Paracontrolled distributions with applications to singular SPDEs

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Some problems in singular SPDEs /I

Define and solve (locally) the following SPDEs:

- Stochastic differential equations (1+0): $u \in [0,T] \to \mathbb{R}^n$

$$\partial_t u(t) = \sum_i f_i(u(t))\xi^i(t)$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.
- Burgers equations (1+1): $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))Du(t,x) + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

Recall that

$$\xi \in \mathscr{C}^{-d/2-}$$

Some problems in singular SPDEs /II

- Generalized Parabolic Anderson model (1+2):
  $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$
  $$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(x)$$
  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.
- Kardar-Parisi-Zhang equation (1+1)
  $$\partial_t h(t,x) = \Delta h(t,x) + \text{"}(Du(t,x))^2 - \infty\text{"} + \xi(t,x)$$
  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Some problems in singular SPDEs /III

Define and solve (locally) the following SPDEs:

- Stochastic quantization equation (1+3)

$$\partial_t u(t,x) = \Delta u(t,x) + \text{"}u(t,x)^3\text{"} + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.

- But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Joint work with P. Imkeller and N. Perkowski.
(Also K. Chouk and R. Catellier for $(\Phi)^4_3$).

Rough differential equation

Consider the simple controlled ODE (η smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^{m} f_i(u(t))\eta^i(t)$$

$u : \mathbb{R} \to \mathbb{R}^d$, $\eta : \mathbb{R} \to \mathbb{R}^d$ and smooth vectorfields $f_i : \mathbb{R}^d \to \mathbb{R}^d$.

Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for $\eta \in \mathscr{C}^{\gamma-1}$ with $\gamma < 1/2$.

Reason: $u \in \mathscr{C}^{\gamma}$ and $\eta \in \mathscr{C}^{\gamma-1}$ cannot be multiplied when $2\gamma - 1 \leqslant 0$. The r.h.s. of the equation is not well defined.

Here $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ is the Holder–Besov space (or a local version).

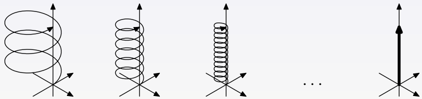
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $\mathscr{C}^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2} \neq I(0,0)(t) = 0$$



The definite integral $I(\cdot,\cdot)(t)$ is **not** a continuous map $\mathscr{C}^\gamma \times \mathscr{C}^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Concept of solution

Goal: Show that Ψ factorizes as

$$\eta \xrightarrow{J} J(\eta) \xrightarrow{\Phi} u$$

▷ *Analytic step:* show that when $\gamma > 1/3$:

$$\Phi : \mathcal{X} \to \mathscr{C}^{\gamma}$$

is continuous. $\mathcal{X} = \overline{\mathrm{Im} J} \subseteq \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ is the space of *enhanced signals* (or rough paths, or models).

But in general J is not a continuous map $\mathscr{C}^{\gamma-1} \to \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$.

▷ *Probabilistic step:* prove that there exists a "reasonable definition" of $J(\xi)$ when ξ is a white noise. $J(\xi)$ is an explicit polinomial in ξ so direct computations are possible.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $\mathscr{C}^\gamma = B^\gamma_{\infty,\infty}$.

$f \in \mathscr{C}^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \leqslant \|f\|_\gamma 2^{-i\gamma}, \qquad i \geqslant -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi)\hat{f}(\xi)$$

where $\rho_i : \mathbb{R}^d \to \mathbb{R}_+$ are smooth functions with support $\simeq 2^i \mathscr{A}$ when $i \geqslant 0$ and form a partition of unity $\sum_{i \geqslant -1} \rho_i(\xi) = 1$ for all $\xi \neq 0$ so that

$$f = \sum_{i \geqslant -1} \Delta_i f$$

in $\mathcal{S}'$.

