Paracontrolled distributions with applications to singular SPDEs

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Some problems in singular SPDEs /I

Define and solve (locally) the following SPDEs:

- Stochastic differential equations (1+0): $u \in [0, T] \to \mathbb{R}^n$

  $$\partial_t u(t) = \sum_i f_i(u(t))\xi^i(t)$$

  with $\xi : \mathbb{R} \to \mathbb{R}^m$ m-dimensional white noise in time.
- Burgers equations (1+1): $u \in [0, T] \times \mathbb{T} \to \mathbb{R}^n$

  $$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))Du(t,x) + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

Recall that

$$\xi \in \mathscr{C}^{-d/2-}$$

Some problems in singular SPDEs /II

- Generalized Parabolic Anderson model (1+2):
  $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

  $$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(x)$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.
- Kardar-Parisi-Zhang equation (1+1)

  $$\partial_t h(t,x) = \Delta h(t,x) + "(Du(t,x))^2 - \infty" + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Some problems in singular SPDEs /III

Define and solve (locally) the following SPDEs:

- Stochastic quantization equation (1+3)

$$\partial_t u(t,x) = \Delta u(t,x) + "u(t,x)^3" + \xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ space-time white noise.

- But (currently) not: Multiplicative SPDEs (1+1)

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(t,x)$$

  with $\xi : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ space-time white noise.

Joint work with P. Imkeller and N. Perkowski.
(Also K. Chouk and R. Catellier for $(\Phi)^4_3$).

Rough differential equation

Consider the simple controlled ODE (η smooth, fixed initial condition)

$$\partial_t u(t) = \sum_{i=1}^{m} f_i(u(t))\eta^i(t)$$

$u : \mathbb{R} \to \mathbb{R}^d$, $\eta : \mathbb{R} \to \mathbb{R}^d$ and smooth vectorfields $f_i : \mathbb{R}^d \to \mathbb{R}^d$.

Problem

The solution map

$$\eta \xrightarrow{\Psi} u$$

is generally **not** continuous for $\eta \in \mathscr{C}^{\gamma-1}$ with $\gamma < 1/2$.

Reason: $u \in \mathscr{C}^{\gamma}$ and $\eta \in \mathscr{C}^{\gamma-1}$ cannot be multiplied when $2\gamma - 1 \leqslant 0$. The r.h.s. of the equation is not well defined.

Here $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ is the Holder–Besov space (or a local version).

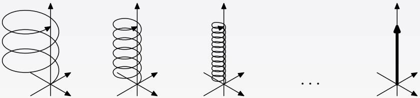
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $\mathscr{C}^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2} \neq I(0,0)(t) = 0$$



The definite integral $I(\cdot,\cdot)(t)$ is **not** a continuous map $\mathscr{C}^\gamma \times \mathscr{C}^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Concept of solution

Goal: Show that Ψ factorizes as

$$\eta \xrightarrow{J} J(\eta) \xrightarrow{\Phi} u$$

$\triangleright$ *Analytic step:* show that when $\gamma > 1/3$:

$$\Phi : \mathcal{X} \to \mathscr{C}^{\gamma}$$

is continuous. $\mathcal{X} = \overline{\mathrm{Im} J} \subseteq \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ is the space of *enhanced signals* (or rough paths, or models).

But in general J is not a continuous map $\mathscr{C}^{\gamma-1} \to \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$.

$\triangleright$ *Probabilistic step:* prove that there exists a "reasonable definition" of $J(\xi)$ when ξ is a white noise. $J(\xi)$ is an explicit polinomial in ξ so direct computations are possible.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $\mathscr{C}^\gamma = B^\gamma_{\infty,\infty}$.

$f \in \mathscr{C}^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \leqslant \|f\|_\gamma 2^{-i\gamma}, \qquad i \geqslant -1.$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho_i(\xi)\hat{f}(\xi)$$

where $\rho_i : \mathbb{R}^d \to \mathbb{R}_+$ are smooth functions with support $\simeq 2^i \mathscr{A}$ when $i \geqslant 0$ and form a partition of unity $\sum_{i \geqslant -1} \rho_i(\xi) = 1$ for all $\xi \neq 0$ so that

$$f = \sum_{i \geqslant -1} \Delta_i f$$

in $\mathcal{S}'$.

