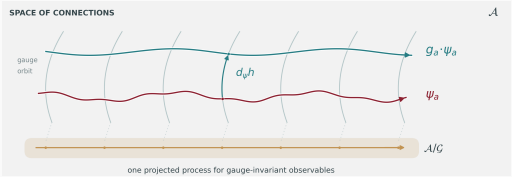
$A_a = \mathcal{K}_a\mathcal{K}_a^*$. Sphere: $\tau(n) = \Delta n + |\nabla n|^2 n$; the metric runs (Ricci flow at leading order, $d = 2$).

Gauge fields. Average in a common frame,

$$(\mathcal{K}_a\omega)(x) = \int \chi_a(x,y)\, \mathrm{Ad}_{h^{xy}(\psi)}\, \omega(y)\, \mathrm{d}y,$$

$A_a = \mathcal{K}_a\mathcal{K}_a^*$, gauge covariant. A drift $\mathrm{d}_\psi h$ tangent to the orbits is removed by an adapted gauge transformation.





Schematic. The field changes the force, the averaging and hence the noise. A programme, not a construction (no 4D Yang–Mills claimed).

What remains a programme

- **Locality of the limiting force**: the heart of every construction.
- **Coherent germs**: replace the terminal condition by asymptotic local data; a forward–backward system for the residual drift. Open in general.
- **Non-gradient forces**: flows for different profiles no longer commute (curvature); Wilson–Itô beyond the Gibbs sector.
- **Geometry**: covariant martingale compatibility for sigma models and gauge fields; Grassmann fields.
- **Analysis**: FBSDEs are hard, but PDE methods apply, as in Parisi–Wu stochastic quantisation. Numerics?

The object is the compatible family

- **Stochastic observables enlarge random variables**: finite resolution information, no terminal representative needed.
- **Renormalisation resolves products into observables**: Brownian deltas and GFF Wick powers, explicit OPEs at fixed resolution.
- **The interacting field solves an FBSDE**: forward field, backward force; the force is an observable.
- **One object, three faces**: Polchinski / Wetterich $\leftrightarrow$ FBSDE $\leftrightarrow$ Boué–Dupuis; DS equations at every resolution.

The ambition

Construct a field theory together with the observables that make it meaningful.

Collaborators



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Ilya Chevyrev



Sarah-Jean Meyer



Peter Paulovics



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Thank you.

Backup

Backup: the product structure

$O \star P$ is defined when $O_t P_t$ is asymptotically (L^1) a martingale.

- **Classical part.** Closed observables $O_t = \mathbb{E}_t Y$, $P_t = \mathbb{E}_t Z$ with $Y, Z \in L^2$: $(O \star P)_t = \mathbb{E}_t[YZ]$. The algebra of random variables sits inside.
- **Algebra.** $\star$ is commutative, with unit $\mathbb{1}$; associative whenever both sides are in the class of $(O_t P_t R_t)_t$.
- **Correlations are expectations of products.** For distinct points, $\langle X(x_1) \star \cdots \star X(x_n) \rangle = \sum_{\text{pairings}} \prod_{\{i,j\}} G(x_i - x_j)$.
- **The domain is geometric.** It is controlled by $\mathbb{E} \int^{\infty} |\mathrm{d}\langle O, P \rangle|$, i.e. by $\dot{Q}$ between the supports.

separated supports
product exists,
$\approx$ pointwise product

→

colliding supports
no product; OPE: singular coefficients $\times$ observables

→

renormalised products
regular part of the OPE,
e.g. $:X^2:$

A commutative partial algebra with a state $\langle \cdot \rangle$, suggesting a factorisation-algebra structure (precise axioms not established).

Backup: the force solves a Burgers equation

Differentiating Polchinski, $F_t = -DV_t$ solves

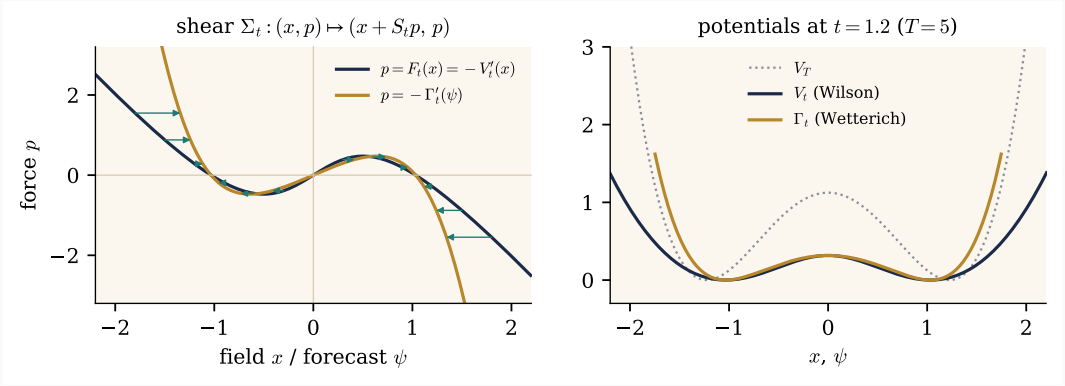
$$\partial_t F_t = -DF_t[\dot{Q}_t F_t] - \tfrac{1}{2}\operatorname{Tr}\big(\dot{Q}_t D^2 F_t\big), \qquad F_T = -DV_T.$$

- Intensive (unlike V_t); Cole–Hopf $F = D\log u$.
- Transport term ← drift of the state: trees. Itô term ← noise: loops.
- In forecast coordinates, $\mathcal{F}_t(\psi) = -F_t(x)$, $x = \psi + S_t\mathcal{F}_t(\psi)$, the transport term disappears:

$$\partial_t \mathcal{F}_t + \tfrac{1}{2}\operatorname{Tr}\big(\mathcal{C}_t D^2 \mathcal{F}_t\big) = 0, \qquad \mathcal{C}_t = H_t \dot{\mathcal{R}}_t H_t^*, \quad H_t = \big(S_t^{-1} + D\mathcal{F}_t\big)^{-1}.$$

- Flows for different covariances commute when DF is symmetric (gradient case); generically not otherwise.