Paraproducts

Deconstruction of a product: $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$f \prec g \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$f \circ g \in \mathscr{C}^{\gamma+\rho} \qquad \text{only if } \gamma + \rho > 0$$

Proof. Recall $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$.

$$i \ll j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A} \qquad i \sim j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^\gamma,$$

while if $\rho < 0$

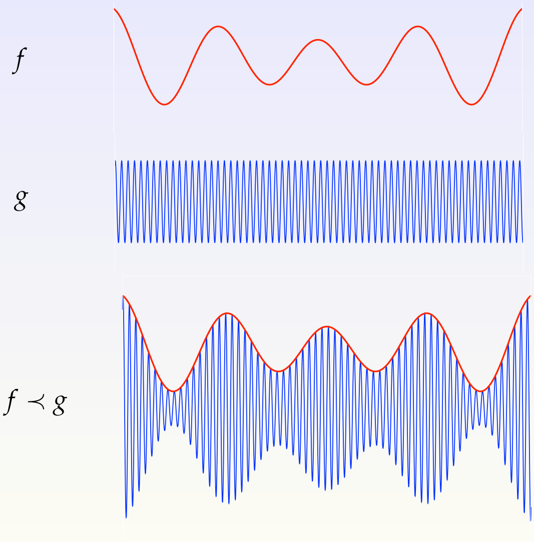
$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j\gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i\gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

Paraproduct as frequency modulation



Small detour : Young integral

Take $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$ with $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{f \prec Dg}_{\mathscr{C}^{\gamma-1}} + \underbrace{f \circ Dg + f \succ Dg}_{\mathscr{C}^{\gamma+\rho-1}}$$

then

$$\int fDg = \underbrace{\int f \prec Dg}_{\mathscr{C}^{\gamma}} + \underbrace{\int (f \circ Dg + f \succ Dg)}_{\mathscr{C}^{\gamma+\rho}}$$

$$= f \prec g + \mathscr{C}^{\gamma+\rho}.$$

Compare with standard estimate for the Young integral in Hölder spaces (valid when $\gamma + \rho > 1$):

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}).$$

Expansion in smalleness of increments vs. Expansion in regularity

The main commutator estimate

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

which however "is" smoother than $f \prec g$ if f or g has positive regularity.

Paraproducts decouple the problem from the source of the problem.

Commutator lemma

The trilinear operator $C(f,g,h) = (f \prec g) \circ h - f(g \circ h)$ satisfies

$$\|C(f,g,h)\|_{\beta+\gamma} \lesssim \|f\|_{\alpha} \|g\|_{\beta} \|h\|_{\gamma}$$

when $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, $\alpha < 1$.

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^{ε}) and φ a smooth function. Then $B \in \mathscr{C}^{\gamma}$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathscr{C}^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathscr{C}^{2\gamma}$$

Then

$$\varphi(B) \circ DB = (\varphi'(B) \prec B) \circ DB + \underbrace{\mathscr{C}^{2\gamma} \circ DB}_{\text{OK}}$$

$$= \varphi'(B)(B \circ DB) + \mathscr{C}^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B) \underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathscr{C}^{3\gamma-1}$$

The Besov area

If $d = 1$ (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2} D(B \circ B)$$

which is well defined and belongs indeed to $\mathscr{C}^{2\gamma-1}$.

In general the Besov area $B \circ DB$ can be defined and studied efficiently using Gaussian arguments:

$$B^\varepsilon \circ DB^\varepsilon \to B \circ DB$$

almost surely in $\mathscr{C}^{2\gamma-1}_{\mathrm{loc}}$ as $\varepsilon \to 0$.

Tools: Besov embeddings $L^p(\Omega; C^\theta) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Controlled paths/distributions

Controlled paths are paths which “looks like” a *given* path which often is random (but not necessarily).

A "good" quantification of this proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

A mix of functional analytic arguments and probabilistic ones.

Basic analogies

- Itô processes
$$\mathrm{d}X_t = f_t \mathrm{d}M_t + g_t \mathrm{d}t$$
- Amplitude modulation
$$f(t) = g(t)\sin(\omega t)$$
with $|\operatorname{supp}\hat{g}| \ll \omega$.