Paraproducts

Deconstruction of a product: $f \in \mathscr{C}^{\rho}, g \in \mathscr{C}^{\gamma}$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = f \prec g + f \circ g + f \succ g$$

$$f \prec g = g \succ f = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$f \prec g \in \mathscr{C}^{\min(\gamma+\rho,\gamma)}$$

$$f \circ g \in \mathscr{C}^{\gamma+\rho}$$ only if $\gamma + \rho > 0$

Proof. Recall $f \in \mathscr{C}^{\rho}, g \in \mathscr{C}^{\gamma}$.

$$i \ll j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{A} \qquad i \sim j \Rightarrow \operatorname{supp}\mathscr{F}(\Delta_i f \Delta_j g) \subseteq 2^j \mathscr{B}$$

So if $\rho > 0$

$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q\gamma}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma},$$

while if $\rho < 0$

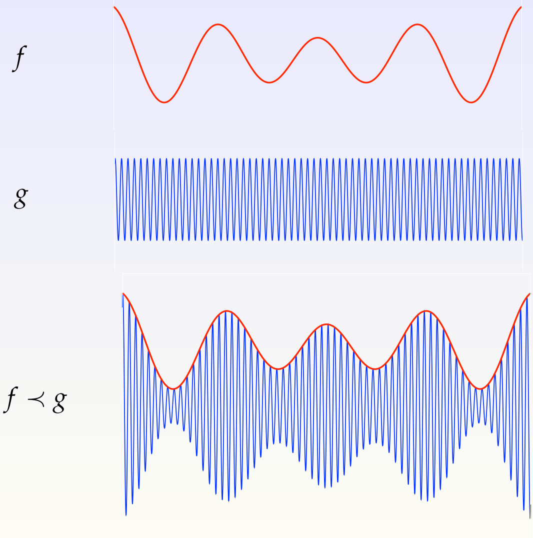
$$\Delta_q(f \prec g) = \sum_{j:j\sim q} \sum_{i:i<j-1} \underbrace{\Delta_q(\Delta_i f \Delta_j g)}_{O(2^{-i\rho - j\gamma})} = O(2^{-q(\gamma+\rho)}) \Rightarrow f \prec g \in \mathscr{C}^{\gamma+\rho}.$$

Finally for the resonant term we have

$$\Delta_q(f \circ g) = \sum_{i\sim j\gtrsim q} \Delta_q(\Delta_i f \Delta_j g) = \sum_{i\gtrsim q} O(2^{-j(\rho+\gamma)}) \Rightarrow f \circ g \in \mathscr{C}^{\gamma+\rho}$$

but *only if* the sum converges.

Paraproduct as frequency modulation



The main commutator estimate

All the difficulty is concentrated in the resonating term

$$f \circ g = \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

which however "is" smoother than $f \prec g$ if f or g has positive regularity.

Paraproducts decouple the problem from the source of the problem.

Commutator lemma

The trilinear operator $C(f, g, h) = (f \prec g) \circ h - f(g \circ h)$ satisfies

$$\|C(f, g, h)\|_{\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

when $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, $\alpha < 1$.

The Good, the Ugly and the Bad

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in \mathscr{C}^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \underbrace{\varphi(B) \prec DB}_{\text{the Bad}} + \underbrace{\varphi(B) \circ DB}_{\text{the Ugly}} + \underbrace{\varphi(B) \succ DB}_{\text{the Good, } \mathscr{C}^{2\gamma-1}}$$

and recall the paralinearization

$$\varphi(B) = \varphi'(B) \prec B + \mathscr{C}^{2\gamma}$$

Then

$$\varphi(B) \circ DB = (\varphi'(B) \prec B) \circ DB + \underbrace{\mathscr{C}^{2\gamma} \circ DB}_{\text{OK}}$$

$$= \varphi'(B)(B \circ DB) + \mathscr{C}^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \varphi(B) \prec DB + \varphi'(B) \underbrace{(B \circ DB)}_{\text{"Besov area"}} + \varphi(B) \succ DB + \mathscr{C}^{3\gamma-1}$$

The Besov area

If $d = 1$ (or by symmetrization) we can perform an integration by parts to get

$$B \circ DB = \frac{1}{2}((B \circ DB) + (DB \circ B)) = \frac{1}{2} D(B \circ B)$$

which is well defined and belongs indeed to $\mathscr{C}^{2\gamma-1}$.

In general the Besov area $B \circ DB$ can be defined and studied efficiently using Gaussian arguments:

$$B^{\varepsilon} \circ DB^{\varepsilon} \to B \circ DB$$

almost surely in $\mathscr{C}^{2\gamma-1}_{\text{loc}}$ as $\varepsilon \to 0$.

Tools: Besov embeddings $L^p(\Omega; C^{\theta}) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Controlled structures and paraproducts

▷ Gubinelli (2004): For $\alpha \in (0,1)$, $g \in C^\alpha$, f is called controlled by g if

$$f(t) - f(s) = f'(s)(g(t) - g(s)) + f^\sharp(s,t), \qquad |f^\sharp(s,t)| \lesssim |t-s|^{2\alpha}.$$

Then $f - f' \prec g \in \mathscr{C}^{2\alpha}$.