Backup: the shear in zero dimensions



Left: Σ_t maps the graph of $F_t = -V_t'$ to the graph of $-\Gamma_t'$ over the forecast. Right: V_T, V_t and Γ_t ($t = 1.2$, $T = 5$). $V_t(x) = \inf_\psi\{\Gamma_t(\psi) + \frac{1}{2}\langle x - \psi, S_t^{-1}(x - \psi)\rangle\}$.

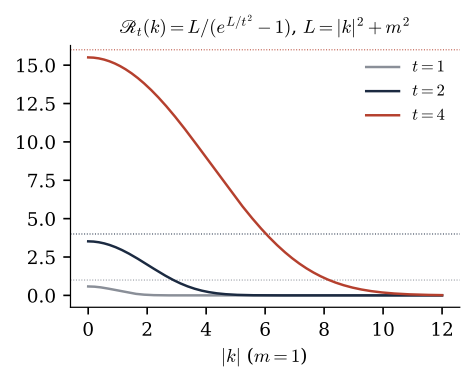
Backup: the regulator of the heat profile

For $T = \infty$, $S_t = G(1 - \mathrm{e}^{-L/t^2})$, $L = m^2 - \Delta$, and

$$\mathcal{R}_t = \frac{L}{\mathrm{e}^{L/t^2} - 1}.$$

- $\approx t^2$ (a mass term) on the modes $|k|^2 + m^2 \ll t^2$;
- exponentially small on $|k| \gg t$.

The exponential regulator of the functional RG.



Backup: perturbation theory and counterterms

Gaussian convolution lowers the Wick ordering. At second order, $S = Q_{T,t}$:

$$V_t = \int \left[\lambda{:}\varphi^4{:}_t + \gamma_T{:}\varphi^2{:}_t\right] - \frac{\lambda^2}{2}\sum_{k=1}^{4}\binom{4}{k}^2 k! \iint S(x,y)^k\, {:}\varphi^{4-k}{:}_t(x)\, {:}\varphi^{4-k}{:}_t(y) + O(\lambda^3).$$

$d = 3$, $T \to \infty$ at fixed t:

- $k = 1, 2$: finite, nonlocal;
- $k = 3$ (sunset): local divergence $-48\lambda^2(\int S^3)\int {:}\varphi^2{:}_t$ (+ finite nonlocal + const), cancelled by $\gamma_T = 48\lambda^2 \int Q_T^3 = \frac{3\lambda^2}{\pi^2}\log T + O(1)$;
- $k = 4$: vacuum energy, cancels in ν_T.

Renormalisation = choosing V_T so that V_t at fixed resolution converges.

Backup: coherent germs lead to a forward–backward problem

Guess the drift through a germ, $f_a = F_a(\varphi_a) + R_a$ ($\sigma = 1$). With $\mathcal{L}_b = b\,\partial_b + \frac{1}{2}\operatorname{Tr}(A_b D^2) + \langle A_b F_b, D\rangle$,

$$\begin{cases} \mathrm{d}\varphi_a = A_a\big(F_a(\varphi_a) + R_a\big)\dfrac{\mathrm{d}a}{a} + A_a^{1/2}\dfrac{\mathrm{d}W_a}{\sqrt{a}}, \\ R_a = \mathbb{E}_a\Big[\displaystyle\int_a^\infty \mathcal{L}_b F_b(\varphi_b)\frac{\mathrm{d}b}{b} + \int_a^\infty DF_b(\varphi_b)A_b R_b \frac{\mathrm{d}b}{b}\Big]. \end{cases}$$

- The repair formula of the first part, applied to the force.
- A small residual drift with integrable scale tail is what makes the reconstruction useful.
- The operator depends on the unknown coefficients: not an explicit formula. Open in general.

Backup: the Dyson–Schwinger martingale

$\mathcal{L}_t = \langle \dot{Q}_t F_t, D\rangle + \frac{1}{2}\operatorname{Tr}(\dot{Q}_t D^2)$; assume $(\partial_t + \mathcal{L}_t)F_t = 0$, $(\partial_t + \mathcal{L}_t)\Theta_t = 0$, $DF_t = (DF_t)^*$, K, Q_t commuting, $Q_0 = 0$, $\Phi_0 = 0$. Then

$$\mathrm{d}\eta_t = (K - J_t DF_t)\dot{Q}_t^{1/2}\,\mathrm{d}W_t.$$

$D_{J_t h}\Theta_t(\Phi_t)$ and $\Theta_t(\Phi_t)\,\eta_t(h)$ have the same drift

$$k_t = D_{\dot{J}_t h}\Theta_t - D\Theta_t\,\dot{Q}_t\,DF_t\,J_t h,$$

since $J_t = J_t^*$, $\dot{Q}_t K = \dot{J}_t$, $DF_t = (DF_t)^*$. Hence

$$\mathcal{M}_t = D_{J_t h}\Theta_t(\Phi_t) - \Theta_t(\Phi_t)\,\eta_t(h)$$

is a local martingale with $\mathcal{M}_0 = 0$, a true one under integrability. For non-symmetric DF a transpose remains and the cancellation fails.