Paracontrolled distributions
with applications to singular SPDEs

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Stochastic Analysis workshop, Tohoku University – October 15th, 2014

Some problems in singular SPDEs /I

Define and solve (locally) the following SPDEs:

- Stochastic differential equations (1+0): $u \in [0,T] \to \mathbb{R}^n$

$$\partial_t u(t) = \sum_i f_i(u(t))\xi^i(t)$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.
- Burgers equations (1+1): $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))Du(t,x) + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

Recall that

$$\xi \in \mathscr{C}^{-d/2-}$$

Some problems in singular SPDEs /II

- Generalized Parabolic Anderson model (1+2):
  $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

  $$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(x)$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.
- Kardar-Parisi-Zhang equation (1+1)

  $$\partial_t h(t,x) = \Delta h(t,x) + "(Dh(t,x))^2 - \infty" + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Some problems in singular SPDEs /III

Define and solve (locally) the following SPDEs:

- Stochastic quantization equation (1+3)

$$\partial_t u(t,x) = \Delta u(t,x) + \text{"}u(t,x)^3\text{"} + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.
- But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Joint work with P. Imkeller and N. Perkowski.
(Also K. Chouk and R. Catellier for $(\Phi)^4_3$).

Rough differential equation

Consider the simple controlled PDE (η smooth, fixed initial condition)

$$\partial_t u(t,x) = \nabla u(t,x) + F(u(t,x))\eta(x)$$

$u : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$, $\eta : \mathbb{T}^d \to \mathbb{R}$ and smooth function $F : \mathbb{R} \to \mathbb{R}$.

Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for $\eta \in \mathscr{C}^{\gamma-2}$ with $\gamma < 1$.

Reason: $u \in \mathscr{C}^{\gamma}$ and $\eta \in \mathscr{C}^{\gamma-2}$ cannot be multiplied when $2\gamma - 2 \leqslant 0$. The r.h.s. of the equation is not well defined.

Here $\mathscr{C}^{\alpha} = C([0,T], B^{\alpha}_{\infty,\infty}(\mathbb{T}^d))$ is the Holder–Besov space (or a local version).

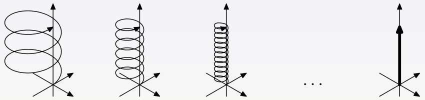
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $\mathscr{C}^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s) \partial_t x^{n,2}(s) \mathrm{d}s \to \frac{t}{2} \neq I(0,0)(t) = 0$$



The definite integral $I(\cdot,\cdot)(t)$ is **not** a continuous map $\mathscr{C}^\gamma \times \mathscr{C}^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

A possible concept of solution

Goal: Show that $\Psi : \eta \mapsto u$ factorizes as

$$\eta \xrightarrow{J} J(\eta) \xrightarrow{\Phi} u$$

▷ *Analytic step:* show that when $\gamma > 1/3$:

$$\Phi : \mathcal{X} \to \mathscr{C}^\gamma$$

is continuous. $\mathcal{X} = \overline{\mathrm{Im} J} \subseteq \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ is the space of *enhanced signals* (or rough paths, or models).

But in general J is not a continuous map $\mathscr{C}^{\gamma-1} \to \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$.

▷ *Probabilistic step:* prove that there exists a "reasonable definition" of $J(\xi)$ when ξ is a white noise. $J(\xi)$ is an explicit polinomial in ξ so direct computations are possible.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $\mathscr{C}^\gamma = B^\gamma_{\infty,\infty}$.

$f \in \mathscr{C}^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \leqslant \|f\|_\gamma 2^{-i\gamma}, \qquad i \geqslant -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi)\hat{f}(\xi)$$

where $\rho_i : \mathbb{R}^d \to \mathbb{R}_+$ are smooth functions with support $\simeq 2^i \mathscr{A}$ when $i \geqslant 0$ and form a partition of unity $\sum_{i\geqslant -1} \rho_i(\xi) = 1$ for all $\xi \neq 0$ so that

$$f = \sum_{i\geqslant -1} \Delta_i f$$

in $\mathcal{S}'$.

