Regularisation by noise in PDEs

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Stochastic Analysis workshop – Tohoku University, October 2014.

Regularization by noise in ODEs/PDEs:

Addition of noise has positive effects on the theory of the equation (in some pathwise sense)

→ ODEs:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}s + W_t$$

where (W_t) is a BM in $\mathbb{R}^d$ and b a less-than-Lipshitz vectorfield. Many results: Veretennikov, Davie, Krylov-Röckner, Flandoli, Attanasio, Fedrizzi, Proske, ... Essentially: bounded b: (in L^∞ or with some particular integrability: LPS condition).

→ Transport equation:

$$\mathrm{d}_t u(t,x) + b(x)\cdot\nabla u(t,x)\mathrm{d}t = \nabla u(t,x)\cdot \mathrm{d}W_t$$

good theory for L^∞ solutions and preservation of regularity. Flandoli–G.–Priola, Flandoli–Attanasio, Flandoli–Maurelli, Flandoli–Beck–G.–Maurelli

→ Some other PDE: Vlasov–Poisson, point vortices in 2d.

We want to provide a deterministic framework to discuss regularization by "perturbations/modulation" for the following model PDEs:

- **Transport equation:** $x \in \mathbb{R}^d$, $t \geqslant 0$, $w: \mathbb{R} \to \mathbb{R}^d$, $b: \mathbb{R}^d \to \mathbb{R}^d$

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) + b(x) \cdot \nabla u(t,x) = 0, \qquad u(0,\cdot) = u_0.$$

- **Non-linear Schrödinger equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$, $w: \mathbb{R} \to \mathbb{R}$

$$\partial_t \varphi(t,x) = i \Delta \varphi(t,x) \dot{w}_t + i |\varphi(t,x)|^{p-2} \varphi(t,x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$, $w: \mathbb{R} \to \mathbb{R}$

$$\partial_t u(t,x) = \partial_x^3 u(t,x) \dot{w}_t + \partial_x (u(t,x))^2.$$

Joint work with Remi Catellier and Khalil Chouk.

Consider the linear transport PDE

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) = f(x), \qquad u(0,\cdot) = 0.$$

Solutions are give explicitly by

$$u(t,x) = \int_0^t f(x + w_s - w_t)\mathrm{d}s = T_t^w f(x - w_t)$$

where given a function $w\colon [0,1] \to \mathbb{R}^d$ we define the **averaging operator**

$$T_t^w f(x) = \int_0^t f(x + w_s)\mathrm{d}s, \qquad T_{t,s}^w f = T_t^w f - T_s^w f$$

acting on functions (or distributions) $f\colon \mathbb{R}^d \to \mathbb{R}$.

Question: What is the relation between w, the (space) regularity of f and that of $u(t,\cdot)$?

If w is smooth we do not expect anything special to happen and u to have the same regularity of f.

▷ $d = 1$, $w_t = t$. Then if $F'(x) = f(x)$ we have $T_t^w f(x) = \int_0^t F'(x+s)\mathrm{d}s = F(x+t) - F(x)$ and $T^w \colon L^\infty \to \mathrm{Lip}$:

$$|T_t^w f(x) - T_t^w f(y)| \leqslant \|f\|_\infty |x-y|, \qquad |T_{t,s}^w f(x)| \leqslant \|f\|_\infty |t-s|$$

▷ Tao–Wright: if w "wiggles enough" then T_t^w maps L^q into $L^{q'}$ with $q' > q$.

▷ Davie: if w is a sample of BM then a.s. (the exceptional set depends on f)

$$|T_{t,s}^w f(x) - T_{t,s}^w f(y)| \leqslant C_w \|f\|_\infty |x-y|^{1-} |t-s|^{1/2-}$$

Problem: study the mapping properties of T^w for w the sample path of a stochastic process.

Consider

$$Y_t^w(\xi) = \int_0^t e^{i\langle \xi, w_s\rangle} \mathrm{d}s$$

then $T_t^w f = \mathcal{F}^{-1}(Y_t^w \mathcal{F}(f))$. Mapping properties of T^w in $(H^s)_{s\in\mathbb{R}}$ spaces can be discussed in terms of Y^w:

$$\|T_{t,s}^w f\|_{H^s} = \left\|(1+\xi^2)^{s/2} Y_{t,s}^w(\xi) \mathcal{F} f(\xi)\right\|_{H_\xi^s}.$$

In our setting more convenient to look at the scale $(\mathcal{F}L^\alpha)_\alpha$:

$$\|f\|_{\mathcal{F}L^\alpha} = \int |f(\xi)|(1+\xi^2)^{\alpha/2} \mathrm{d}\xi$$

since $C^\alpha \subseteq \mathcal{F}L^\alpha$.