▷ Hairer (2013): For $\gamma > 0$, $f : \mathbb{R}^d \to T$ is called modelled, $f \in \mathscr{D}^\gamma$, if

$$|f_x - \Gamma_{x,y} f_y|_\beta \lesssim |x-y|^{\gamma-\beta}.$$

If $\mathscr{R}$ denotes the reconstruction operator, then $\mathscr{R}f - P(f,\Pi) \in C^\gamma$, where

$$\begin{aligned} P(f,\Pi)(x) &= \sum_{j<k-1} \int K_j(x-z) K_k(x-y) \Pi_z f_z(y) \mathrm{d}y \mathrm{d}z \\ &= \sum_{j<k-1} \int K_{j,x}(z) \Pi_z f_z(K_{k,x}) \mathrm{d}z. \end{aligned}$$

Paracontrolled distributions

Use the paraproduct to *define* a controlled structure. We say $y \in \mathscr{D}_x^{\rho}$ if $x \in \mathscr{C}^{\gamma}$

$$y = y^x \prec x + y^{\sharp}$$

with $y^x \in C^{\rho-\gamma}$ and $y^{\sharp} \in C^{\rho}$.

Theorem

If $\alpha + \beta + \gamma > 0$*,* $h \in \mathscr{C}^{\gamma}$*,* $f \in \mathscr{D}_g^{\alpha+\beta}$*, and* $g \circ h \in \mathscr{C}^{\gamma+\beta}$ *is given, then* fh *can be constructed continuously as*

$$fh = \Phi(f', f^{\sharp}, g, h, g \circ h).$$

Moreover, fh *is paracontrolled by* h*:*

$$fh - f \prec h \in \mathscr{C}^{\beta+\gamma}$$

Operations on paracontrolled distributions

▷ **Paralinearization.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in \mathscr{C}^{\gamma}$, $\gamma > 0$. Then

$$\varphi(x) = \varphi'(x) \prec x + \mathscr{C}^{2\gamma}$$

▷ Another commutator: $f, g \in \mathscr{C}^{\rho-\gamma}$, $x \in \mathscr{C}^{\gamma}$

$$f \prec (g \prec h) = (fg) \prec h + \mathscr{C}^{\rho}$$

▷ **Stability.** $(\rho \leqslant 2\gamma)$

$$\varphi(y) = (\varphi'(y) y^{x}) \prec x + \mathscr{C}^{\rho}$$

so we can take $\varphi(y)^{x} = \varphi'(y) y^{x}$.

RDEs - I - the r.h.s.

$u : \mathbb{R} \to \mathbb{R}^d$, $\xi \in \mathscr{C}^{-1/2-}$ is (an approx. to) 1d white noise. We want to solve

$$\partial_t u = f(u)\xi = f(u) \prec \xi + f(u) \circ \xi + f(u) \succ \xi$$

▷ Paracontrolled ansatz. Take $\partial_t X = \xi$, $X \in \mathscr{C}^{1/2-}$ and assume that $u \in \mathscr{D}_X^{1-}$:

$$u = u^X \prec X + u^\sharp$$

with $u^\sharp \in \mathscr{C}^{1-}$ and $u^X \in \mathscr{C}^{1/2-}$.

▷ Paralinearization:

$$f(u) = f'(u) \prec u + \mathscr{C}^{1-} = (f'(u)u^X) \prec X + \mathscr{C}^{1-}$$

▷ Commutator lemma:

$$f(u) \circ \xi = ((f'(u)u^X) \prec X) \circ \xi + \mathscr{C}^{1-} \circ \xi$$

$$= \underbrace{(f'(u)u^X)(X \circ \xi)}_{\in \mathscr{C}^{0-}} + \underbrace{C(f'(u)u^X, X, \xi) + \mathscr{C}^{1-} \circ \xi}_{\in \mathscr{C}^{1/2-}}$$

if we *assume* that $(X \circ \xi) \in \mathscr{C}^{0-}$.

RDEs - II - the l.h.s.

So if u is paracontrolled by X:

$$u = u^X \prec X + u^\sharp$$

and if $X \circ \xi \in \mathscr{C}^{0-}$ we have a control on the r.h.s. of the equation:

$$f(u)\xi = \underline{f(u) \prec \xi} + f'(u)u^X(X \circ \xi) + \mathscr{C}^{1/2-}$$

What about the l.h.s.?

$$\partial_t u = \partial_t u^X \prec X + \underline{u^X \prec \xi} + \partial_t u^\sharp$$

so letting $u^X = f(u)$ we have

$$\partial_t u^\sharp = -\partial_t f(u) \prec X + f'(u)f(u)(X \circ \xi) + \mathscr{C}^{1/2-}$$

RDEs - III - the paracontrolled fixed point.