Paraproducts

Deconstruction of a product: $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$f \prec g \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$f \circ g \in \mathscr{C}^{\gamma+\rho} \qquad \text{only if } \gamma + \rho > 0$$

Proof. Recall $f \in \mathscr{C}^\rho, g \in \mathscr{C}^\gamma$.

$$i \ll j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A} \qquad i \sim j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^\gamma,$$

while if $\rho < 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j\gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i \gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

Small detour : Young integral

Take $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$ with $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{f \prec Dg}_{\mathscr{C}^{\gamma-1}} + \underbrace{f \circ Dg + f \succ Dg}_{\mathscr{C}^{\gamma+\rho-1}}$$

then

$$\int fDg = \underbrace{\int f \prec Dg}_{\mathscr{C}^{\gamma}} + \underbrace{\int (f \circ Dg + f \succ Dg)}_{\mathscr{C}^{\gamma+\rho}}$$
$$= f \prec g + \mathscr{C}^{\gamma+\rho}.$$

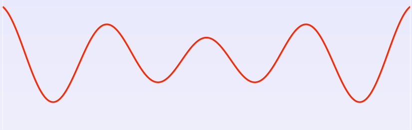
Compare with standard estimate for the Young integral in Hölder spaces (valid when $\gamma + \rho > 1$):

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}).$$

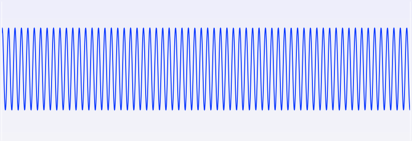
$\boxed{\text{Expansion in smalleness of increments vs. Expansion in regularity}}$

Paraproduct as frequency modulation

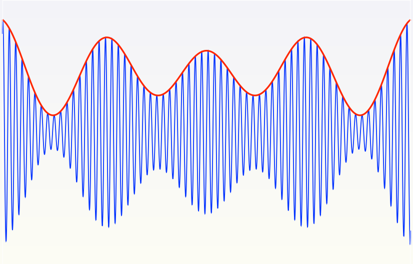
f



g



$f \prec g$



The main commutator estimate

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

which however "is" smoother than $f \prec g$ if f or g has positive regularity.

Paraproducts decouple the problem from the source of the problem.

Commutator lemma

The trilinear operator $C(f,g,h) = (f \prec g) \circ h - f(g \circ h)$ satisfies

$$\|C(f,g,h)\|_{\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

when $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, $\alpha < 1$.

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in \mathscr{C}^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathscr{C}^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathscr{C}^{2\gamma}$$

Then

$$\varphi(B) \circ DB = (\varphi'(B) \prec B) \circ DB + \underbrace{\mathscr{C}^{2\gamma} \circ DB}_{\text{OK}}$$

$$= \varphi'(B)(B \circ DB) + \mathscr{C}^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B)\underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathscr{C}^{3\gamma-1}$$

The Besov area

If $d = 1$ (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2} D(B \circ B)$$

which is well defined and belongs indeed to $\mathscr{C}^{2\gamma-1}$.

In general the Besov area $B \circ DB$ can be defined and studied efficiently using Gaussian arguments:

$$B^{\varepsilon} \circ DB^{\varepsilon} \to B \circ DB$$

almost surely in $\mathscr{C}^{2\gamma-1}_{\mathrm{loc}}$ as $\varepsilon \to 0$.

Tools: Besov embeddings $L^p(\Omega; C^{\theta}) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Paracontrolled distributions

Use the paraproduct to *define* a controlled structure. We say $y \in \mathscr{D}_x^\rho$ if $x \in \mathscr{C}^\gamma$

$$y = y^x \prec x + y^\sharp$$

with $y^x \in C^{\rho-\gamma}$ and $y^\sharp \in C^\rho$.

▷ **Paralinearization.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^\gamma$, $\gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ Another commutator: $f, g \in \mathscr{C}^{\rho-\gamma}$, $x \in \mathscr{C}^\gamma$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^\rho$$

▷ **Stability.** ($\rho \leqslant 2\gamma$)

$$\varphi(y) = (\varphi'(y) y^x) \prec x + \mathscr{C}^\rho$$

so we can take $\varphi(y)^x = \varphi'(y) y^x$.

Homogeneisation of a random potential

▷ Consider the linear heat equation with a small random time-independent (Gaussian) potential V

$$\partial_t U(t,x) = \Delta U(t,x) + \varepsilon^{2-\alpha} V(x) U(t,x)$$

on $(\mathbb{T}/\varepsilon)^d$ and where ε is a small parameter and $\alpha < 2$.

▷ Introduce macroscopic variables $u_\varepsilon(t,x) = U(t/\varepsilon^2, x/\varepsilon)$ with parabolic rescaling, then

$$\partial_t u_\varepsilon(t,x) = \Delta u_\varepsilon(t,x) + V_\varepsilon(x) u_\varepsilon(t,x)$$

on $\mathbb{T}$ and where $V_\varepsilon(x) = \varepsilon^{-\alpha} V(x/\varepsilon)$.