Definition 1 **(Catellier–G.)** *We say that w is (ρ,γ)–irregular if there exists a constant K for which*

$$|Y_{t,s}^w(\xi)| \leqslant K(1+|\xi|)^{-\rho}|t-s|^\gamma$$

for $\xi \in \mathbb{R}^d$ and $0 \leqslant s \leqslant t \leqslant 1$.

Theorem 2 *The fBM of Hurst index H is ρ–irregular for any $\rho < 1/2H$.*

$\Rightarrow$ there exists functions of arbitrarily high irregularity and arbitrarily L^∞-near any given continuous function.

Lemma 3 *An irregular function cannot be too regular.*

Proof. If $w \in C^\theta$ with $\alpha\theta + \gamma > 1$ and $\alpha \in [0,1]$, using the Young integral, we find

$$|t-s| = |e^{ia}(t-s)| = \left| \int_s^t \underbrace{e^{ia - iaw_r}}_{C^{\alpha\theta}} \mathrm{d}_r \underbrace{Y_r^w(a)}_{C^\gamma} \right|$$

$$\leqslant C K_w (|t-s|^\gamma + |t-s|^{\alpha\theta+\gamma} |a|^\alpha) \|w\|_\theta (1+|a|)^{-\rho} \rightarrow 0$$

if $t > s$ and $\alpha < \rho$. This implies that is not possible that $\theta > (1-\gamma)/\rho$.

▷ Not easy to say if a function is irregular.

▷ In $d=1$ smooth functions are (ρ,γ) irregular for $\rho+\gamma=1$. In particular if we insist on $\gamma>1/2$ we have $\rho<1/2$.

▷ For $d>1$ smooth functions are not irregular: if $|t-s|\ll 1$

$$\int_s^t e^{i\langle a,w_r\rangle}\mathrm{d}r \simeq \int_s^t e^{i\langle a,w_s'\rangle(t-s)}\mathrm{d}r \simeq (1+|\langle a,w_s'\rangle|)^{-1} \not\lesssim (1+|a|)^{-\rho}.$$

▷ If w is ρ–irregular and φ is a C^1 perturbation then $w+\varphi$ is at least $\rho-(1-\gamma)$ irregular since:

$$Y_{t,s}^{w+\varphi}(\xi) = \int_s^t e^{i\langle \xi, w_r+\varphi_r\rangle}\mathrm{d}r = \int_s^t e^{i\langle \xi,\varphi_r\rangle}\mathrm{d}_r Y_{s,r}^{w}(\xi)$$

and we can use Young integral estimates.

▷ If W is a fBM and Φ an adapted smooth perturbation then $W+\Phi$ is as irregular as W (via Girsanov theorem).

Theorem 4 *If w is ρ–irregular then*

$$T^w\colon H^s \to H^{s+\rho}$$

and

$$T^w\colon \mathcal{F}L^\alpha \to \mathcal{F}L^{\alpha+\rho}.$$

Proof. Indeed

$$\|T^w_{t,s} f\|_{\mathcal{F}L^{\alpha+\rho}} = \int \mathrm{d}\xi\, (1+|\xi|)^{\alpha+\rho} |Y^w_{t,s}(\xi)(\mathcal{F}f)(\xi)|$$

$$\leqslant K_w |t-s|^\gamma \int \mathrm{d}\xi\, (1+|\xi|)^\alpha |(\mathcal{F}f)(\xi)| = K_w |t-s|^\gamma \|f\|_{\mathcal{F}L^\alpha}.$$

Remark 5 More difficult to understand the mapping properties in other spaces, for example Hölder spaces C^α. Only partial results available.

▷ Consider the transport equation with a perturbation:

$$\partial_t u(t,x) + \dot{w}_t \cdot \nabla u(t,x) + b(x) \cdot \nabla u(t,x) = 0, \qquad u(0,\cdot) = u_0.$$

▷ In the Lipshitz case there is only one solution u given by the method of characteristics:

$$u(t,x) = u_0(\phi_t^{-1}(x))$$

where $\phi_t(x) = x_t$ is the flow of the ODE

$$\begin{cases} \dot{x}_t = b(x_t) + \dot{w}_t \\ x_0 = x \end{cases}$$

▷ Uniqueness of solutions is related to the uniqueness (and smothness) theory of the flow.