The RDE

$$\partial_t u = f(u)\xi$$

is equivalent to the system

$$\begin{aligned}
\partial_t X &= \xi \\
\partial_t u^\sharp &= (f'(u)f(u))(X \circ \xi) - \underbrace{\partial_t f(u) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f,u,X,\xi)}_{\in \mathscr{C}^{1/2-}} \circ \xi \\
u &= f(u) \prec X + u^\sharp
\end{aligned}$$

▷ The system can be solved by fixed point (for small time) in the space $\mathscr{D}_X^{1-}$ if we assume that

$$X \in \mathscr{C}^{1/2-}, \qquad (X \circ \xi) \in \mathscr{C}^{0-}.$$

Structure of the solution

▷ When ξ smooth, the solution to

$$\partial_t u = f(u)\xi, \qquad u(0) = u_0$$

is given by $u = \Phi(u_0, \xi, X \circ \xi)$ where

$$\Phi : \mathbb{R}^d \times \mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1} \to \mathscr{C}^\gamma$$

is continuous for any $\gamma > 1/3$ and $z = \Phi(u_0, \xi, \varphi)$ is given by

$$\begin{cases} z = f(z) \prec X + z^\sharp \\ \partial_t z^\sharp = (f'(z)f(z))\varphi - \underbrace{\partial_t f(z) \prec X}_{\in \mathscr{C}^{0-}} + \underbrace{R(f, z, X, \xi) \circ \xi}_{\in \mathscr{C}^{1/2-}} \end{cases}$$

▷ If $(\xi^n, X^n \circ \xi^n) \to (\xi, \eta)$ in $\mathscr{C}^{\gamma-1} \times \mathscr{C}^{2\gamma-1}$ and

$$\partial_t u^n = f(u^n)\xi^n, \qquad u(0) = u_0$$

then

$$u^n \to u = \Phi(u_0, \xi, \eta).$$

Relaxed form of the RDE

▷ Note that in general we can have $\xi^{1,n} \to \xi$, $\xi^{2,n} \to \xi$ and

$$\lim_n X^{1,n} \circ \xi^{1,n} \neq \lim_n X^{2,n} \circ \xi^{2,n}$$

▷ Take ξ^n, ξ smooth but $\xi^n \to \xi$ in $\mathscr{C}^{\gamma-1}$. It can happen that

$$\lim_n X^n \circ \xi^n = X \circ \xi + \varphi \in \mathscr{C}^{2\gamma-1}$$

In this case $u^n \to u$ and $u = \Phi(\xi, X \circ \xi + \varphi)$ solves the equation

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

The limit procedure generates correction terms to the equation.

The original equation **relaxes** to another form in which additional terms are generated.

"Itô" form of the RDE

In the smooth setting $u = \Phi(\xi, X \circ \xi + \varphi)$ solves

$$\partial_t u = f(u)\xi + f'(u)f(u)\varphi.$$

If we choose $\varphi = -X \circ \xi$ then

$$v = \Phi(\xi, X \circ \xi + \varphi) = \Phi(\xi, 0)$$

solves

$$\partial_t v = f(v)\xi - f'(v)f(v)X \circ \xi$$

and has the particular property of being a continuous map of $\xi \in \mathscr{C}^{\gamma-1}$ alone.

The discrete parabolic Anderson model

- Stochastic heat equation on $\mathbb{Z}^d$:

$$\partial_t u(t,x) = \Delta_{\mathbb{Z}^d} u(t,x) + F(u(t,x))\eta(x);$$

  with potential landscape of i.i.d. random variables $(\eta(x))_{x\in\mathbb{Z}^d}$;
- linear version with $F(u) = u$ is model for many phenomena in physics, e.g. growth of magnetic fields in young stars;
- mathematical interest in long time behavior of PAM: simple model which exhibits intermittency (largest part of the mass concentrated in few small "islands");
- countless results since early 90s, different universality classes depending on distribution of η.

Conjectured scaling limit

- To study long time behavior, and to obtain universality for different potentials η, would be interested in scaling limit:

$$\partial_t v^n(t,x) = \Delta_{\mathbb{Z}^d} v^n(t,x) + n^{d/2-2} F(v^n(t,x))\eta(x);$$
$$u^n(t,x) = v^n(n^2 t, nx).$$

- Natural conjecture: limit solves

$$\mathcal{L}u(t,x) = F(u(t,x))\xi(x),$$

for spatial white noise ξ.

