The infinitesimal generator of the stochastic Burgers equation

(handout version)

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Motivation: Study the martingale problem of certain singular SPDEs.

Outline

- Our example: Burgers' equation in $d=1$.
- Singular SPDEs: difficulties with the definition of the dynamics.
- Existence of solutions to the cylinder martingale problem.
- Uniqueness: study of the Kolmogorov backward equation.
- Domain? The idea of controlled functions.
- Cauchy problem and apriori estimates.

☞ **The infinitesimal generator of the stochastic Burgers equation** | joint work with N. Perkowski. PTRF 178, 2020.

☞ **Hyperviscous stochastic Navier–Stokes equations with white noise invariant measure** | joint work with M. Turra. Stoch. & Dyn. 20, 2020.

☞ Lukas Gräfner, **Fractional stochastic quasi-geostrophic equations on the two-dimensional torus**, Master thesis Humboldt-Universität zu Berlin, 2021.

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Goal: probabilistic well–posedness for (almost) stationary solutions to

$$\partial_t u(t,x) = \Delta u(t,x) + \partial_x(u(t,x)^2) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, \mathbb{R}, \quad t \geqslant 0$$

$u(0) \sim \mu$ and μ white noise on $\mathbb{T}$ with zero mean, ξ space–time white noise.

$$\mathbb{E}[u(t,x)u(t,y)] \approx \delta(x-y)$$

Singular equations, related to KPZ ($h = \partial_x u$), well-posedness via rough paths, regularity structures or paracontrolled distributions.

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- (Stroock–Vadadhan) Characterisation of the diffusion u by requiring that for a "large" class of functions φ

$$\varphi(t,u(t)) = \varphi(0,u(0)) + \int_0^t (\partial_s + \mathscr{L})\varphi(s,u(s))\mathrm{d}s + M^\varphi(t)$$

with M^φ a martingale. $\mathscr{L}$ is called the generator, usually unbounded $(\mathscr{L}, D(\mathscr{L}))$.

$$\mathscr{L}\varphi(u) = \underbrace{\int \partial_x^2 u(x) \mathrm{D}_x\varphi(u)\mathrm{d}x + \frac{1}{2}\mathrm{Tr}[\partial_x \otimes \partial_x \mathrm{D}^2\varphi(u)]}_{\mathscr{L}_0 \text{ linear part}} + \underbrace{\int (\partial_x u(x)^2)\mathrm{D}_x\varphi(u)\mathrm{d}x}_{\mathscr{G} \text{ non-linear drift}}$$

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Cylinder functions

$$\varphi(u) = \Phi(u(f_1), \dots, u(f_n)) \quad \Rightarrow \quad D_x\varphi(u) = \sum_{k=1}^{n} \partial_k \Phi(u(f_1), \dots, u(f_n)) f_k(x)$$

Linear part (OU operator)

$$\mathscr{L}_0 \varphi(u) = \sum_{k=1}^{n} \partial_k \Phi(u(f_1), \dots, u(f_n)) u(\Delta f_k) + \frac{1}{2} \sum_{k,\ell=1}^{n} \partial_k \partial_\ell \Phi(u(f_1), \dots, u(f_n)) \langle \partial_x f_k, \partial_x f_\ell \rangle$$

Number operator (commutes with $\mathscr{L}_0$, both diagonalized via chaos expansion)

$$\mathscr{N} \varphi(u) = \sum_{k=1}^{n} \partial_k \Phi(u(f_1), \dots, u(f_n)) u(f_k) + \frac{1}{2} \sum_{k,\ell=1}^{n} \partial_k \partial_\ell \Phi(u(f_1), \dots, u(f_n)) \langle f_k, f_\ell \rangle$$

singular drift

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$$\mathscr{G}\varphi(u) = -\sum_{k=1}^{n} \partial_k \Phi(u(f_1), \ldots, u(f_n)) \int u(x)^2 \partial_x f_k(x) \mathrm{d}x$$

- Problem: $u^2(\partial_x f)$ is not a well-defined random variable – not even tested with $\partial_x f$.

$$\mathbb{E}[u^2(f)u^2(f)] \text{ ''='' } \int \delta(x-y)^2 f(x) f(y) \mathrm{d}x\mathrm{d}y \quad ?????$$

Indeed, it is a "distribution" on $L^2(\mu)$:

$$(1-\mathscr{L}_0)^{-1/2}\mathscr{G}\varphi \in L^2(\mu).$$

☞ *diffusion with singular drift & regularisation by noise*

[Assing ('03) (pre-generator), Flandoli-Russo-Wolf ('03), Delarue-Diel ('16), Allez-Chouk, Cannizzaro-Chouk]

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- By Galerkin approximation we can construct a stationary process $(u_t^m)_t$ such that

$$\varphi(u^m(t)) = \varphi(u^m(0)) + \int_0^t \mathscr{L}^m \varphi(u^m(s))\mathrm{d}s + M_t^{m,\varphi}$$

- Compactness by energy solution methods [Gonçalves–Jara] [Gubinelli–Jara].