ODEs and the averaging operator

In order to exploit the averaging properties of w in the study of the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

we rewrite it in order to make the action of the averaging operator explicit: let $\theta_t = x_t - w_t$:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s)\mathrm{d}s = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

where $G_s(x) = T_s^w b(x)$ so that $\mathrm{d}_s G_s(x) = f(w_s + x)$.

If we assume that G is C^γ in time ($\gamma > 1/2$) with values in a space of regular enough functions we can study this equation as a Young type equation for $\theta \in C^\gamma$.

▷ **Non-linear Young integral:**

$$\int_0^t (\mathrm{d}_s G_s)(\theta_s) = \lim_{\Pi} \sum_i G_{t_{i+1}, t_i}(\theta_{t_i})$$

This limit exists if $\theta \in C_t^\gamma$ and $G \in C_t^\gamma C_x^\nu$ with $\gamma(1 + \nu) > 1$. The integral is in C_t^γ.

Theorem 6 *The integral equation*

$$\theta_t = \theta_0 + \int_0^t (\mathrm{d}_s G_s)(\theta_s)$$

is well defined for $\theta \in C^\gamma$ *and* $G \in C_t^\gamma C_{x,\mathrm{loc}}^\nu$ *with* $(1+\nu)\gamma > 1$.

- *Existence of global solutions if* G *of linear growth.*
- *Uniqueness if* $G \in C_t^\gamma C_{x,\mathrm{loc}}^{\nu+1}$ *and differentiable flow.*
- *Smooth flow if* $G \in C_t^\gamma C_x^{\nu+k}$.

Theorem 7 *The equation*

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

has a unique solution for w ρ*–irregular and* $b \in \mathcal{F}L^\alpha$ *for* $\alpha > 1 - \rho$*. In this case we can take* $\theta \in C^1$ *above and the condition for uniqueness (and Lipshitz flow) is* $G \in C_t^\gamma C_x^{3/2}$.

▷ Say that x is controlled by w if $\theta = x - w \in C^{\gamma}$. In this case we have

$$I_x(b) = \int_0^t b(x_s)\mathrm{d}s = \int_0^t (\mathrm{d}_s T_s^w b)(\theta_s)$$

and the r.h.s. is well defined as soon as $T^w b \in C_t^{\gamma} C_x^{\nu}$.

▷ If w is ρ irregular and $b \in \mathcal{F}L^{\alpha}$ then $T^w b \in C_t^{\gamma} \mathcal{F}L_x^{\alpha+\rho}$ so if $\alpha + \rho \geqslant \nu$ we have $T^w b \in C_t^{\gamma} C_x^{\nu}$.

In this case $I_x(b)$ can be extended by continuity to all $b \in \mathcal{F}L^{\alpha}$ and in particular we have given a meaning to