Homogeneisation of a random potential (II)

The covariance of the macroscopic noise is

$$\mathbb{E}[V_\varepsilon(x)V_\varepsilon(y)] = \varepsilon^{-2\alpha}C((x-y)/\varepsilon)$$

Theorem

If $d > 2\alpha$ then $V_\varepsilon \to 0$ in $\mathscr{C}^{-\alpha-}$. While if $d = 2\alpha$ then V_ε converges to the space white noise on $\mathbb{T}$.

So we are let to the study of the stability properites of the equation

$$\mathcal{L}u = \eta u$$

with $\eta \in \mathscr{C}^{-\alpha}$. This stability is easy to estabilish when $2 - 2\alpha > 0$ by standard estimates in Besov spaces. We are concerned then with the case $\alpha = 1$.

Transformation of PAM

▷ In order to understand the difficulties, let us perfom a change of variable by letting $u = e^X v$ with $\mathcal{L}X = \eta$. We get

$$\mathcal{L}u = v\mathcal{L}e^X + e^X\mathcal{L}v - \partial_x e^X \partial_x v$$

$$= ve^X\mathcal{L}X - ve^X(\partial_x X)^2 + e^X\mathcal{L}v - e^X\partial_x X\partial_x v$$

so v solves

$$\mathcal{L}v = (\partial_x X)^2 v + \partial_x X \partial_x v.$$

Let $\gamma = 2 - \alpha -$ the regularity of X.

▷ If we *assume* that $(\partial_x X)^2 \in \mathscr{C}^{2\gamma-2}$ then we see that this equation can be solved for $v \in \mathscr{C}^{2\gamma}$ since in this case $\partial_x X \partial_x v \in \mathscr{C}^{\gamma-1}$ and we have a continous map

$$(X, (\partial_x X)^2) \in \mathscr{C}^\gamma \times \mathscr{C}^{2\gamma-2} \mapsto v \in \mathscr{C}^\gamma$$

Homogeneisation

When $\eta = V_\varepsilon$:

Theorem

Assume $d > 2$ and $\alpha = 1$ and let $\mathcal{L} X_\varepsilon = V_\varepsilon$ (+ technical conditions on the covariance C), then $(\partial_x X_\varepsilon)^2 \to \sigma^2$ in $\mathscr{C}^{0-}$.

▷ If $d > 2$ writing $u_\varepsilon = e^{X_\varepsilon} v_\varepsilon$ we obtain that v_ε converges to the solution of the PDE

$$\mathcal{L} v = \sigma^2 v$$

and so does u since $X \to 0$ in $\mathscr{C}^\gamma$.

▷ Now

$$\mathcal{L} u_\varepsilon = V_\varepsilon u_\varepsilon \not\to \mathcal{L} u = 0 * u$$

but $\mathcal{L} u = \sigma^2 u$ with $\sigma^2 \neq 0$. Lack of continuity of the problem wrt the data V_ε in the $\mathscr{C}^{\gamma-2}$ topology if $\gamma - 2 < -1$.

Renormalization

When $d = 2$, $\alpha = 1$:

Theorem

Let $\gamma = 1-$, then $V_\varepsilon \to \xi$ (white noise on $\mathbb{T}^2$) in $\mathscr{C}^{\gamma-2}$ and $\mathcal{L}X_\varepsilon = V_\varepsilon$ (+ technical conditions on the covariance C), then there exists a sequence $c_\varepsilon \to +\infty$ such that $(\partial_x X_\varepsilon)^2 - c_\varepsilon \to (\partial_x X)^{\diamond 2}$ in $\mathscr{C}^{2\gamma-2}$.

Here, formally, $\sigma^2 = +\infty$, so there is not a well defined limit for u_ε.

Consider $\tilde{u}_\varepsilon(t,x) = e^{-c_\varepsilon t}u(t,x)$ which solves

$$\mathcal{L}\tilde{u}_\varepsilon = V_\varepsilon u_\varepsilon - c_\varepsilon u_\varepsilon$$

then for $\tilde{v}_\varepsilon = e^{-X_\varepsilon}\tilde{u}_\varepsilon$ we have the equation

$$\mathcal{L}\tilde{v}_\varepsilon = [(\partial_x X_\varepsilon)^2 - c_\varepsilon]\tilde{v}_\varepsilon + \partial_x X_\varepsilon \partial_x \tilde{v}_\varepsilon$$

which behaves well in the limit $\varepsilon \to 0$.

Paracontrolled ansatz

$\triangleright$ **Question:** What is the equation satisfied by $\tilde{u} = \lim_{\varepsilon\to 0} \tilde{u}_\varepsilon$?

It should be someting like $\mathcal{L}\tilde{u} = "\tilde{u}\xi - \infty\tilde{u}" = \tilde{u} \diamond \xi$ (in which sense?)

