



Weak universality and singular SPDEs



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- The analysis of scaling limits of stochastic non-linear diffusion problems generates irregular random fields which should be described by *universal* non-linear SPDEs (i.e. independent of specific details of the microscopic model).
- The combination of irregularity and non-linearity is problematic and can generate unexpected phenomena which escape a purely analytical control: the statistical structure of the noise has to be carefully taken into account.
- Hairer's regularity structures provide a general tool to do so: they allow to describe the local features of the random fields in terms of simpler objects. The effect of non-linear operations is then more easily understood given the improved description.
- In a parallel work, G.-Imkeller-Perkowski exploited tools from harmonic analysis to perform a similar kind of analysis (Fourier-space counterpart of the regularity structure business)
- *Paracontrolled distributions* are not as general as regularity structures but they provide an alternative approach in many relevant cases: KPZ, $\Phi_{2,3}^4$, Parabolic/Hamiltonian Anderson model in $2d$.

Talk based on joint work with: R. Catellier, K. Chouk, P. Imkeller, N. Perkowski, M. Furlan.

- ▷ G.-Imkeller-Perkowski, *Paracontrolled distributions and singular PDEs* (2012)
- ▷ G.-Perkowski, *Lectures on singular stochastic PDEs* (2015), *KPZ reloaded* (2016), *An introduction to singular SPDEs* (2017)
- ▷ Catellier-Chouk, *Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation* (2013)

Other relevant literature (not exhaustive!):

- ▷ Cannizzaro-Chouk (2015). Diffusions with singular random drift via singular paracontrolled martingale problems.
- ▷ Allez-Chouk (2016). Paracontrolled analysis of the Anderson Hamiltonian in $d = 2$.
- ▷ Work of Bailleul and Bernicot on higher order paracontrolled calculus, Weber and Mourrat on weak universality and global solution to SQE.
- ▷ Hairer's Inventiones paper.
- ▷ Hairer–Quastel (2015). Weak universality for **KPZ**.
- ▷ Hairer–Xu (2016). Weak universality for Φ_3^4 .

▷ Scalar diffusion equation with slow reaction term

$$\mathcal{L}\psi(t, x) = \varepsilon^\gamma F(\psi(t, x)) + \eta(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}_\varepsilon^d$$

with $\mathbb{T}_\varepsilon = \mathbb{T}/\varepsilon$, F odd (typically $F = -\partial U$ for some potential U) and $\mathcal{L} = \partial_t - \Delta$.

▷ Parabolic rescaling $\psi_\varepsilon(t, x) = \varepsilon^{-\alpha} \psi(t/\varepsilon^2, x/\varepsilon)$ and $\xi_\varepsilon(t, x) = \varepsilon^{-d/2-1} \eta(t/\varepsilon^2, x/\varepsilon)$

$$\mathcal{L}\psi_\varepsilon(t, x) = \varepsilon^{\gamma-\alpha-2} F(\varepsilon^\alpha \psi_\varepsilon(t, x)) + \varepsilon^{(d-2)/2-\alpha} \xi_\varepsilon(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}_\varepsilon^d$$

Note that $\xi_\varepsilon \rightarrow \xi$ the space–time white noise.

▷ Let $\alpha = (d-2)/2$ to keep a noisy evolution:

$$\mathcal{L}\psi_\varepsilon = \varepsilon^{\gamma-d/2-1} F(\varepsilon^{(d-2)/2} \psi_\varepsilon) + \xi_\varepsilon$$

In the following we will concentrate on the $d = 3$ case.

▷ **Linear approximation:** assume $\psi_\varepsilon \simeq X_\varepsilon$ with $\mathcal{L}X_\varepsilon = \xi_\varepsilon$

$$\varepsilon^{\gamma-d/2-1}F(\varepsilon^{(d-2)/2}\psi_\varepsilon) \simeq \varepsilon^{\gamma-d/2-1}F(\varepsilon^{(d-2)/2}X_\varepsilon)$$

▷ Explicit Gaussian computations shows that

$$\varepsilon^{-(d-2)/2}F(\varepsilon^{(d-2)/2}X_\varepsilon) \rightarrow \mu X \quad (\text{as space-time distributions})$$

where $\mu = \mathbb{E}[GF(G)]$ and $G \sim \mathcal{N}(0, c)$ and $\mathcal{L}X = \xi$. Then

$$\varepsilon^{\gamma-d/2-1}F(\varepsilon^{(d-2)/2}\psi_\varepsilon) \simeq \varepsilon^{\gamma-2}\mu X \simeq \varepsilon^{\gamma-2}\mu \psi$$

▷ If $\mu \neq 0$ we need to take $\gamma = 2$. The limiting equation is

$$\mathcal{L}\psi = \mu\psi + \xi$$

Result: we *expect* Gaussian fluctuations.