Controlled structures and paraproducts

▷ Gubinelli (2004): For $\alpha \in (0,1)$, $g \in C^\alpha$, f is called controlled by g if

$$f(t) - f(s) = f'(s)(g(t) - g(s)) + f^\sharp(s,t), \qquad |f^\sharp(s,t)| \lesssim |t-s|^{2\alpha}.$$

Then $f - f' \prec g \in \mathscr{C}^{2\alpha}$.

▷ Hairer (2013): For $\gamma > 0$, $f : \mathbb{R}^d \to T$ is called modelled, $f \in \mathscr{D}^\gamma$, if

$$|f_x - \Gamma_{x,y} f_y|_\beta \lesssim |x-y|^{\gamma-\beta}.$$

If $\mathscr{R}$ denotes the reconstruction operator, then $\mathscr{R}f - P(f,\Pi) \in C^\gamma$, where

$$P(f,\Pi)(x) = \sum_{j<k-1} \int K_j(x-z) K_k(x-y) \Pi_z f_z(y) \mathrm{d}y \mathrm{d}z$$
$$= \sum_{j<k-1} \int K_{j,x}(z) \Pi_z f_z(K_{k,x}) \mathrm{d}z.$$

Paracontrolled distributions

Use the paraproduct to *define* a controlled structure. We say $y \in \mathscr{D}_x^\rho$ if $x \in \mathscr{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with $y^x \in C^{\rho-\gamma}$ and $y^\sharp \in C^\rho$.

Theorem

If $\alpha + \beta + \gamma > 0$, $h \in \mathscr{C}^\gamma$, $f \in \mathscr{D}_g^{\alpha+\beta}$, and $g \circ h \in \mathscr{C}^{\gamma+\beta}$ is given, then fh can be constructed continuously as

$$fh = \Phi(f', f^\sharp, g, h, g \circ h).$$

Moreover, fh is paracontrolled by h:

$$fh - f \prec h \in \mathscr{C}^{\beta+\gamma}$$

Operations on paracontrolled distributions

▷ **Paralinearization.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^{\gamma}$, $\gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ Another commutator: $f, g \in \mathscr{C}^{\rho-\gamma}$, $x \in \mathscr{C}^{\gamma}$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^{\rho}$$

▷ **Stability.** $(\rho \leqslant 2\gamma)$

$$\varphi(y) = (\varphi'(y) y^{x}) \prec x + \mathscr{C}^{\rho}$$

so we can take $\varphi(y)^{x} = \varphi'(y) y^{x}$.

RDEs - I - the r.h.s.

$u : \mathbb{R} \to \mathbb{R}^d$, $\xi \in \mathscr{C}^{-1/2-}$ is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz. Take $\partial_t X = \xi$, $X \in \mathscr{C}^{1/2-}$ and assume that $u \in \mathscr{D}_X^{1-}$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{1-}$ and $u^X \in \mathscr{C}^{1/2-}$.

▷ Paralinearization:

$$f(u) = f'(u) \prec u + \mathscr{C}^{1-} = (f'(u)u^X) \prec X + \mathscr{C}^{1-}$$

▷ Commutator lemma:

$$f(u) \circ \xi = ((f'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{1-} \circ \xi$$

$$= \underbrace{(f'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{0-}} + \underbrace{C(f'(u)u^X, X, \xi) + \mathscr{C}^{1-} \circ \xi}_{\in \mathscr{C}^{1/2-}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{0-}$.

RDEs - II - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{0-}$ we have a control on the r.h.s. of the equation:

$$f(u)\xi = \underline{f(u) \prec \xi} + f'(u)u^X(X \circ \xi) + \mathscr{C}^{1/2-}$$

What about the l.h.s.?

$$\partial_t u = \partial_t u^X \prec X + \underline{u^X \prec \xi} + \partial_t u^\sharp$$

so letting $u^X = f(u)$ we have

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f'(u)f(u)(X \circ \xi) + \mathscr{C}^{1/2-}$$

RDEs - III - the paracontrolled fixed point.