Continuous PAM

$$\mathcal{L}u(t,x) = F(u(t,x))\xi(x)$$

Equation is ill posed for $d > 1$, needs some form of renormalization.

Existing solutions only work in linear case and use Wick products (e.g. Hu (2002)):

$$\mathcal{L}u(t,x) = :u(t,x)\xi(x):$$

Obtain existence and uniqueness of solutions for $d \leqslant 3$.

- apply formal chaos expansion to the solution;
- formally obtain solution as chaos series, $u = \sum_n I_n(f_n)$ for suitable deterministic f_n;
- see that for $d < 4$, the series indeed converges.

Problem: Discrete PAM not formulated in terms of Wick products! How does the Wick product transform the equation? Scaling limit?

Continuous PAM and paracontrolled distributions

$$\mathcal{L}u(t,x) = F(u(t,x)) \diamond \xi(x)$$

Paracontrolled distributions can handle the equation in the general case for $d \leqslant 2$, in the linear case for $d = 3$ (in principle...); agrees with Wick product solution in the linear case. Advantages:

- renormalization $u(t,x) \diamond \xi(x)$ of $u(t,x)\xi(x)$ is very transparent and we can apply the same renormalization in the discrete model;
- solution depends pathwise continuously on suitably extended data.

The solution is scaling limit of renormalized discrete system (work in progress by Perkowski, Chouk, Gairing):

- show weak convergence of $n^{d/2-2}\eta(n\cdot)$ to ξ and of renormalized extended data;
- use pathwise continuous dependence of solution on extended data in combination with Skorokhod representation to obtain weak convergence of solutions.

Generalized Parabolic Anderson Model on $\mathbb{T}^2$

$\mathcal{L} = \partial_t - D^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$, $\xi \in \mathscr{C}^{-1-}(\mathbb{T}^2)$ space white noise.

$$\mathcal{L}u = f(u)\xi$$

▷ Paracontrolled ansatz $\qquad \mathcal{L}X = \xi$ so $X \in C([0,T], \mathscr{C}^{1-})$

$$u = f(u) \prec X + u^\sharp$$

▷ Paralinearization: $\qquad f(u) = (f'(u)f(u)) \prec X + R(f,u,X)$

$$f(u) \circ \xi = (f'(u)f(u))(X \circ \xi) + C(f'(u)f(u), X, \xi) + R(f,u,X) \circ \xi$$

A problem

If ξ is the space white noise we have

$$\xi \in \mathscr{C}^{-1-}, \qquad X \in C([0,T];\mathscr{C}^{1-})$$

and

$$X \circ \xi = X \circ \mathcal{L}X = \frac{1}{2}\mathcal{L}(X \circ X) + \frac{1}{2}(DX \circ DX)$$
$$= \frac{1}{2}\mathcal{L}(X \circ X) - (DX \prec DX) + \frac{1}{2}(DX)^2$$

But now

$$\frac{1}{2}(DX)^2 = c + C\mathscr{C}^{0-}$$

with $c = +\infty$!.

No obvious definition of $X \circ \xi$ can be given. But there exists c_ε such that

$$X_\varepsilon \circ \xi_\varepsilon - c_\varepsilon \to \text{"}X \diamond \xi\text{"} \qquad \text{in } C\mathscr{C}^{0-}.$$

A first renormalization

To cure the problem we add a suitable counterterm to the equation

$$\mathcal{L}u = f(u) \diamond \xi = f(u)\xi - c(f'(u)f(u))$$

this defines a new product, denote by $\diamond$. Now

$$f(u)\circ\xi - c(f'(u)f(u)) = (f'(u)f(u))(X\circ\xi - c) + C(f'(u)f(u), X, \xi) + R(f, u, X)\circ\xi$$

▷ The renormalized gPAM is equivalent to the equation

$$\mathcal{L}u^\sharp = -\mathcal{L}f(u) \prec X + \mathrm{D}f(u) \prec \mathrm{D}X + (f'(u)f(u))(X \circ \xi - c)$$
$$+ C(f'(u)f(u), X, \xi) + R(f, u, X) \circ \xi$$

together with $u = f(u) \prec X + u^\sharp$ and where

$$X \in \mathscr{C}^{1-}, \qquad X \diamond \xi = (X \circ \xi - c) \in \mathscr{C}^{0-}, \quad u^\sharp \in \mathscr{C}^{2-}.$$

KPZ and its siblings:

Besides the generalized PAM, the following equations have been solved using the paracontrolled approach (joint work with N. Perkowski)

$\mathcal{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- KPZ equation: h: $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

- Burgers equation: $u = \partial_x h$;