$\triangleright$ Note that (by paralinearization)

$$u = e^X v = e^X \prec v + e^X \succeq v = (e^X \prec X) \prec v + \mathscr{C}^{2\gamma} = u \prec X + \mathscr{C}^{2\gamma}$$

so u is controlled by X: $u \in \mathscr{D}^{2\gamma}_X$. Similarly $\tilde{u}_\varepsilon \in \mathscr{D}^{2\gamma}_{X_\varepsilon}$. Then

$$\tilde{u}_\varepsilon V_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon \prec V_\varepsilon + \tilde{u}_\varepsilon \circ V_\varepsilon + \tilde{u}_\varepsilon \succ V_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon$$

$$= \tilde{u}_\varepsilon \prec V_\varepsilon + (\tilde{u}_\varepsilon \prec X_\varepsilon) \circ V_\varepsilon + \tilde{u}^\sharp_\varepsilon \circ V_\varepsilon + \tilde{u}_\varepsilon \succ V_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon$$

$$= \tilde{u}_\varepsilon \prec V_\varepsilon + \tilde{u}_\varepsilon (X_\varepsilon \circ V_\varepsilon - c_\varepsilon) + \mathrm{C}(\tilde{u}_\varepsilon, X_\varepsilon, V_\varepsilon) + \tilde{u}^\sharp_\varepsilon \circ V_\varepsilon + \tilde{u}_\varepsilon \succ V_\varepsilon$$

Paracontrolled ansatz (II)

▷ So in the limit $\varepsilon \to 0$ we have

$$\tilde{u}_\varepsilon V_\varepsilon - c_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon \prec V_\varepsilon + \tilde{u}_\varepsilon (X_\varepsilon \circ V_\varepsilon - c_\varepsilon) + C(\tilde{u}_\varepsilon, X_\varepsilon, V_\varepsilon) + \tilde{u}^\sharp_\varepsilon \circ V_\varepsilon + \tilde{u}_\varepsilon \succ V_\varepsilon$$

$$\to \tilde{u} \prec \xi + \tilde{u}(X \diamond \xi) + C(\tilde{u}, X, \xi) + \tilde{u}^\sharp \circ \xi + \tilde{u} \succ \xi$$

$$=: \tilde{u} \diamond \xi = \Phi(\tilde{u}, \tilde{u}^\sharp, X, X \diamond \xi)$$

where $X \diamond \xi := \lim_{\varepsilon \to 0}(X_\varepsilon \circ V_\varepsilon - c_\varepsilon)$.

▷ **Question:** What is the equation satisfied by $\tilde{u} = \lim_{\varepsilon \to 0} \tilde{u}_\varepsilon$?

Indeed

$$\mathcal{L}\tilde{u} = \text{"}\tilde{u}\xi - \infty\tilde{u}\text{"} = \tilde{u} \diamond \xi = \Phi(\tilde{u}, \tilde{u}^\sharp, X, X \diamond \xi).$$

Where the r.h.s. is well defined since $\tilde{u}$ is paracontrolled.

gPAM - I - the r.h.s.

$u : \mathbb{R}_+ \hat{\mathrm{E}} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{\gamma-2}$, $\gamma = 1-$. We want to solve (have uniform bounds for)

$$\mathcal{L}u = F(u)\xi = F(u) \prec \xi + F(u) \circ \xi + F(u) \succ \xi.$$

▷ Paracontrolled ansatz. Take $\mathcal{L}X = \xi$, $X \in \mathscr{C}^\gamma$ and assume that $u \in \mathscr{D}_X^{2\gamma}$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{2\gamma}$ and $u^X \in \mathscr{C}^\gamma$.

▷ Paralinearization:

$$F(u) = F'(u) \prec u + \mathscr{C}^{2\gamma} = (F'(u)u^X) \prec X + \mathscr{C}^{2\gamma}$$

▷ Commutator lemma:

$$F(u) \circ \xi = ((F'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{2\gamma} \circ \xi$$

$$= \underbrace{(F'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{2\gamma-2}} + \underbrace{C(F'(u)u^X, X, \xi) + \mathscr{C}^{2\gamma} \circ \xi}_{\in \mathscr{C}^{3\gamma-2}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{2\gamma-2}$.

gPAM - II - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{2\gamma-2}$ we have a control on the r.h.s. of the equation:

$$F(u)\xi = \underline{F(u) \prec \xi} + F'(u)u^X(X \circ \xi) + \mathscr{C}^{3\gamma-2}$$

What about the l.h.s.?

$$\mathcal{L}u = \mathcal{L}u^X \prec X + \underline{u^X \prec \xi} + \mathcal{L}u^\sharp - \partial_x u^X \prec \partial_x X$$

so letting $u^X = F(u)$ we have

$$\mathcal{L}u^\sharp = -\mathcal{L}F(u) \prec X + F'(u)F(u)(X \circ \xi) + \mathbb{C}^{2\gamma-2}$$

gPAM - III - the paracontrolled fixed point.