The RDE

$$\partial_t u = f(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X &= \xi \\
\partial_t u^\sharp &= (f'(u)f(u))(X \circ \xi) - \underbrace{\partial_t f(u) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f,u,X,\xi)}_{\in \mathscr{C}^{1/2-}} \circ \xi \\
u &= f(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^{1-}$ if we assume that

$$X \in \mathscr{C}^{1/2-}, \qquad (X \circ \xi) \in \mathscr{C}^{0-}.$$

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = f(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1} \to \mathscr{C}^{\gamma}$$

is continuous for any $\gamma > 1/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by

$$\begin{cases} z = f(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (f'(z)f(z))\color{red}{\varphi}\color{black} - \underbrace{\partial_t f(z) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f, z, X, \xi) \circ \xi}_{\in \mathscr{C}^{1/2-}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then

$$u^n \to u = \Phi(u_0, \xi, \eta).$$

Relaxed form of the RDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-1}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

"Itô" form of the RDE

In the smooth setting $u = \Phi(\xi, X \circ \xi + \varphi)$ solves

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

If we choose $\varphi = -X \circ \xi$ then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\partial_t v = f(v)\xi - f'(v)f(v)X \circ \xi$$

and has the particular property of being a continuous map of $\xi \in \mathscr{C}^{\gamma-1}$ alone.

The discrete parabolic Anderson model

- Stochastic heat equation on $\mathbb{Z}^d$:

$$\partial_t u(t,x) = \Delta_{\mathbb{Z}^d} u(t,x) + F(u(t,x))\eta(x);$$

  with potential landscape of i.i.d. random variables $(\eta(x))_{x\in\mathbb{Z}^d}$;
- linear version with $F(u) = u$ is model for many phenomena in physics, e.g. growth of magnetic fields in young stars;
- mathematical interest in long time behavior of PAM: simple model which exhibits intermittency (largest part of the mass concentrated in few small "islands");
- countless results since early 90s, different universality classes depending on distribution of η.

Conjectured scaling limit

- To study long time behavior, and to obtain universality for different potentials η, would be interested in scaling limit:

$$\partial_t v^n(t,x) = \Delta_{\mathbb{Z}^d} v^n(t,x) + n^{d/2-2} F(v^n(t,x))\eta(x);$$
$$u^n(t,x) = v^n(n^2 t, nx).$$

- Natural conjecture: limit solves

$$\mathcal{L}u(t,x) = F(u(t,x))\xi(x),$$

for spatial white noise ξ.

Continuous PAM

$$\mathcal{L}u(t,x) = F(u(t,x))\xi(x)$$

Equation is ill posed for $d > 1$, needs some form of renormalization.

Existing solutions only work in linear case and use Wick products (e.g. Hu (2002)):

$$\mathcal{L}u(t,x) =: u(t,x)\xi(x):$$

Obtain existence and uniqueness of solutions for $d \leqslant 3$.

- apply formal chaos expansion to the solution;
- formally obtain solution as chaos series, $u = \sum_n I_n(f_n)$ for suitable deterministic f_n;
- see that for $d < 4$, the series indeed converges.

Problem: Discrete PAM not formulated in terms of Wick products! How does the Wick product transform the equation? Scaling limit?

Continuous PAM and paracontrolled distributions

$$\mathcal{L}u(t,x) = F(u(t,x)) \diamond \xi(x)$$

Paracontrolled distributions can handle the equation in the general case for $d \leqslant 2$, in the linear case for $d = 3$ (in principle...); agrees with Wick product solution in the linear case. Advantages:

- renormalization $u(t,x) \diamond \xi(x)$ of $u(t,x)\xi(x)$ is very transparent and we can apply the same renormalization in the discrete model;
- solution depends pathwise continuously on suitably extended data.