Ito trick:

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \varphi(u^m(s))\mathrm{d}s \right|^p \lesssim (T^{p/2} \vee T^p) \| c_{2p}^{\mathscr{N}} (1 - \mathscr{L}_0)^{-1/2} \varphi \|_{L^2(\mu)}^p.$$

- Any limit point $(u_t)_{t\geqslant 0}$ is a solution to the cylinder martingale problem.

cylinder martingale problem for $\mathscr{L}$

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$(u_t)_{t \geqslant 0}$ is a solution to the cylinder martingale problem for $\mathscr{L}$ iff

1. $(u_t)_{t \geqslant 0}$ is incompressible, i.e. for all T

$$\sup_{t \in [0,T]} \mathbb{E}[|\varphi(u_t)|] \leqslant C_T \|\varphi\|_{L^2(\mu)};$$

2. Satifies the Ito trick;

3. For any φ cylindrical

$$M_t^{\varphi} = \varphi(u_t) - \varphi(u_0) - \int_0^t \mathscr{L}\varphi(u_s) \mathrm{d}s$$

   is a continuous martingale with quadratic variation $\langle M^{\varphi} \rangle_t = \int_0^t \mathscr{E}(\varphi)(u_s) \mathrm{d}s$ with $\mathscr{E}(\varphi)(u) = \int |\partial_x D_x \varphi(u)|^2 \mathrm{d}x$.

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- Duality with the backward equation

$$\mathbb{E}[\varphi(u_t)\psi(u_s)] = \mathbb{E}\Big[\Big(\varphi(t-s,u_t) + \int_s^t (\partial_r + \mathscr{L})\varphi(t-r,u_r)\mathrm{d}r\Big)\psi(u_s)\Big] = \mathbb{E}[\varphi(t-s,u_s)\psi(u_s)]$$

Need: classical solutions $\varphi \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{L})) \cap C^1(\mathbb{R}_+, L^2(\mu))$ of the Cauchy problem

$$\partial_t \varphi = \mathscr{L}\varphi = \mathscr{L}_0\varphi + \mathscr{G}\varphi$$

Problem: what is a suitable domain $\mathscr{D}(\mathscr{L})$ of $\mathscr{L}$??

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- Idea: use $\mathscr{L}_0\varphi$ to compensate for $\mathscr{G}\varphi$: we look for "controlled" φ such that

$$\mathscr{L}_0\varphi \approx -\mathscr{G}\varphi$$

We don't need to be greedy.

$$\mathscr{G}^{\succ} := \mathbb{1}_{|\mathscr{L}_0| \geqslant L\mathscr{N}^{\alpha}}\mathscr{G}, \qquad \mathscr{G}^{\prec} = \mathscr{G} - \mathscr{G}^{\succ}$$

$\mathscr{G}^{\succ}$ models the large momentum behaviour of $\mathscr{G}$. L is a cutoff to be chosen later.

$$\varphi = -\mathscr{L}_0^{-1}\mathscr{G}^{\succ}\varphi + \varphi^{\#}, \qquad \varphi = \mathscr{K}\varphi^{\#}$$

$$\mathscr{L}\varphi = \mathscr{L}_0\varphi + \mathscr{G}\varphi = \mathscr{L}_0\varphi^{\#} + \mathscr{G}^{\prec}\varphi$$

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- For $\gamma \in (1/4, 1/2]$

$$\| w(\mathscr{N})(-\mathscr{L}_0)^{\gamma-1}\mathscr{G}^{\succ}\varphi \| \lesssim \varepsilon |w| \, \|(-\mathscr{L}_0)^{\gamma} w(\mathscr{N})\varphi\|$$

$$\|(-\mathscr{L}_0)^{\gamma} w(\mathscr{N})\mathscr{K}\varphi^{\#}\| + (|w|\varepsilon)^{-1}\|(-\mathscr{L}_0)^{\gamma} w(\mathscr{N})(\mathscr{K}\varphi^{\#} - \varphi^{\#})\| \lesssim \|(-\mathscr{L}_0)^{\gamma} w(\mathscr{N})\varphi^{\#}\|$$