Rigorous analysis. Let $\psi_\varepsilon = X_\varepsilon + \theta_\varepsilon$ then (recall $d=3, \gamma=2$)

$$\mathcal{L}\theta_\varepsilon = \varepsilon^{-1/2}F(\varepsilon^{1/2}X_\varepsilon + \varepsilon^{1/2}\theta_\varepsilon)$$

▷ Taylor expansion gives the approximate equation

$$\mathcal{L}\theta_\varepsilon = \mu\tilde{X}_\varepsilon + \tilde{\mu}_\varepsilon\theta_\varepsilon + \varepsilon^{1/2}R_\varepsilon(\theta_\varepsilon)$$

with

$$\mu\tilde{X}_\varepsilon = \varepsilon^{-1/2}F(\varepsilon^{1/2}X_\varepsilon), \quad \tilde{\mu}_\varepsilon = F'(\varepsilon^{1/2}X_\varepsilon), \quad R_\varepsilon(\theta_\varepsilon) = \theta_\varepsilon^2 \int_0^1 F''(\varepsilon^{1/2}X_\varepsilon + \tau\varepsilon^{1/2}\theta_\varepsilon)(1-\tau)d\tau$$

▷ Since $\|\varepsilon^{1/2}X_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-0-}$ establish easily that

$$\varepsilon^{1/2}R_\varepsilon(\theta_\varepsilon) \xrightarrow{L^\infty} 0$$

▷ Direct Gaussian computations show

$$\tilde{\mu}_\varepsilon \xrightarrow{\mathcal{L}^{0-}} \mu \quad X_\varepsilon, \tilde{X}_\varepsilon \xrightarrow{\mathcal{L}^{-1/2-}} X \quad \varepsilon^{1/2}R_\varepsilon(\theta_\varepsilon) \xrightarrow{L^\infty} 0$$

$$\mathcal{L}\theta_\varepsilon = \mathfrak{M}_\varepsilon(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) = \mu\tilde{X}_\varepsilon + \tilde{\mu}_\varepsilon\theta_\varepsilon + \varepsilon^{1/2}R_\varepsilon(\theta_\varepsilon) \in \mathcal{C}^{-1/2-}$$

▷ **Parabolic estimates** gives $\theta_\varepsilon \in \mathcal{C}^{3/2-}$

▷ **Continuity of the product** in $\mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^{\alpha \wedge \beta}$ when $\alpha + \beta > 0$ results in continuity for

$$\tilde{\mu}_\varepsilon \times \theta_\varepsilon \in \mathcal{C}^{-0-} \times \mathcal{C}^{3/2-} \rightarrow \tilde{\mu}_\varepsilon\theta_\varepsilon$$

Hence, the family of maps \mathfrak{M}_ε

$$(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) \in \mathcal{C}^{-1/2-} \times \mathcal{C}^{-0-} \times \mathcal{C}^{3/2-} \mapsto \mathfrak{M}_\varepsilon(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) \in \mathcal{C}^{-1/2-}$$

is continuous (and even locally Lipschitz). From this easy to deduce that

$$\begin{array}{ccc} \theta_\varepsilon \xrightarrow[\mathcal{C}^{3/2-}]{} \theta & & \mathcal{L}\theta = \mu X + \mu\theta \\ \psi_\varepsilon = X_\varepsilon + \theta_\varepsilon \xrightarrow[\mathcal{C}^{-1/2-}]{} \psi = X + \theta & & \mathcal{L}\psi = \mu\psi + \xi \end{array}$$

Suppose now that we choose F such that $\mu = 0$. In this case by symmetry

$$\mathbb{E}[H_{c,2}(G)F(G)] = 0$$

but if

$$\lambda = \mathbb{E}[H_{c,3}(G)F(G)] \neq 0$$

we have

$$\varepsilon^{-3(d-2)/2} F(\varepsilon^{(d-2)/2} X_\varepsilon) \xrightarrow[\varepsilon^{-3/2-}]{} \lambda X^{*3} \quad (\text{as space-time distributions})$$

where $X^{*3} = \lim_{\varepsilon \rightarrow 0} H_{\gamma_\varepsilon, 3}(X_\varepsilon)$.