$$\int_0^t b(x_s)\mathrm{d}s$$

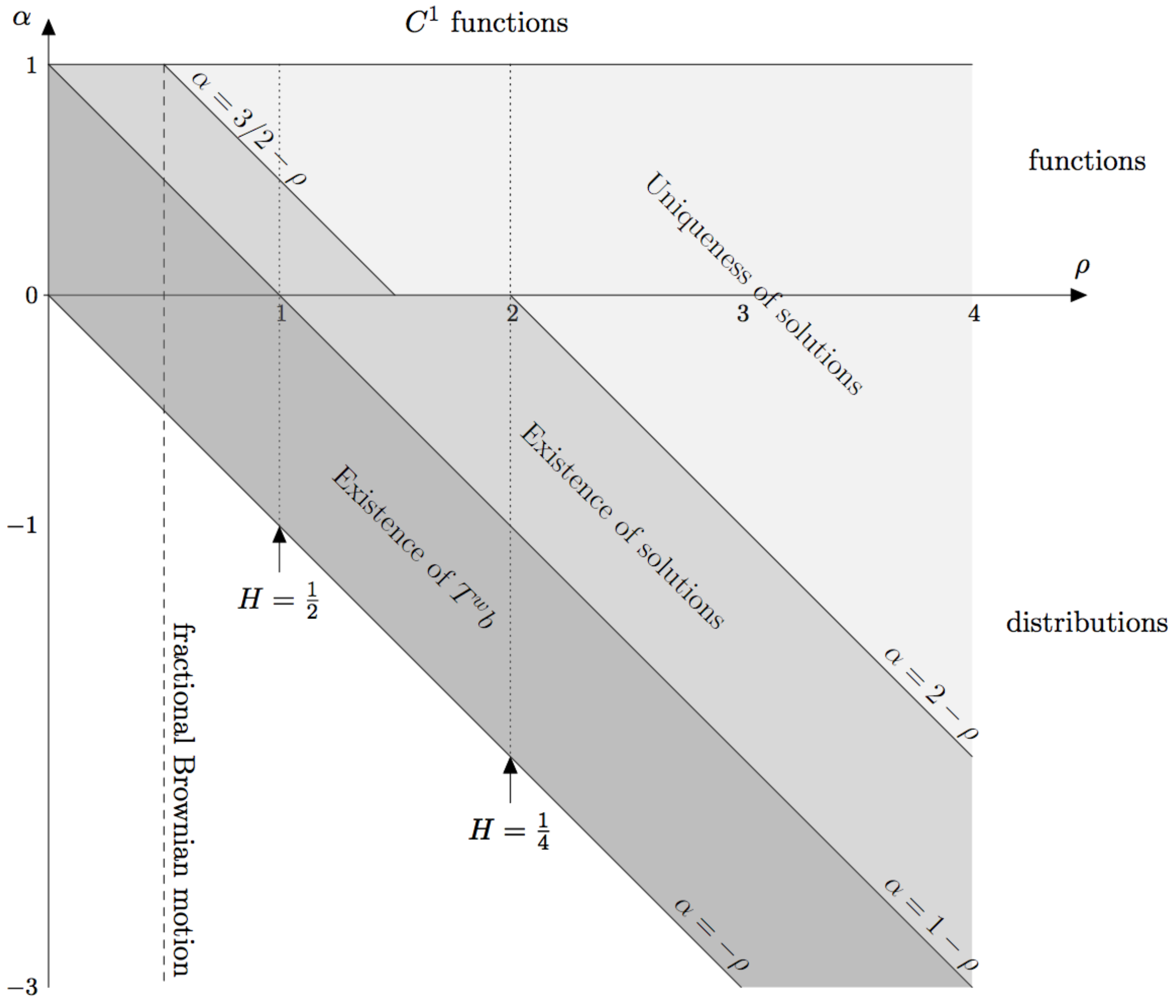
when b is a distribution *provided* x is controlled by a ρ-irregular path.

▷ For controlled paths the ODE

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

make sense even for certain distributions b as a Young equation for θ.

Regularization of ODEs at a glance



(joint work with R. Catellier)

We want to give a meaning and study the uniqueness issue for the transport equation

$$(\partial_t + b(x) \cdot \nabla + \dot{w}_t \cdot \nabla) u(t, x) = 0$$

for $u \in L^\infty$ and $w \in C^\sigma$ with $\sigma > 1/3$ such that $(w, \mathbb{W})$ is a geometric σ-Hölder rough path such that w is ρ-irregular. For the moment only in the case $\operatorname{div} b = 0$.

▷ **Weak formulation:** We consider u as a distribution: $u_t(\varphi) = \int \mathrm{d}x \, \varphi(x) \, u(t, x)$ for all $\varphi \in L^1(\mathbb{R}^d)$. The integral formulation of the equation is

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi)) \mathrm{d}r + \int_s^t u_r(\nabla \varphi) \mathrm{d}_r w_r$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $0 \leqslant s \leqslant t$.

We need to give a meaning to such an integral equation in order to discuss the regularization by noise phenomenon. (No way out!)

▷ It is possible via the theory of **controlled rough paths** (G. JFA 2004).

Let $(X, \mathbb{X})$ be a σ-Hölder rough path with $\sigma > 1/3$:

$$\mathbb{X}_{t,s} = \mathbb{X}_{t,u} + \mathbb{X}_{u,s} + (X_t - X_u) \otimes (X_u - X_s), \qquad |X_t - X_s| + |\mathbb{X}_{s,t}|^{1/2} = O(|t-s|^\sigma)$$

▷ We say that $y \in C_t^\sigma$ is **controlled by $\boldsymbol{X}$** if there exists $y^X \in C_t^\sigma$ such that

$$y_t - y_s - y_s^X (X_t - X_s) =: y_{s,t}^\sharp = O(|t-s|^{2\sigma}).$$

▷ For a controlled path y we can define the integral against X by compensated Riemman sums:

$$I_t = \int_0^t y_s \mathrm{d}X_s := \lim_\Pi \sum_i y_{t_i}(X_{t_{i+1}} - X_{t_i}) + y_{t_i}^X \mathbb{X}_{t_{i+1},t_i}$$

▷ This integral is the only function (up to constants) which has the following property

$$I_t - I_s = y_s(X_t - X_s) + y_s^X \mathbb{X}_{t,s} + O(|t-s|^{3\sigma}).$$

In particular, the integral is itself controlled by X and $I^X = y$.