The solution is scaling limit of renormalized discrete system (work in progress by Perkowski, Chouk, Gairing):

- show weak convergence of $n^{d/2-2}\eta(n\cdot)$ to ξ and of renormalized extended data;
- use pathwise continuous dependence of solution on extended data in combination with Skorokhod representation to obtain weak convergence of solutions.

Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - \mathrm{D}^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{-1-}(\mathbb{T}^2)$ space white noise.

$$\mathcal{L}u = f(u)\xi$$

▷ Paracontrolled ansatz $\qquad \mathcal{L}X = \xi$ so $X \in C([0,T], \mathscr{C}^{1-})$

$$u = f(u) \prec X + u^\sharp$$

▷ Paralinearization: $\qquad f(u) = (f'(u)f(u)) \prec X + R(f,u,X)$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f,u,X) \circ \xi$$

A problem

If ξ is the space white noise we have

$$\xi \in \mathscr{C}^{-1-}, \qquad X \in C([0,T];\mathscr{C}^{1-})$$

and

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$
$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2$$

But now

$$\frac{1}{2}(DX)^2 = c + C\mathscr{C}^{0-}$$

with $c = +\infty$!.

No obvious definition of $X \circ \xi$ can be given. But there exists c_ε such that

$$X_\varepsilon \circ \xi_\varepsilon - c_\varepsilon \to \text{"}X \diamond \xi\text{"} \qquad \text{in } C\mathscr{C}^{0-}.$$

A first renormalization

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denote by $\diamond$. Now

$$f(u)\circ\xi - c(f'(u)f(u)) = (f'(u)f(u))(X\circ\xi - c) + C(f'(u)f(u), X, \xi) + R(f,u,X)\circ\xi$$

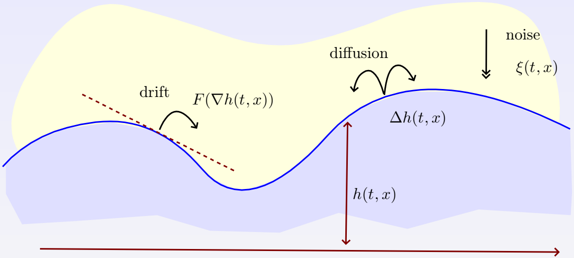
▹ The renormalized gPAM is equivalent to the equation

$$\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X\circ\xi - c)$$
$$+C(f'(u)f(u), X, \xi) + R(f,u,X)\circ\xi$$

together with $u = f(u) \prec X + u^\sharp$ and where

$$X \in \mathscr{C}^{1-}, \qquad X \diamond \xi = (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^\sharp \in \mathscr{C}^{2-}.$$

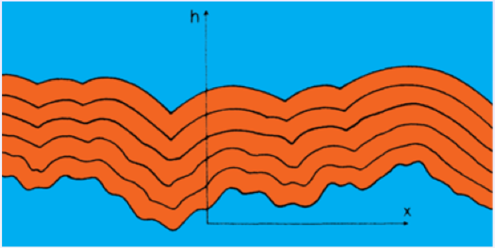
Fluctuations of a growing interface

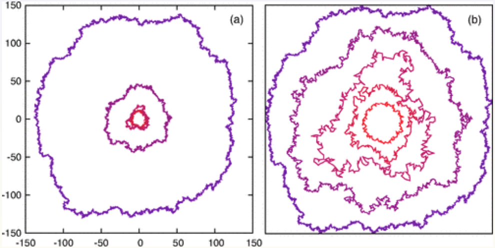


A model for random interface growth (think e.g. expansion of colony of bacteria): $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{F(\partial_x h(t,x))}_{\text{slope-dependent growth}} + \underbrace{\eta(t,x)}_{\text{noise with microscopic correlations}}$$

Fluctuations of a growing interface





The Kardar–Parisi–Zhang equation

- Kardar–Parisi–Zhang ’84: slope-dependent growth given by $F(\partial_x h)$, in a certain scaling regime of small gradients:

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + F''(0)(\partial_x h)^2 + \ldots$$