Now we guess that

$$\varepsilon^{\gamma-d/2-1} F(\varepsilon^{(d-2)/2} \psi_\varepsilon) \simeq \varepsilon^{d-4+\gamma} \lambda X^{*3}$$

so something non-trivial can (maybe) be obtained if $\gamma = 4 - d = 1$.

$$\mathcal{L}\varphi_\varepsilon = \varepsilon^{-3/2}F(\varepsilon^{1/2}\varphi_\varepsilon) + \xi_\varepsilon$$

▷ Taylor expansion with $\varphi_\varepsilon = X_\varepsilon + \theta_\varepsilon$

$$\mathcal{L}\theta_\varepsilon = \varepsilon^{-3/2}F(\varepsilon^{1/2}X_\varepsilon) + \varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon + \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon^2 + F'''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon^3 + \varepsilon^{1/2}R_\varepsilon$$

$$\varepsilon^{-3/2}F(\varepsilon^{1/2}X_\varepsilon) \xrightarrow{\mathcal{C}^{-3/2-}} \lambda X^{*3}, \quad \varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon) \xrightarrow{\mathcal{C}^{-1-}} \lambda X^{*2}, \quad \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon) \xrightarrow{\mathcal{C}^{-1/2-}} \lambda X$$

Problem: the previous strategy will not work since

$$\mathcal{L}\theta_\varepsilon \in \mathcal{C}^{-3/2-} \Rightarrow \theta_\varepsilon \in \mathcal{C}^{1/2-}$$

and the products

$$\varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon, \quad \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon$$

are then **not** under control.

▷ Need better understanding of the structure of the solution...

Consider the case $F(x) = \lambda(x^3 - c_\varepsilon x)$:

$$\mathcal{L}\varphi_\varepsilon = \lambda(\varphi_\varepsilon^3 - \varepsilon^{-1}c_\varepsilon\varphi_\varepsilon) + \xi_\varepsilon$$

and let $\varphi_\varepsilon = X_\varepsilon + \lambda Y_\varepsilon + \lambda\varphi_\varepsilon^Q$, ($c_\varepsilon = 3c_{1,\varepsilon} + 9\lambda c_{2,\varepsilon}$)

$$\mathcal{L}Y_\varepsilon + \mathcal{L}\varphi_\varepsilon^Q = X_\varepsilon^{*3} + 3\lambda X_\varepsilon^{*2}(Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda^2 X_\varepsilon(Y_\varepsilon + \varphi_\varepsilon^Q)^2 + \lambda^3(Y_\varepsilon + \varphi_\varepsilon^Q)^3 - 9\lambda c_{2,\varepsilon}\varphi_\varepsilon$$

$$X_\varepsilon^{*3} = X_\varepsilon^3 - 3c_{1,\varepsilon}X_\varepsilon \in \mathcal{O}^{-3/2-} \quad X_\varepsilon^{*2} = X_\varepsilon^2 - c_{1,\varepsilon} \in \mathcal{O}^{-1-}$$

Choosing $\mathcal{L}Y_\varepsilon = X_\varepsilon^{*3}$ we get rid of the first term.

▷ $\mathcal{L}\varphi_\varepsilon^Q \in \mathcal{O}^{-1-} \Rightarrow \varphi_\varepsilon^Q \in \mathcal{O}^{1-}$ and $\mathcal{L}Y_\varepsilon \in \mathcal{O}^{-3/2-} \Rightarrow Y_\varepsilon \in \mathcal{O}^{1/2-}$.