Definition 8 *We say that u is a function controlled by w if for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$u_t(\varphi) - u_s(\varphi) = u_s^w(\varphi)(w_t - w_s) + u_{t,s}^\sharp(\varphi)$$

where $u_\cdot^w(\varphi) \in C^\sigma$ and $|u_{t,s}^\sharp(\varphi)| \lesssim |t-s|^{2\sigma}$.

Definition 9 *If u is controlled we say that it is a L^∞ solution of the rough transport equation (RTE) if*

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))\mathrm{d}r + \int_s^t u_r(\nabla\varphi)\mathrm{d}_r w_r$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $0 \leqslant s \leqslant t$.

Remark: If $\sigma > 1/2$ we can just assume that $u_t(\nabla\varphi) \in C_t^\sigma$ so that the rough integral becomes a Young integral.

Equivalently, u is a solution to the RTE iff

$$u_t(\varphi) - u_s(\varphi) = \int_s^t u_r(\nabla \cdot (b\varphi))\mathrm{d}r + u_s(\nabla\varphi)(w_t - w_s) + u_s(\nabla^2\varphi)\mathbb{W}_{t,s} + O(|t-s|^{3\sigma})$$

Lemma 10 *If b is Lipshitz there exists a solution to the RTE given by $u(t,x) = u_0(\phi_t^{-1}(x))$.*

Proof. The proof proceed by approximation of $(w, \mathbb{W})$ by $(w^\varepsilon, \mathbb{W}^\varepsilon)$ and by stability of the flow. Let ϕ^ε be the approximate flow, then $u_t^\varepsilon(\varphi) = \int_{\mathbb{R}^d} u_0(\phi_t^{\varepsilon,-1}(x))\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^\mathrm{d}} u_0(x)\varphi(\phi_t^\varepsilon(y))\mathrm{d}y$. Taylor expansion gives

$$\varphi(\phi_t^\varepsilon(y)) = \varphi(\phi_s^\varepsilon(y)) + \int_s^t \nabla\varphi(\phi_r^\varepsilon(y))b(\phi_r^\varepsilon(y))\mathrm{d}r + \nabla\varphi(\phi_s^\varepsilon(y))(w_t^\varepsilon - w_s^\varepsilon) + O_\varphi(|t-s|^{2\sigma})$$

That is $u_t^\varepsilon(\varphi) = u_s^\varepsilon(\varphi) + u_s^\varepsilon(\nabla\varphi)(w_t^\varepsilon - w_s^\varepsilon) + O_\varphi(|t-s|^{2\sigma})$. By weak compactness it is possible to pass to the limit (along a subsequence) in this equation and obtain a controlled path $u = \lim_{\varepsilon_k} u_{\varepsilon_k}$.

Uniqueness is proven by showing via a direct computation that

$$t \mapsto \int_{\mathbb{R}^d} u(t, \phi_t(x))\rho(x)\mathrm{d}x = u_t(\rho \circ \phi_t^{-1})$$

is a constant function of t for all $\rho \in \mathcal{S}(\mathbb{R}^d)$. This implies that $u(t, \phi_t(x)) = u_0(x)$. Uniqueness depends only on the Lipschitz property of the flow.

Theorem 11 *Let $b \in \mathcal{F}L^{\alpha}$ for $\alpha > 0$ and $\alpha + \rho > 3/2$ and let w be ρ-irregular. Then there exists a unique solution to the RTE given by the method of characteristics.*

Proof. Approximate b by b_{ε}, then by the previous theorem there exists a unique solution u_{ε} to the RTE. Analysis of the approximate flow ϕ_{ε} shows that this solution converges to a controlled solution u of the RTE with vectorfield b. Since ϕ is Lipschitz we can prove again uniqueness. □

Remark 12 The above result is path-wise. In particular b can depend on w.

Remark 13 If $b \in C^{\alpha}$, b deterministic and w is a fBm of Hurst index H then the uniqueness holds almost surely when $\alpha > 1 - 1/(2H)$ and $\alpha > 0$. This recovers the results of Flandoli–Gubinelli–Priola for the Brownian case but extend them well beyond the Brownian context.