- KPZ equation is the universal model for random interface growth

$$\partial_t h(t,x) = \underbrace{\kappa\Delta h(t,x)}_{\text{relaxation}} + \underbrace{\lambda[(\partial_x h(t,x))^2 - \infty]}_{\text{renormalized growth}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- This derivation is highly problematic since $\partial_x h$ is a distribution. But: Hairer, Quastel (2014, unpublished) justify it rigorously via scaling of smooth models and small gradients.
- KPZ equation is suspected to be universal scaling limit for random interface growth models, random polymers, and many particle systems;
- contrary to Brownian setting: KPZ has fluctuations of order $t^{1/3}$; large time limit distribution of $t^{-1/3}h(t, t^{2/3}x)$ is expected to be universal in a sense comparable only to the Gaussian distribution.

KPZ and its siblings:

- KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

  $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- Burgers equation:

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x);$$

  solution is (formally) given by derivative of the KPZ equation: $u = \partial_x h$;

- solution to KPZ (formally) given by Cole-Hopf transform of the stochastic heat equation: $h = \log w$, where w solves

$$\mathcal{L}w(t,x) = w(t,x)\xi(t,x).$$

- All three are universal objects, that are expected to be scaling limits of a wide range of particle systems.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion /I

▷ Let $u = X + u_1$ then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▷ Let $X^{\vee}$ be the solution to

$$\mathcal{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

▷ Define $\mathcal{L}X^{\curlyvee} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\curlyvee} + u_3$ then $X^{\curlyvee} \in \mathscr{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\curlyvee}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\Yleft} + U$$

$$\mathcal{L}U = 2\partial_x(UX) + 2\partial_x(X^{\Yleft}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\Yleft}+U)X^{\vee}) + \partial_x(2X^{\Yleft}+U)^2$$

$$= {\color{red}2\partial_x(UX)} + \mathcal{L}(2X^{\Yleft\Yleft} + X^{\vee\vee}) + 2\partial_x((2X^{\Yleft}+U)X^{\vee}) + \partial_x(2X^{\Yleft}+U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	$X^{\Yleft}$	$X^{\Yleft\Yleft}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\Yleft}+U)X^{\vee}) + \partial_x(2X^{\Yleft}+U)^2$$

are well defined.

The remaining problem is to deal with ${\color{red}2\partial_x(UX)}$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Q + U^\sharp$. Then

$$\mathcal{L}U = \mathcal{L}U' \prec Q + U' \prec \mathcal{L}Q - \partial_x U' \prec \partial_x Q + LU^\sharp$$

while

$$\mathcal{L}U = 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\Ythree} + X^{\Ybal}) + 2\partial_x((2X^{\Ytwo} + U)X^{\vee}) + \partial_x(2X^{\Ytwo} + U)^2}_{R(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

so we can set $U' = 2U$ and $\mathcal{L}Q = \partial_x X$ and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Q + \partial_x U' \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

▷ Observe that $Q, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▹ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▹ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Q) \circ X + U^\sharp \circ X = 2U(Q \circ X) + \underbrace{C(2U, Q, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▹ A stochastic estimate shows that $Q \circ X \in \mathscr{C}^{0-}$

Paracontrolled solution to SBE

▷ The final system reads

$$u = X + X^{\vee} + 2X^{\vee_{\vee}} + U$$

$$U = U' \prec Q + U^{\sharp}, \qquad U' = 2X^{\vee_{\vee}} + 2U$$

$$\mathcal{L}U^{\sharp} = 4\partial_x(U\underline{(Q \circ X)}) + 4\partial_x C(U, Q, X) + 2\partial_x(U^{\sharp} \circ X) - 2\mathcal{L}U \prec Q$$
$$+2\partial_x U \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + R(U)$$

▷ This equation has a (local in time) solution $U = \Phi(J(\xi))$ which is a continuous function of the data $J(\xi)$ given by a collection of multilinear functions of ξ:

$$J(\xi) = (X, X^{\vee}, X^{\vee_{\vee}}, X^{\vee_{\vee_{\vee}}}, X^{\vee\vee}, X \circ Q)$$