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x);$$

- stochastic heat equation: $h = \log w$

$$\mathcal{L}w(t,x) = w(t,x)\xi(t,x).$$

Other applications

- Gubinelli, Imkeller, P. (2012): Multidimensional extension of Hairer's (2011) generalized Burgers equation ($\sigma - d/2 > 1/3$):

$$\partial_t u(t,x) = -(-\Delta)^\sigma u(t,x) + G(u(t,x)) D_x u(t,x) + \xi(t,x);$$

- Catellier, Chouk (2013): Stochastic quantization equation ϕ_3^4 ($d = 3$):

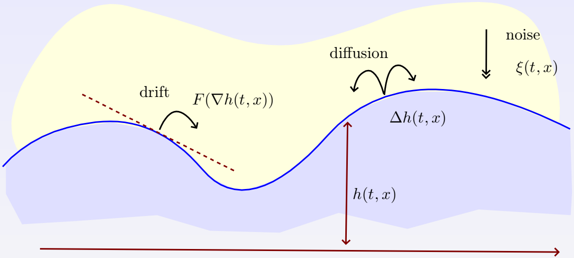
$$\mathcal{L}u(t,x) = -u(t,x)^{\diamond 3} + \xi(t,x);$$

- Furlan (2014): Stochastic Navier Stokes equation ($d = 3$):

$$\mathcal{L}u(t,x) = -P((u(t,x) \cdot \nabla) u(t,x)) + \xi(t,x).$$

Thanks

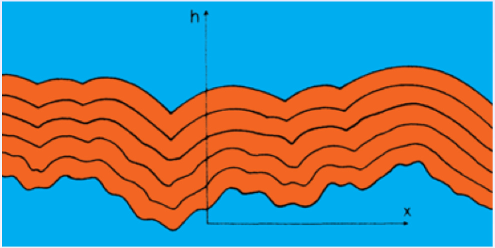
Fluctuations of a growing interface

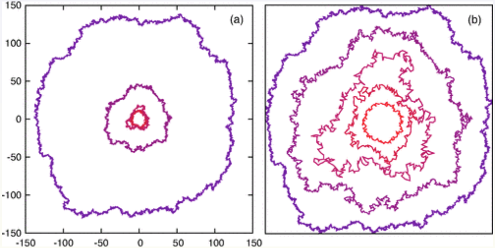


A model for random interface growth (think e.g. expansion of colony of bacteria): $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{F(\partial_x h(t,x))}_{\text{slope-dependent growth}} + \underbrace{\eta(t,x)}_{\text{noise with microscopic correlations}}$$

Fluctuations of a growing interface





The Kardar–Parisi–Zhang equation

- Kardar–Parisi–Zhang '84: slope-dependent growth given by $F(\partial_x h)$, in a certain scaling regime of small gradients:

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + F''(0)(\partial_x h)^2 + \dots$$

- KPZ equation is the universal model for random interface growth

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{relaxation}} + \underbrace{\lambda[(\partial_x h(t,x))^2 - \infty]}_{\text{renormalized growth}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

- This derivation is highly problematic since $\partial_x h$ is a distribution. But: Hairer, Quastel (2014, unpublished) justify it rigorously via scaling of smooth models and small gradients.
- KPZ equation is suspected to be universal scaling limit for random interface growth models, random polymers, and many particle systems;
- contrary to Brownian setting: KPZ has fluctuations of order $t^{1/3}$; large time limit distribution of $t^{-1/3}h(t, t^{2/3}x)$ is expected to be universal in a sense comparable only to the Gaussian distribution.

KPZ and its siblings:

- KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x);$$

  $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{L} = \partial_t - \Delta$ heat operator, ξ space-time white noise;

- Burgers equation:

$$\mathcal{L}u(t,x) = \partial_x(u(t,x)^2) + \partial_x \xi(t,x);$$

  solution is (formally) given by derivative of the KPZ equation: $u = \partial_x h$;

- solution to KPZ (formally) given by Cole-Hopf transform of the stochastic heat equation: $h = \log w$, where w solves

$$\mathcal{L}w(t,x) = w(t,x)\xi(t,x).$$

- All three are universal objects, that are expected to be scaling limits of a wide range of particle systems.