The PDE

$$\mathcal{L}u = F(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X =& \xi \\
\partial_t u^\sharp =& (F'(u)F(u))(X \circ \xi) - \underbrace{\mathcal{L}f(u) \prec X}_{"\in"\mathscr{C}^{2\gamma-2}} + \underbrace{R(f, u, X, \xi)}_{\in\mathscr{C}^{3\gamma-2}} \circ \xi \\
u =& F(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^{2\gamma}$ if we assume that

$$X \in \mathscr{C}^\gamma, \qquad (X \circ \xi) \in \mathscr{C}^{2\gamma-2}.$$

Paracontrolled solutions to gPAM

Theorem

Let $d = 2$, $\alpha = 1$, $\gamma = 1-$ and small $T > 0$. There exist constants c_ε such that letting u_ε the solution to

$$\mathcal{L}u_\varepsilon = V_\varepsilon F(u_\varepsilon) - c_\varepsilon F'(u_\varepsilon)$$

then $u_\varepsilon \to u$ in $\mathbb{C}^\gamma$ as $\varepsilon \to 0$ and $u \in \mathscr{D}_X^{2\gamma}$ is the unique weak solution in $\mathscr{D}_X^{2\gamma}$ to the equation

$$\mathcal{L}u = \xi \diamond F(u) = F(u) \prec \xi + F'(u)(X \diamond \xi) + G(u^X, u^\sharp, X)$$

where

$$\xi = \lim_{\varepsilon \to 0} V_\varepsilon, \qquad X \diamond \xi = \lim_{\varepsilon \to 0} X_\varepsilon \circ V_\varepsilon - c_\varepsilon$$

in $\mathbb{C}^{\gamma-2}$ and $\mathbb{C}^{2\gamma-2}$ resp. and ξ has the law of the white noise on $\mathbb{T}^2$.

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = F(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2} \to \mathscr{C}^{\gamma}$$

is continuous for any $\gamma > 2/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by

$$\begin{cases} z = F(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (F'(z)F(z))\color{red}{\varphi}\color{black} - \underbrace{\mathcal{L}F(z) \prec X}_{"\in"\mathscr{C}^{2\gamma-2}} + \underbrace{R(F, z, X, \xi) \circ \xi}_{\in\mathscr{C}^{3\gamma-2}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-2} \times \mathscr{C}^{2\gamma-2}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then $u^n \to u = \Phi(u_0, \xi, \eta)$.

Relaxed form of the RDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-2}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\mathcal{L}u = F(u)\xi + F'(u)F(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

"Itô" form of the RDE

In the smooth setting $u = \Phi(\xi, X \circ \xi + \varphi)$ solves

$$\mathcal{L}u = F(u)\xi + F'(u)F(u)\varphi.$$

If we choose $\varphi = -X \circ \xi$ then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\mathcal{L}v = F(v)\xi - F'(v)F(v)X \circ \xi$$

and has the particular property of being a continuous map of $\xi \in \mathscr{C}^{\gamma-2}$ alone.

The renormalization problem

If ξ is the space white noise we have

$$\xi \in \mathscr{C}^{-1-}, \qquad X \in C([0,T];\mathscr{C}^{1-})$$

and

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$
$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2$$

But now

$$\frac{1}{2}(DX)^2 = c + C\mathscr{C}^{0-}$$

with $c = +\infty$!.

No obvious definition of $X \circ \xi$ can be given. But there exists c_ε such that

$$X_\varepsilon \circ \xi_\varepsilon - c_\varepsilon \to "X \diamond \xi" \qquad \text{in } C\mathscr{C}^{0-}.$$

The renormalized gPAM

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denoted by $\diamond$. Now

$$f(u) \circ \xi - c(f'(u)f(u)) = (f'(u)f(u))(X \circ \xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c)$$
$$+ C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

together with $u = f(u) \prec X + u^\sharp$ and where

$$X \in \mathscr{C}^{1-}, \qquad X \diamond \xi = (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^\sharp \in \mathscr{C}^{2-}.$$

KPZ and its siblings:

Besides the generalized PAM, the following equations have been solved using the paracontrolled approach (joint work with N. Perkowski)

$\mathcal{L} = \partial_t - \Delta$ heat operator on $\mathbb{T}$, ξ space-time white noise;