(joint work with K. Chouk)

Two simple dispersive models with ρ-irregular modulation w:

- **Non-linear Schödinger equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$

$$\partial_t \varphi(t,x) = i\Delta\varphi(t,x)\partial_t w_t + i|\varphi(t,x)|^{p-2}\varphi(t,x).$$

- **Korteweg–de Vries equation:** $x \in \mathbb{T}, \mathbb{R}$, $t \geqslant 0$

$$\partial_t u(t,x) = \partial_x^3 u(t,x)\partial_t w_t + \partial_x (u(t,x))^2.$$

To be compared to the non-modulated setting where $\partial_t w_t = 1$ and studied in the scale of $(H^s)_s$ spaces.

The equations are understood in the mild formulation

$$u(t) = \mathcal{U}_t^w u(0) + \int_0^t \mathcal{U}_t^w (\mathcal{U}_s^w)^{-1} \partial_x (u(s))^2 \mathrm{d}s.$$

with $\mathcal{U}_t^w = e^{i w_t \partial_x^3}$. (similarly for NLS). Here w can be an arbitrary continuous function.

Rewrite the mild formulation as

$$v(t) = (\mathcal{U}_t^w)^{-1} u(t) = u(0) + \int_0^t (\mathrm{d}_s X_s)(v(s))$$

where X is the bi-linear operator

$$X_t(\varphi) = X_t(\varphi, \varphi) = \int_0^t (\mathcal{U}_s^w)^{-1} \partial_x (\mathcal{U}_s^w \varphi)^2 \mathrm{d}s.$$

If w is ρ irregular then $X \in C^\gamma \mathrm{Lip}_{\mathrm{loc}}(H^\alpha)$ for $\alpha > -\rho$ and $\rho > 3/4$.

The above equation has local solutions for initial conditions in H^α with locally Lipshitz flow. Uniqueness in $C^\gamma H^\alpha$ (for v).

$\Rightarrow$ Regularization by modulation. In the non-modulated case it is known that there cannot be continous flow for $\alpha \leqslant -1/2$ on $\mathbb{T}$ and $\alpha \leqslant -3/4$ on $\mathbb{R}$.

$\triangleright$ Global solutions thanks to the L^2 conservation and smoothing for $\alpha > 0$ or an adaptation of the I-method for $-3/2 \leqslant \alpha < 0$ and $\alpha > -\rho/(3 - 2\gamma)$.

$\triangleright$ **NLS:** global solutions for $\alpha \geqslant 0$ and $\rho > 1/2$.

A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same ρ-irregularity assumption.

Theorem 14 *Let $T > 0$, $p \in (2, 5]$,$\rho > \min\left(\frac{3}{2} - \frac{2}{p}, 1\right)$ then there exists a finite constant $C_{w,T} > 0$ and $\gamma^\star(p) > 0$ such that the following inequality holds:*

$$\left|\left|\int_0^\cdot U_\cdot (U_s)^{-1}\,\psi_s\,d\,s\right|\right|_{L^p([0,T],L^{2\,p}(\mathbb{R}))} \leq C_w T^{\gamma^\star(p)} ||\,\psi\,||_{L^1([0,T],L^2(\mathbb{R}))}$$

for all $\psi \in L^1([0,T], L^2(\mathbb{R}))$.

As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity $i\,e$: $\mathcal{N}(\phi) = |\phi|^{\mu}\,\phi$: (Debussche–de Bouard, Debussche–Tsutsumi)

Theorem 15 *Let $\mu \in (1, 4]$, $p = \mu + 1$, $\rho > \min\left(1, 3/2 - \frac{2}{p}\right)$ and $u^0 \in L^2(\mathbb{R})$ then there exists $T^\star > 0$ and a unique $u \in L^p([0,T], L^{2p}(\mathbb{R}))$ such that the following equality holds:*

$$u_t = U_t\, u^0 + i \int_0^t U_t (U_s)^{-1} \left(|u_s|^{\mu}\, u_s \right) d\,s$$

for all $t \in [0, T^\star]$. Moreover we have that $||\,u_t||_{L^2(\mathbb{R})} = ||\,u_0||_{L^2(\mathbb{R})}$ and then we have a global unique solution $u \in L^p_{loc}([0,+\infty), L^{2p}(\mathbb{R}))$ and $u \in C([0,+\infty), L^2(\mathbb{R}))$. If $u^0 \in H^1(\mathbb{R})$ then $u \in C([0,\infty), H^1(\mathbb{R}))$.

Thanks.