Stochastic Burgers equation

Take $u = Dh$

$$\mathcal{L}u = D\xi + Du^2$$

to obtain the stochastic Burgers equation (SBE) with additive noise.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)X(s,y)] = p_{|t-s|}(x-y).$$

Almost surely $X(t,\cdot) \in \mathscr{C}^\gamma$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Expansion /I

- ▹ Let $u = X + u_1$ then

$$\mathcal{L}u_1 = \partial_x(u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x(u_1 X) + \partial_x u_1^2$$

- ▹ Let $X^{\vee}$ be the solution to

$$\mathcal{L}X^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$\mathcal{L}u_2 = \underbrace{2\partial_x(X^{\vee}X)}_{-3/2-} + 2\partial_x(u_2 X) + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

- ▹ Define $\mathcal{L}X^{\curlyvee} = 2\partial_x(X^{\vee}X)$ and $u_2 = X^{\curlyvee} + u_3$ then $X^{\curlyvee} \in \mathscr{C}^{1/2-}$

$$\mathcal{L}u_3 = \underbrace{2\partial_x(u_3 X)}_{-3/2-} + \underbrace{2\partial_x(X^{\curlyvee}X)}_{-3/2-} + \underbrace{\partial_x(X^{\vee}X^{\vee})}_{-1-} + 2\partial_x(u_2 X^{\vee}) + \partial_x(u_2)^2$$

Expansion /II

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2X^{\Psi} + U$$

$$\mathcal{L}U = 2\partial_x(UX) + 2\partial_x(X^{\Psi}X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

$$= 2\partial_x(UX) + \mathcal{L}(2X^{\Psi} + X^{\vee\vee}) + 2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	X^{Ψ}	$X^{\Psi'}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms

$$2\partial_x((2X^{\Psi} + U)X^{\vee}) + \partial_x(2X^{\Psi} + U)^2$$

are well defined.

The remaining problem is to deal with $2\partial_x(UX)$.

Paracontrolled ansatz for SBE

▷ Make the following ansatz $U = U' \prec Q + U^\sharp$. Then

$$\mathcal{L}U = \mathcal{L}U' \prec Q + U' \prec \mathcal{L}Q - \partial_x U' \prec \partial_x Q + LU^\sharp$$

while

$$\mathcal{L}U = 2\partial_x(UX) + \underbrace{\mathcal{L}(2X^{\overset{\vee}{\vee}} + X^{\overset{\vee\vee}{\vee}}) + 2\partial_x((2X^{\overset{\vee}{\vee}} + U)X^{\vee}) + \partial_x(2X^{\overset{\vee}{\vee}} + U)^2}_{R(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

so we can set $U' = 2U$ and $\mathcal{L}Q = \partial_x X$ and get the equation

$$\mathcal{L}U^\sharp = -\mathcal{L}U' \prec Q + \partial_x U' \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + R(U)$$

▷ Observe that $Q, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

Commutator

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Q) \circ X + U^\sharp \circ X = 2U(Q \circ X) + \underbrace{C(2U, Q, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Q \circ X \in \mathscr{C}^{0-}$

Paracontrolled solution to SBE

▷ The final system reads

$$u = X + X^{\vee} + 2X^{\curlyvee} + U$$

$$U = U' \prec Q + U^{\sharp}, \qquad U' = 2X^{\curlyvee} + 2U$$

$$\mathcal{L}U^{\sharp} = 4\partial_x(U\underline{(Q \circ X)}) + 4\partial_x C(U, Q, X) + 2\partial_x(U^{\sharp} \circ X) - 2\mathcal{L}U \prec Q$$
$$+2\partial_x U \prec \partial_x Q + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + R(U)$$

▷ This equation has a (local in time) solution $U = \Phi(J(\xi))$ which is a continuous function of the data $J(\xi)$ given by a collection of multilinear functions of ξ:

$$J(\xi) = (X, X^{\vee}, X^{\curlyvee}, X^{\curlyvee\!\!\!\curlyvee}, X^{\vee\!\vee}, X \circ Q)$$

Burgers equation and paracontrolled distributions

$$\mathcal{L}u(t,x) = \partial_x u^2(t,x) + \partial_x \xi(t,x), \qquad u(0) = u_0.$$

Paracontrolled Ansatz

$u \in \mathscr{P}_{\mathrm{rbe}}$ if $u = X + X^{\vee} + 2X^{\vee\!\!\!\vee} + u^Q$ with

$$u^Q = \pi_<(u', Q) + u^\sharp.$$

- Paracontrolled structure: Can define u^2 continuously as long as $(Q \circ X) \in C([0,T], \mathscr{C}^{0-})$ is given (together with tree data $X, X^{\vee}, X^{\vee\!\!\!\vee}, X^{\vee\!\!\!\vee\!\!\!\vee}, X^{\vee\vee}$).
- Obtain local existence and uniqueness of paracontrolled solutions. Solution depends pathwise continuously on extended data $J(\xi) = (\xi, X, X^{\vee}, X^{\vee\!\!\!\vee}, X^{\vee\!\!\!\vee\!\!\!\vee}, X^{\vee\vee}, Q \circ X)$.