Burgers equation and paracontrolled distributions

$$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled Ansatz

$u \in \mathscr{P}_{\mathrm{rbe}}$ if $u = X + X^{\vee} + 2X^{\vee\!\!\!\vee} + u^Q$ with

$$u^Q = \pi_<(u', Q) + u^\sharp.$$

- Paracontrolled structure: Can define u^2 continuously as long as $(Q \circ X) \in C([0,T], \mathscr{C}^{0-})$ is given (together with tree data $X, X^{\vee}, X^{\vee\!\!\!\vee}, X^{\vee\!\!\!\vee\!\!\!\vee}, X^{\vee\vee}$).
- Obtain local existence and uniqueness of paracontrolled solutions. Solution depends pathwise continuously on extended data $J(\xi) = (\xi, X, X^{\vee}, X^{\vee\!\!\!\vee}, X^{\vee\!\!\!\vee\!\!\!\vee}, X^{\vee\vee}, Q \circ X)$.

KPZ equation

KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x), \qquad h(0) = h_0.$$

Expect $h(t) \in \mathscr{C}^{1/2-}$, so $\partial_x h(t) \in \mathscr{C}^{-1/2-}$ and $(\partial_x h(t))^2$ not defined.
But: expand

$$u = Y + Y^{\vee} + 2Y^{\check{\vee}} + h^P,$$

where $\mathcal{L}Y = \xi$, $\mathcal{L}Y^{\vee} = \partial_x Y \partial_x Y, \ldots$ In general: $\partial_x Y^{\tau} = X^{\tau}$. Make paracontrolled ansatz for h^P:

$$h^P = \pi_<(h', P) + h^\sharp$$

with $h' \in C([0,T], \mathscr{C}^{1/2-})$, $h^\sharp \in C([0,T], \mathscr{C}^{2-})$, $\mathcal{L}P = X$. Write $h \in \mathscr{P}_{\mathrm{kpz}}$.

Can define $(\partial_x h(t))^2$ for $h \in \mathscr{P}_{\mathrm{kpz}}$ and obtain local existence and uniqueness of solutions.

KPZ and Burgers equation

$h \in \mathscr{P}_{\text{kpz}}$ if

$$h = Y + Y^{\vee} + 2Y^{\Yup} + h^P, \qquad h^P = h' \prec P + h^{\sharp}.$$

$u \in \mathscr{P}_{\text{rbe}}$ if

$$u = X + X^{\vee} + 2X^{\Yup} + u^Q, \qquad u^Q = u' \prec Q + u^{\sharp}.$$

- If $h \in \mathscr{P}_{\text{kpz}}$, then $\partial_x h \in \mathscr{P}_{\text{rbe}}$.
- If h solves KPZ equation, then $u = \partial_x h$ solves Burgers equation with initial condition $u(0) = \partial_x h_0$.
- If $u \in \mathscr{P}_{\text{rbe}}$, then any solution h of $\mathcal{L}h = u^2 + \xi$ is in $\mathscr{P}_{\text{kpz}}$.
- If u solves Burgers equation with initial condition $u(0) = \partial_x h_0$, and h solves $\mathcal{L}h = u^2 + \xi$ with initial condition $h(0) = h_0$, then h solves KPZ equation.

KPZ and heat equation

Heat equation:

$$\mathcal{L}w(t,x) = w(t,x) \diamond \xi(t,x) = w(t,x)\xi(t,x) - w(t,x)\cdot\infty, \quad w(0) = w_0.$$

Paracontrolled ansatz: $w \in \mathscr{P}_{\mathrm{rhe}}$ if

$$w = e^{Y + Y^{\vee} + 2Y^{\checkmark}} w^P, \qquad w^P = \pi_<(w', P) + w^\sharp$$

(comes from Cole-Hopf transform).