- For all $\gamma \geqslant 0$ $\delta > 0$

$$\| w(\mathscr{N})(-\mathscr{L}_0)^{\gamma}\mathscr{G}^{\prec}\varphi \| \lesssim \| w(\mathscr{N})(1+\mathscr{N})^{9/2+7\gamma}(-\mathscr{L}_0)^{1/4+\delta}\varphi^{\#}\|$$

so $\mathscr{L}\varphi = \mathscr{L}_0\varphi^{\#} + \mathscr{G}^{\prec}\varphi$ is well defined for controlled functions.

$$\mathscr{D}_w(\mathscr{L}) = \{\varphi = \mathscr{K}\varphi^{\#}: \| w(\mathscr{N})(-\mathscr{L}_0)\varphi^{\#}\| + \| w(\mathscr{N})(1+\mathscr{N})^{9/2}(-\mathscr{L}_0)^{1/2}\varphi^{\#}\|\}$$

is *dense* in $w(\mathscr{N})^{-1}\Gamma H$ and $\mathscr{D}(\mathscr{L}) = \mathscr{D}_1(\mathscr{L})$

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- Densely defined operator

$$(\mathscr{L}, \mathscr{D}(\mathscr{L}))$$

- For $\mathscr{L}^{(\lambda)} = \mathscr{L}_0 + \lambda \mathscr{G}$ with $\lambda \in \mathbb{R}$ similar construction: $\mathscr{D}(\mathscr{L}^{(\lambda)}) \cap \mathscr{D}(\mathscr{L}^{(\lambda')}) = \{\text{constants}\} \ldots$

- $\mathscr{L}$ is dissipative

$$\langle \varphi, \mathscr{L}\varphi \rangle = -\|(-\mathscr{L}_0)^{1/2}\varphi\|^2 \leqslant 0, \qquad \varphi \in \mathscr{D}(\mathscr{L})$$

$$\langle \psi, \mathscr{L}\varphi \rangle = \langle \mathscr{L}^{(-1)}\psi, \varphi \rangle, \qquad \varphi \in \mathscr{D}(\mathscr{L}), \psi \in \mathscr{D}(\mathscr{L}^{(-1)})$$

[ ]

- $\mathscr{L}^m$ Galerkin approximation for $\mathscr{L}$, $(T_t^m)_t$ Markov semigroup

$$\partial_t \varphi^m(t) = \mathscr{L}^m \varphi^m(t)$$

- To pass to the limit we need to control the growth of solutions in weighted spaces

$$\frac{1}{2}\partial_t \| w(\mathscr{N})\varphi^m(t)\|^2 + \| w(\mathscr{N})(-\mathscr{L}_0)^{1/2}\varphi^m(t)\|^2 = \langle \varphi^m(t), w(\mathscr{N})^2 \mathscr{G}^m \varphi^m(t)\rangle$$

- We have for $\gamma > 1/4$ and uniformly in m

$$\| w(\mathscr{N})(-\mathscr{L}_0)^{-\gamma}\mathscr{G}_+^m \psi\| \lesssim \| w(\mathscr{N})\mathscr{N}(-\mathscr{L}_0)^{3/4-\gamma}\psi\| \qquad \textit{(roughly)}$$

$$\langle \varphi^m(t), w(\mathcal{N})^2 \mathcal{G}^m \varphi^m(t) \rangle = \langle \varphi^m(t), w(\mathcal{N})^2 (\mathcal{G}^m_+ + \mathcal{G}^m_-) \varphi^m(t) \rangle$$

$$= \langle \varphi^m(t), w(\mathcal{N})^2 \mathcal{G}^m_+ \varphi^m(t) \rangle + \langle \varphi^m(t), \mathcal{G}^m_- w(\mathcal{N}+1)^2 \varphi^m(t) \rangle$$

$$= \langle \varphi^m(t), [w(\mathcal{N})^2 - w(\mathcal{N}+1)^2] \mathcal{G}^m_+ \varphi^m(t) \rangle \approx \left\langle \varphi^m(t), w(\mathcal{N}) \underbrace{w'(\mathcal{N})}_{\approx w(\mathcal{N})\mathcal{N}^{-1}} \mathcal{G}^m_+ \varphi^m(t) \right\rangle$$

$$\lesssim \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t) \|^2 + c_\delta \| w(\mathcal{N})(-\mathcal{L}_0)^{-1/2} \mathcal{N}^{-1} \mathcal{G}^m_+ \varphi^m(t) \|^2$$

$$\lesssim \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t) \|^2 + c_\delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/4} \varphi^m(t) \|^2$$

$$\boxed{\frac{1}{2} \partial_t \| w(\mathcal{N}) \varphi^m(t) \|^2 + \delta \| w(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(t) \|^2 \lesssim_\delta \| w(\mathcal{N}) \varphi^m(t) \|^2}$$