- KPZ equation: $h\colon \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$,

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

- Burgers equation: $u = \partial_x h$;

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x);$$

- Stochastic Heat equation: $h = \log w$

$$\mathcal{L}w(t,x) = w(t,x)\xi(t,x).$$

Other applications

- Gubinelli, Imkeller, P. (2012): Multidimensional extension of Hairer's (2011) generalized Burgers equation ($\sigma - d/2 > 1/3$):

$$\partial_t u(t,x) = -(-\Delta)^\sigma u(t,x) + G(u(t,x))D_x u(t,x) + \xi(t,x);$$

- Catellier, Chouk (2013): Stochastic quantization equation ϕ^4_3 ($d = 3$):

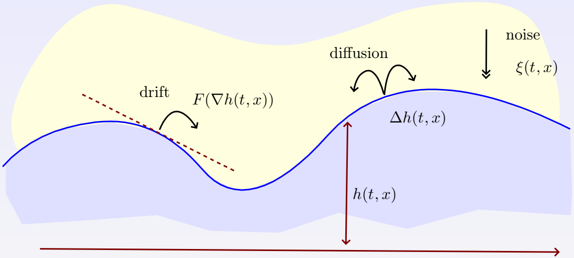
$$\mathcal{L}u(t,x) = -u(t,x)^{\diamond 3} + \xi(t,x);$$

- Furlan (2014): Stochastic Navier Stokes equation ($d = 3$):

$$\mathcal{L}u(t,x) = -P((u(t,x) \cdot \nabla)u(t,x)) + \xi(t,x).$$

Thanks

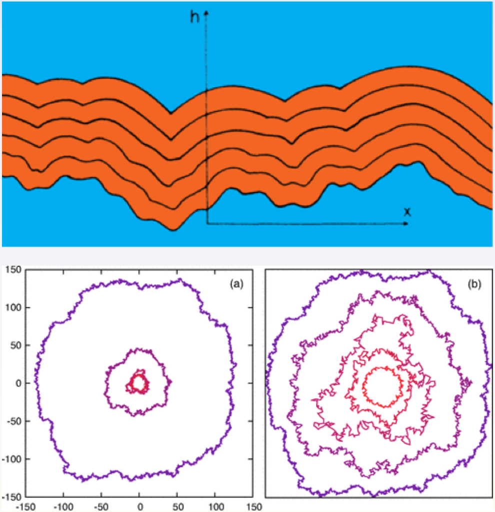
Fluctuations of a growing interface



A model for random interface growth (think e.g. expansion of colony of bacteria): $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{F(\partial_x h(t,x))}_{\text{slope-dependent growth}} + \underbrace{\eta(t,x)}_{\text{noise with microscopic correlations}}$$

Fluctuations of a growing interface



The Kardar–Parisi–Zhang equation

- Kardar–Parisi–Zhang '84: slope-dependent growth given by $F(\partial_x h)$, in a certain scaling regime of small gradients:

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + F''(0)(\partial_x h)^2 + \dots$$

- KPZ equation is the universal model for random interface growth

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{\lambda[(\partial_x h(t,x))^2 - \infty]}_{\text{renormalized growth}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- This derivation is highly problematic since $\partial_x h$ is a distribution. But: Hairer, Quastel (2014, unpublished) justify it rigorously via scaling of smooth models and small gradients.
- KPZ equation is suspected to be universal scaling limit for random interface growth models, random polymers, and many particle systems;
- contrary to Brownian setting: KPZ has fluctuations of order $t^{1/3}$; large time limit distribution of $t^{-1/3}h(t, t^{2/3}x)$ is expected to be universal in a sense comparable only to the Gaussian distribution.

KPZ and its siblings:

- KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

  $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- Burgers equation:

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x);$$

  solution is (formally) given by derivative of the KPZ equation: $u = \partial_x h$;

- solution to KPZ (formally) given by Cole-Hopf transform of the stochastic heat equation: $h = \log w$, where w solves

$$\mathcal{L}w(t,x) = w(t,x)\xi(t,x).$$

- All three are universal objects, that are expected to be scaling limits of a wide range of particle systems.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion /I

▷ Let $u = X + u_1$ then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

▷ Let $X^{\vee}$ be the solution to

$$\mathcal{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

▷ Define $\mathcal{L}X^{\Yup} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\Yup} + u_3$ then $X^{\Yup} \in \mathscr{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\Yup}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\curlyvee} + U$$

$$\mathcal{L}U = 2\partial_x(UX) + 2\partial_x(X^{\curlyvee}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\curlyvee}+U)X^{\vee}) + \partial_x(2X^{\curlyvee}+U)^2$$

$$= {\color{red}2\partial_x(UX)} + \mathcal{L}(2X^{\curlyvee\curlyvee} + X^{\vee\vee}) + 2\partial_x((2X^{\curlyvee} + U)X^{\vee}) + \partial_x(2X^{\curlyvee} + U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	$X^{\curlyvee}$	$X^{\curlyvee\curlyvee}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\curlyvee} + U)X^{\vee}) + \partial_x(2X^{\curlyvee} + U)^2$$

are well defined.