- Slightly cheat to make sense of product $w \diamond \xi$ for $w \in \mathscr{P}_{\mathrm{rhe}}$:

$$w \diamond \xi = \mathcal{L}w - e^{Y+Y^{\vee}+2Y^{\checkmark}}\left[\mathcal{L}w^P - [\mathcal{L}(Y^{\vee} + Y^{\checkmark}) + (\partial_x(Y + Y^{\vee} + 2Y^{\checkmark}))^2]w^P\right]$$
$$+ 2e^{Y+Y^{\vee}+2Y^{\checkmark}}\partial_x(Y + Y^{\vee} + 2Y^{\checkmark})\partial_x w^P;$$

  (agrees with renormalized pointwise product $w \diamond \xi$ in smooth case and with Itô integral in white noise case, continuous in extended data).
- Obtain global existence and uniqueness of solutions.
- One-to-one correspondence between $\mathscr{P}_{\mathrm{kpz}}$ and strictly positive elements of $\mathscr{P}_{\mathrm{rhe}}$.
- Any solution of KPZ gives solution of heat equation. Any strictly positive solution of heat equation gives solution of KPZ equation.

Further applications

Besides KPZ, Burgers, heat equation, and continuous PAM, the following equations have been solved using the paracontrolled approach:

- Gubinelli, Imkeller, P. (2012): Multidimensional extension of Hairer's (2011) generalized Burgers equation ($\sigma - d/2 > 1/3$):

$$\partial_t u(t,x) = -(-\Delta)^{\sigma} u(t,x) + G(u(t,x)) D_x u(t,x) + \xi(t,x);$$

- Catellier, Chouk (2013): Stochastic quantization equation ϕ^4_3 ($d = 3$):

$$\mathcal{L} u(t,x) = -u(t,x)^{\diamond 3} + \xi(t,x);$$

- Furlan (2014): Stochastic Navier Stokes equation ($d = 3$):

$$\mathcal{L} u(t,x) = -P((u(t,x) \cdot \nabla) u(t,x)) + \xi(t,x).$$

Para-modelled distributions

Let $\gamma > 0$ and (T, Π, Γ) regularity structure. Say f is para-modelled, $f \in \mathscr{P}^\gamma$, if there exists $f^\pi \in \mathscr{D}^\gamma$, with

$$f - \pi_<(f^\pi, \Pi) \in C^\gamma.$$

Example: $\mathscr{R} f^\pi \in \mathscr{P}^\gamma$.
Consider rough path model, say $T = \mathrm{span}(\Xi, \mathscr{I}(\Xi)\Xi, \mathscr{I}(\mathscr{I}(\Xi)\Xi)\Xi, \mathbf{1}, \mathscr{I}(\Xi), \mathscr{I}(\mathscr{I}(\Xi)\Xi))$. Try to solve $\partial_t u = F(u)\xi$.
(Simplified) para-modelled ansatz: $u = \mathscr{R} u^\pi = \pi_<(u^\pi, \Pi) + u^\sharp$ with $u^\pi \in \mathscr{D}^{3\alpha}$. Equation for $u^\sharp$:

$$\partial_t u^\sharp = -\partial_t \pi_<(u^\pi, \Pi) + F(u)\xi = \pi_<(u^\pi, D\Pi) - \pi_<(F(u^\pi) \star \xi^\pi, \Pi) + \text{smooth}.$$

To have $u^\sharp \in C^{3\alpha}$: choose expansion u^π so that all coefficients for terms of homogeneity $< 3\alpha - 1$ cancel. Obtain a priori bounds on $\|u^\sharp\|_{3\alpha}$ and then on $\|u^\pi\|_{\mathscr{D}^{3\alpha}}$. Thus at least local existence of solutions.

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3$: $\xi \in C^{-5/2-}$, $u \in C^{-1/2-}$,
$u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 - c_1$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects:
$(u_1^2 - c_1), (u_1^3 - 3c_1 u_1), (3\lambda(X \circ (u_1^2 - c_1)) - c_2), (u_2 u_1), (u_2^2 u_1)$

Thanks