[ ]

- To pass to the limit in the Kolmogorov equation we need further regularity to put $\lim_m \varphi^m$ in the domain of $\mathscr{L}$. We need control of

$$\varphi^{m,\#}(t) = \varphi^m(t) + \mathscr{L}_0^{-1} \mathscr{G}^{m,\succ} \varphi^m(t)$$

- The equation for $\varphi^{m,\#}$ gives the required apriori estimates

$$\partial_t \varphi^{m,\#}(t) = \mathscr{L}^m \varphi^m(t) + \mathscr{L}_0^{-1} \mathscr{G}^{m,\succ} \partial_t \varphi^m(t) = \mathscr{L}_0 \varphi^{m,\#}(t) + \mathscr{G}^{m,\prec} \varphi^m(t) + \mathscr{L}_0^{-1} \mathscr{G}^{m,\succ} \partial_t \varphi^m(t)$$

For $\gamma \in (3/8, 5/8)$, exists $p(\alpha)$ s.t.

$$\| (1+\mathscr{N})^{\alpha} (-\mathscr{L}_0)^{1+\gamma} \varphi^{m,\#}(t) \| + \| (1+\mathscr{N})^{\alpha} (-\mathscr{L}_0)^{\gamma} \partial_t \varphi^{m,\#}(t) \|$$

$$\lesssim \| (1+\mathscr{N})^{p(\alpha)} (-\mathscr{L}_0)^{1+\gamma} \varphi^{m,\#}(0) \|$$

Kolmogorov backward equation

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- Given

$$\|(1+\mathcal{N})^{p(\alpha)}(-\mathscr{L}_0)^{1+\gamma}\varphi(0)\| < \infty$$

with $\alpha > 9/2$ and $\gamma \in (3/8, 5/8)$ then

$$\partial_t \varphi(t) = \mathscr{L}\varphi(t)$$

has a solution

$$\varphi \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{L})) \cap C^1(\mathbb{R}_+, \Gamma H)$$

- Unique by dissipativity but we cannot define flow $e^{t\mathscr{L}}$.

(However see Gräfner for improved strategy)

[ ]

- Multi-component Burgers eq. [Funaki-Hoshino '17, Kupiainen-Marcozzi '17]

$$\partial_t u^i = \Delta u^i + \sum_{j,k} \Gamma^i_{jk} \partial_x (u^j u^k) + \partial_x \xi^i$$

under "trilinear condition" [Funaki-Hoshino '17]: $\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^j_{ki}$.

- Fractional Burgers eq. [G.-Jara '13]

$$\partial_t u = -(-\Delta)^\theta u + \partial_x u^2 + (-\Delta)^{\theta/2} \xi$$

for $\theta > 3/4$; note that $\theta = 3/4$ is critical, ∞ expansion in reg. str.!

- 2d NS with small hyperdissipation and energy invariant measure (G., Turra) $\kappa > 0$

$$\partial_t u = -(-\Delta)^{1+\kappa} u + u \cdot \nabla u + (-\Delta)^{(1+\kappa)/2} \xi, \qquad u: \mathbb{T}^2 \to \mathbb{R}^2.$$

- Weak universality for fractional Burgers [Sethuraman '16, Gonçalves-Jara '18] and multi-component Burgers [Bernardin-Funaki-Sethuraman '19+]

Outlook

- Probabilistic theory for singular SPDEs $\leftrightarrow$ ∞-dim singular operator $\mathscr{L} = \mathscr{L}_0 + \mathscr{G}$.
- Existence for martingale problem via Galerkin approximation.
- Construct $\mathscr{D}(\mathscr{L})$ via ideas from paracontrolled distributions.
- Existence for backward equation $\partial_t \varphi = \mathscr{L} \varphi$ via energy estimates.
- Duality gives uniqueness for martingale prob. and backward eq.
- (multi-component, fractional) Burgers, down to criticality.

Open problems

- Need Gaussian measure. beyond: unclear.