▷▷ **Problem:** slightly better situation but still not good:

$$\underbrace{X_\varepsilon^{*2}}_{\mathcal{O}^{-1-}} \underbrace{(Y_\varepsilon + \varphi_\varepsilon^Q)}_{\mathcal{O}^{1/2-} + \mathcal{O}^{1-}} \quad \underbrace{X_\varepsilon}_{\mathcal{O}^{-1/2-}} \underbrace{(Y_\varepsilon + \varphi_\varepsilon^Q)^2}_{\mathcal{O}^{1/2-} + \mathcal{O}^{1-}}$$

Decomposition of a product into *paraproducts* and *resonant term*

$$fg = f \langle g + f \circ g + f \rangle g$$

Theorem (Bony, Meyer)

$$(f, g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow f \langle g = g \rangle f \in \mathcal{C}^{\beta + \alpha \wedge 0}, \quad \alpha, \beta \in \mathbb{R} \setminus \mathbb{N}$$

$$(f, g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow f \circ g \in \mathcal{C}^{\alpha + \beta}, \quad \alpha + \beta > 0$$

Paralinearization:

$$f \in \mathcal{C}^\alpha \rightarrow R(f) = G(f) - G'(f) \langle f \rangle f \in \mathcal{C}^{2\alpha}, \quad \alpha > 0$$

A single new *key ingredient*:

Lemma (G-Imkeller-Perkowski)

$$(f, g, h) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \rightarrow C(f, g, h) = (f \langle g \rangle \circ h - f(g \circ h)) \in \mathcal{C}^{\alpha + \beta + \gamma}, \quad \alpha + \beta + \gamma > 0$$

▷ Paralinearization:

$$(Y_\varepsilon + \varphi_\varepsilon^Q)^2 = 2(Y_\varepsilon + \varphi_\varepsilon^Q) \underbrace{<(Y_\varepsilon + \varphi_\varepsilon^Q)>}_{\mathcal{L}^{1-}} + R(Y_\varepsilon + \varphi_\varepsilon^Q)$$

▷ Decomposition

$$X_\varepsilon(Y_\varepsilon + \varphi_\varepsilon^Q)^2 = X_\varepsilon <(Y_\varepsilon + \varphi_\varepsilon^Q)^2> + X_\varepsilon \circ (Y_\varepsilon + \varphi_\varepsilon^Q)^2 + X_\varepsilon \triangleright (Y_\varepsilon + \varphi_\varepsilon^Q)^2$$

$$X_\varepsilon \circ (Y_\varepsilon + \varphi_\varepsilon^Q)^2 = 2 X_\varepsilon \circ [(Y_\varepsilon + \varphi_\varepsilon^Q) \underbrace{<(Y_\varepsilon + \varphi_\varepsilon^Q)>}_{\mathcal{L}^{1-}}] + X_\varepsilon \circ R(Y_\varepsilon + \varphi_\varepsilon^Q)$$

▷ Commutator lemma

$$= 2(Y_\varepsilon + \varphi_\varepsilon^Q) \underbrace{X_\varepsilon \circ (Y_\varepsilon + \varphi_\varepsilon^Q)}_{\mathcal{L}^{1/2-}} + 2C(Y_\varepsilon + \varphi_\varepsilon^Q, Y_\varepsilon, X_\varepsilon) + X_\varepsilon \circ R(Y_\varepsilon + \varphi_\varepsilon^Q)$$

$$= 2(Y_\varepsilon + \varphi_\varepsilon^Q) \underbrace{X_\varepsilon \circ Y_\varepsilon}_{\mathcal{L}^{1/2-}} + 2(Y_\varepsilon + \varphi_\varepsilon^Q)(X_\varepsilon \circ \varphi_\varepsilon^Q) + 2C(Y_\varepsilon + \varphi_\varepsilon^Q, Y_\varepsilon, X_\varepsilon) + X_\varepsilon \circ R(Y_\varepsilon + \varphi_\varepsilon^Q)$$

▷ Assume that we can control $X_\varepsilon \circ Y_\varepsilon$ in \mathcal{L}^{-0-} .

$$3\lambda X_\varepsilon^{*2}(Y_\varepsilon + \varphi_\varepsilon^Q) - 9\lambda c_{2,\varepsilon} \varphi_\varepsilon = 3\lambda X_\varepsilon^{*2} \succ (Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda X_\varepsilon^{*2} \prec (Y_\varepsilon + \varphi_\varepsilon^Q) \\ + 3\lambda X_\varepsilon^{*2} \circ Y_\varepsilon - 3c_{2,\varepsilon} X_\varepsilon + 3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^Q - 9\lambda^2 c_{2,\varepsilon} (Y_\varepsilon + \varphi_\varepsilon^Q)$$

▷ Need of a further renormalization: the quantity $X_\varepsilon^{*2} \diamond Y_\varepsilon = X_\varepsilon^{*2} \circ Y_\varepsilon - 3c_{2,\varepsilon} X_\varepsilon$ is under control in $\mathcal{C}^{-1/2-}$.