The remaining problem is to deal with ${\color{red}2\partial_x(UX)}$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Q + U^\sharp$. Then

$$\mathcal{L}U = \mathcal{L}U' \prec Q + U' \prec \mathcal{L}Q - \partial_x U' \prec \partial_x Q + LU^\sharp$$

while

$$\mathcal{L}U = 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\text{Ⴤ}} + X^{\text{ⴸ}}) + 2\partial_x((2X^{\text{Ⴤ}} + U)X^{\vee}) + \partial_x(2X^{\text{Ⴤ}} + U)^2}_{R(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

so we can set $U' = 2U$ and $\mathcal{L}Q = \partial_x X$ and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Q + \partial_x U' \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

▷ Observe that $Q, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Q) \circ X + U^\sharp \circ X = 2U(Q \circ X) + \underbrace{C(2U, Q, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Q \circ X \in \mathscr{C}^{0-}$

Paracontrolled solution to SBE

▷ The final system reads

$$u = X + X^{\vee} + 2X^{\Yright} + U$$

$$U = U' \prec Q + U^{\sharp}, \qquad U' = 2X^{\Yright} + 2U$$

$$\mathcal{L}U^{\sharp} = 4\partial_x(U\underline{(Q \circ X)}) + 4\partial_x C(U, Q, X) + 2\partial_x(U^{\sharp} \circ X) - 2\mathcal{L}U \prec Q$$
$$+2\partial_x U \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + R(U)$$

▷ This equation has a (local in time) solution $U = \Phi(J(\xi))$ which is a continuous function of the data $J(\xi)$ given by a collection of multilinear functions of ξ:

$$J(\xi) = (X, X^{\vee}, X^{\Yright}, X^{\Yright\Yright}, X^{\vee\vee}, X \circ Q)$$

Burgers equation and paracontrolled distributions

$$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled Ansatz

$u \in \mathscr{P}_{\mathrm{rbe}}$ if $u = X + X^{\vee} + 2X^{\vee\!\!\vee} + u^Q$ with

$$u^Q = \pi_<(u', Q) + u^\sharp.$$

- Paracontrolled structure: Can define u^2 continuously as long as $(Q \circ X) \in C([0,T], \mathscr{C}^{0-})$ is given (together with tree data $X, X^{\vee}, X^{\vee\!\!\vee}, X^{\vee\!\!\vee\!\!\vee}, X^{\vee\vee}$).
- Obtain local existence and uniqueness of paracontrolled solutions. Solution depends pathwise continuously on extended data $J(\xi) = (\xi, X, X^{\vee}, X^{\vee\!\!\vee}, X^{\vee\!\!\vee\!\!\vee}, X^{\vee\vee}, Q \circ X)$.

KPZ equation

KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x), \qquad h(0) = h_0.$$

Expect $h(t) \in \mathscr{C}^{1/2-}$, so $\partial_x h(t) \in \mathscr{C}^{-1/2-}$ and $(\partial_x h(t))^2$ not defined.
But: expand

$$u = Y + Y^{\vee} + 2Y^{\curlyvee} + h^P,$$

where $\mathcal{L}Y = \xi$, $\mathcal{L}Y^{\vee} = \partial_x Y \partial_x Y, \ldots$ In general: $\partial_x Y^\tau = X^\tau$. Make paracontrolled ansatz for h^P:

$$h^P = \pi_<(h', P) + h^\sharp$$

with $h' \in C([0,T], \mathscr{C}^{1/2-})$, $h^\sharp \in C([0,T], \mathscr{C}^{2-})$, $\mathcal{L}P = X$. Write $h \in \mathscr{P}_{\mathrm{kpz}}$.

Can define $(\partial_x h(t))^2$ for $h \in \mathscr{P}_{\mathrm{kpz}}$ and obtain local existence and uniqueness of solutions.