▷ **Paracontrolled ansatz:** $\varphi_\varepsilon^Q = \varphi_\varepsilon^* \ll Q_\varepsilon + \varphi_\varepsilon^\#$ with $Q_\varepsilon \in \mathcal{C}^{1-}$, $\varphi_\varepsilon^* \in \mathcal{C}^{1/2-}$ and $\varphi_\varepsilon^\# \in \mathcal{C}^{3/2-}$.

$$3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^Q - 9\lambda^2 c_{2,\varepsilon} (Y_\varepsilon + \varphi_\varepsilon^Q) = 3\lambda X_\varepsilon^{*2} \circ (\varphi_\varepsilon^* \ll Q_\varepsilon) - 9\lambda^2 c_{2,\varepsilon} (Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^\# \\ = 3\lambda \varphi_\varepsilon^* X_\varepsilon^{*2} \circ Q_\varepsilon - 9\lambda^2 c_{2,\varepsilon} (Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda C_{\ll}(\varphi_\varepsilon^*, Q_\varepsilon, X_\varepsilon^{*2}) + 3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^\#$$

How to choose φ_ε' and Q_ε ?

Note that

$$\mathcal{L}\varphi_\varepsilon^Q = 3\lambda X_\varepsilon^{*2} \triangleright (Y_\varepsilon + \varphi_\varepsilon^Q) + \dots$$

but also

$$\mathcal{L}\varphi_\varepsilon^Q = \mathcal{L}(\varphi_\varepsilon^* \ll Q_\varepsilon + \varphi_\varepsilon^\#) = \varphi_\varepsilon' \ll \mathcal{L}Q_\varepsilon + \mathcal{L}\varphi_\varepsilon^\# + \dots$$

so a natural choice is $\varphi_\varepsilon^* = 3\lambda(Y_\varepsilon + \varphi_\varepsilon^Q)$ and $\mathcal{L}Q_\varepsilon = X_\varepsilon^{*2}$. In this case

$$3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^Q - 9\lambda^2 c_{2,\varepsilon} (Y_\varepsilon + \varphi_\varepsilon^Q) = 3\lambda \varphi_\varepsilon^* X_\varepsilon^{*2} \circ Q_\varepsilon - c_{2,\varepsilon} + 3\lambda C_{\ll}(\varphi_\varepsilon^*, Q_\varepsilon, X_\varepsilon^{*2}) + 3\lambda X_\varepsilon^{*2} \circ \varphi_\varepsilon^\#$$

and we can show that $X_\varepsilon^{*2} \diamond Q_\varepsilon = X_\varepsilon^{*2} \circ Q_\varepsilon - c_{2,\varepsilon}$ is under control in \mathcal{C}^{0-} .

Conclusion: starting from the equation

$$\mathcal{L}\varphi_\varepsilon^Q = 3\lambda X_\varepsilon^{*2}(Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda^2 X_\varepsilon(Y_\varepsilon + \varphi_\varepsilon^Q)^2 + \lambda^3(Y_\varepsilon + \varphi_\varepsilon^Q)^3 - 9\lambda c_{2,\varepsilon}\varphi_\varepsilon$$

and performing the change of variables

$$\varphi_\varepsilon^Q = \varphi_\varepsilon^* \ll Q_\varepsilon + \varphi_\varepsilon^\#, \quad \varphi_\varepsilon^* = 3\lambda(Y_\varepsilon + \varphi_\varepsilon^Q)$$

we get an equation for $\varphi_\varepsilon^\#$ which reads

$$\mathcal{L}\varphi_\varepsilon^\# = \Phi_\varepsilon^\#(\mathbb{X}_\varepsilon, \varphi_\varepsilon^Q, \varphi_\varepsilon^\#)$$

where the r.h.s. depends continuously on $\varphi_\varepsilon^Q, \varphi_\varepsilon^\#$ and on the datum of \mathbb{X}_ε :

$$\mathbb{X}_\varepsilon = (X_\varepsilon, X_\varepsilon^{*2}, X_\varepsilon^{*3}, X_\varepsilon \circ Y_\varepsilon, X_\varepsilon^{*2} \diamond Y_\varepsilon, X_\varepsilon^{*2} \diamond Q_\varepsilon)$$

The limit as $\varepsilon \rightarrow 0$ can be now established via standard PDE estimates (for short times) once it is known that $\mathbb{X}_\varepsilon \rightarrow \mathbb{X}$ in a suitable topology.

Another way to state the previous analysis is to say that to each decomposition

$$\psi = X + Y + \psi^* \ll Q + \psi^\#$$

of a function ψ we can associate a corresponding decomposition of the power ψ^3

$$\psi^3 = G(\mathbb{X}, \psi^*, \psi^\#) = X^3 + 3(Y + \psi^* \ll Q + \psi^\#) \prec X^2 + \dots$$

$$\mathbb{X} = (X, X^2, X^3, Y \circ X, Y \circ X^2, Q \circ X^2)$$

in such a way that, for any $a, b \in \mathbb{R}$

$$\psi^3 - 3a\psi - 3bX - 3b\psi^* = G(R_{a,b}\mathbb{X}, \psi^*, \psi^\#)$$

with $R_{a,b}\mathbb{X} = (X, X^2 - a, X^3 - 3a, Y \circ X, Y \circ (X^2 - a) - bX, Q \circ (X^2 - a) - b)$

Now if ψ is a distribution for which

$$\psi = X + Y + \psi^* \ll Q + \psi^\#$$

we can set

$$\psi_\varepsilon = \rho_\varepsilon * \psi = \rho_\varepsilon * X + \rho_\varepsilon * Y + \psi^* \ll \rho_\varepsilon * Q + \rho_\varepsilon * \psi^\# + [\psi^* \ll, \rho_\varepsilon *]Q$$

so that $X_\varepsilon = \rho_\varepsilon * X$

$$\psi_\varepsilon^3 - 3a_\varepsilon \psi_\varepsilon - 3bX_\varepsilon - 3b\psi^* = G(R_{a_\varepsilon, b_\varepsilon} \mathbb{X}_\varepsilon, \psi^*, \rho_\varepsilon * \psi^\# + [\psi^* \ll, \rho_\varepsilon *]Q)$$

$$\mathbb{X}_\varepsilon = (X_\varepsilon, X_\varepsilon^2, X_\varepsilon^3, Y_\varepsilon \circ X_\varepsilon, Y_\varepsilon \circ X_\varepsilon^2, Q_\varepsilon \circ X_\varepsilon^2)$$

with $Z_\varepsilon = \rho_\varepsilon * Z$ for $Z = X, Y, Q$. And if $R_{a_\varepsilon, b_\varepsilon} \mathbb{X}_\varepsilon \rightarrow \mathbb{X}_{\text{ren}}$ as $\varepsilon \rightarrow 0$ then

$$\psi_\varepsilon^3 - 3a_\varepsilon \psi_\varepsilon - 3bX_\varepsilon - 3b\psi^* \rightarrow \psi_{\text{ren}}^3 = G(\mathbb{X}_{\text{ren}}, \psi^*, \psi^\#)$$

At this point we are capable of defining solutions of the singular SPDE which we expect in the limit of the reaction diffusion equation.

Given the noise ξ we construct its associated “model”

$$R_{c_{1,\varepsilon}, c_{2,\varepsilon}} \mathbb{X}_\varepsilon \rightarrow \mathbb{X}(\xi) = (X, X^{*2}, X^{*3}, Y \diamond X, Y \diamond X^{*2}, Q \diamond X^{*2})$$

by passing to the limit in the canonical model $R_{c_{1,\varepsilon}, c_{2,\varepsilon}} \mathbb{X}_\varepsilon$ for ξ_ε with suitable renormalization constants $c_{1,\varepsilon}, c_{2,\varepsilon}$.

Then for fixed \mathbb{X} we say that a distribution φ is a solution to

$$\mathcal{L}\varphi = \lambda\varphi^{*3} + \xi$$

if $\varphi = X + \lambda Y + \lambda\varphi^* \ll Q + \lambda\varphi^\#$ and

$$\mathcal{L}\varphi = \lambda G(\mathbb{X}, \lambda\varphi^*, \lambda\varphi^\#) + \xi$$

This implies in particular that φ^* can be chosen to be equal to $3\lambda(Y + \lambda\varphi^* \ll Q + \lambda\varphi^\#)$

Thanks!