KPZ and Burgers equation

$h \in \mathscr{P}_{\mathrm{kpz}}$ if

$$h = \Upsilon + \Upsilon^{\vee} + 2\Upsilon^{\vee\!\!\vee} + h^P, \qquad h^P = h' \prec P + h^\sharp.$$

$u \in \mathscr{P}_{\mathrm{rbe}}$ if

$$u = X + X^{\vee} + 2X^{\vee\!\!\vee} + u^Q, \qquad u^Q = u' \prec Q + u^\sharp.$$

- If $h \in \mathscr{P}_{\mathrm{kpz}}$, then $\partial_x h \in \mathscr{P}_{\mathrm{rbe}}$.
- If h solves KPZ equation, then $u = \partial_x h$ solves Burgers equation with initial condition $u(0) = \partial_x h_0$.
- If $u \in \mathscr{P}_{\mathrm{rbe}}$, then any solution h of $\mathcal{L}h = u^2 + \xi$ is in $\mathscr{P}_{\mathrm{kpz}}$.
- If u solves Burgers equation with initial condition $u(0) = \partial_x h_0$, and h solves $\mathcal{L}h = u^2 + \xi$ with initial condition $h(0) = h_0$, then h solves KPZ equation.

KPZ and heat equation

Heat equation:

$$\mathcal{L}w(t,x) = w(t,x) \diamond \xi(t,x) = w(t,x)\xi(t,x) - w(t,x)\cdot\infty, \quad w(0) = w_0.$$

Paracontrolled ansatz: $w \in \mathscr{P}_{\text{rhe}}$ if

$$w = e^{Y + Y^{\vee} + 2Y^{\vee\!\!\vee}} w^P, \qquad w^P = \pi_<(w', P) + w^\sharp$$

(comes from Cole-Hopf transform).

- Slightly cheat to make sense of product $w \diamond \xi$ for $w \in \mathscr{P}_{\text{rhe}}$:

$$w \diamond \xi = \mathcal{L}w - e^{Y+Y^{\vee}+2Y^{\vee\!\!\vee}}\left[\mathcal{L}w^P - [\mathcal{L}(Y^{\vee} + Y^{\vee\!\!\vee}) + (\partial_x(Y + Y^{\vee} + 2Y^{\vee\!\!\vee}))^2]w^P\right]$$
$$+ 2e^{Y+Y^{\vee}+2Y^{\vee\!\!\vee}}\partial_x(Y + Y^{\vee} + 2Y^{\vee\!\!\vee})\partial_x w^P;$$

  (agrees with renormalized pointwise product $w \diamond \xi$ in smooth case and with Itô integral in white noise case, continuous in extended data).
- Obtain global existence and uniqueness of solutions.
- One-to-one correspondence between $\mathscr{P}_{\text{kpz}}$ and strictly positive elements of $\mathscr{P}_{\text{rhe}}$.
- Any solution of KPZ gives solution of heat equation. Any strictly positive solution of heat equation gives solution of KPZ equation.

Para-modelled distributions

Let $\gamma > 0$ and (T, Π, Γ) regularity structure. Say f is para-modelled, $f \in \mathscr{P}^\gamma$, if there exists $f^\pi \in \mathscr{D}^\gamma$, with

$$f - \pi_<(f^\pi, \Pi) \in C^\gamma.$$

Example: $\mathscr{R}f^\pi \in \mathscr{P}^\gamma$.
Consider rough path model, say
$T = \mathrm{span}(\Xi, \mathscr{I}(\Xi)\Xi, \mathscr{I}(\mathscr{I}(\Xi)\Xi)\Xi, \mathbf{1}, \mathscr{I}(\Xi), \mathscr{I}(\mathscr{I}(\Xi)\Xi))$. Try to solve $\partial_t u = F(u)\xi$.
(Simplified) para-modelled ansatz: $u = \mathscr{R}u^\pi = \pi_<(u^\pi, \Pi) + u^\sharp$ with $u^\pi \in \mathscr{D}^{3\alpha}$. Equation for $u^\sharp$:

$$\partial_t u^\sharp = -\partial_t \pi_<(u^\pi, \Pi) + F(u)\xi = \pi_<(u^\pi, D\Pi) - \pi_<(F(u^\pi) \star \xi^\pi, \Pi) + \text{smooth}.$$

To have $u^\sharp \in C^{3\alpha}$: choose expansion u^π so that all coefficients for terms of homogeneity $< 3\alpha - 1$ cancel. Obtain a priori bounds on $\|u^\sharp\|_{3\alpha}$ and then on $\|u^\pi\|_{\mathscr{D}^{3\alpha}}$. Thus at least local existence of solutions.

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3$: $\xi \in C^{-5/2-}$, $u \in C^{-1/2-}$,
$u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 - c_1$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects:
$(u_1^2 - c_1), (u_1^3 - 3c_1 u_1), (3\lambda(X \circ (u_1^2 - c_1)) - c_2), (u_2 u_1), (u_2^2 u_1)$