KPZ equation

KPZ equation:

$$\mathcal{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x), \qquad h(0) = h_0.$$

Expect $h(t) \in \mathscr{C}^{1/2-}$, so $\partial_x h(t) \in \mathscr{C}^{-1/2-}$ and $(\partial_x h(t))^2$ not defined.
But: expand

$$u = Y + Y^{\vee} + 2Y^{\vee\vee} + h^P,$$

where $\mathcal{L}Y = \xi$, $\mathcal{L}Y^{\vee} = \partial_x Y \partial_x Y, \ldots$ In general: $\partial_x Y^{\tau} = X^{\tau}$. Make paracontrolled ansatz for h^P:

$$h^P = \pi_{<}(h', P) + h^{\sharp}$$

with $h' \in C([0,T], \mathscr{C}^{1/2-})$, $h^{\sharp} \in C([0,T], \mathscr{C}^{2-})$, $\mathcal{L}P = X$. Write $h \in \mathscr{P}_{\text{kpz}}$.

Can define $(\partial_x h(t))^2$ for $h \in \mathscr{P}_{\text{kpz}}$ and obtain local existence and uniqueness of solutions.

KPZ and Burgers equation

$h \in \mathscr{P}_{\text{kpz}}$ if

$$h = Y + Y^{\vee} + 2Y^{\vee\!\!\vee} + h^P, \qquad h^P = h' \prec P + h^\sharp.$$

$u \in \mathscr{P}_{\text{rbe}}$ if

$$u = X + X^{\vee} + 2X^{\vee\!\!\vee} + u^Q, \qquad u^Q = u' \prec Q + u^\sharp.$$

- If $h \in \mathscr{P}_{\text{kpz}}$, then $\partial_x h \in \mathscr{P}_{\text{rbe}}$.
- If h solves KPZ equation, then $u = \partial_x h$ solves Burgers equation with initial condition $u(0) = \partial_x h_0$.
- If $u \in \mathscr{P}_{\text{rbe}}$, then any solution h of $\mathcal{L}h = u^2 + \xi$ is in $\mathscr{P}_{\text{kpz}}$.
- If u solves Burgers equation with initial condition $u(0) = \partial_x h_0$, and h solves $\mathcal{L}h = u^2 + \xi$ with initial condition $h(0) = h_0$, then h solves KPZ equation.

KPZ and heat equation

Heat equation:

$$\mathcal{L}w(t,x) = w(t,x) \diamond \xi(t,x) = w(t,x)\xi(t,x) - w(t,x)\cdot\infty, \quad w(0) = w_0.$$

Paracontrolled ansatz: $w \in \mathscr{P}_{\text{rhe}}$ if

$$w = e^{Y + Y^{\vee} + 2Y^{\curlyvee}} w^P, \qquad w^P = \pi_<(w', P) + w^\sharp$$

(comes from Cole-Hopf transform).

- Slightly cheat to make sense of product $w \diamond \xi$ for $w \in \mathscr{P}_{\text{rhe}}$:

  $$w \diamond \xi = \mathcal{L}w - e^{Y+Y^{\vee}+2Y^{\curlyvee}}\left[\mathcal{L}w^P - [\mathcal{L}(Y^{\vee} + Y^{\curlyvee}) + (\partial_x(Y + Y^{\vee} + 2Y^{\curlyvee}))^2]w^P\right] + 2e^{Y+Y^{\vee}+2Y^{\curlyvee}}\partial_x(Y + Y^{\vee} + 2Y^{\curlyvee})\partial_x w^P;$$

  (agrees with renormalized pointwise product $w \diamond \xi$ in smooth case and with Itô integral in white noise case, continuous in extended data).
- Obtain global existence and uniqueness of solutions.
- One-to-one correspondence between $\mathscr{P}_{\text{kpz}}$ and strictly positive elements of $\mathscr{P}_{\text{rhe}}$.
- Any solution of KPZ gives solution of heat equation. Any strictly positive solution of heat equation gives solution of KPZ equation.

Para-modelled distributions

Let $\gamma > 0$ and (T, Π, Γ) regularity structure. Say f is para-modelled, $f \in \mathscr{P}^\gamma$, if there exists $f^\pi \in \mathscr{D}^\gamma$, with

$$f - \pi_<(f^\pi, \Pi) \in C^\gamma.$$

Example: $\mathscr{R} f^\pi \in \mathscr{P}^\gamma$.
Consider rough path model, say
$T = \mathrm{span}(\Xi, \mathscr{I}(\Xi)\Xi, \mathscr{I}(\mathscr{I}(\Xi)\Xi)\Xi, \mathbf{1}, \mathscr{I}(\Xi), \mathscr{I}(\mathscr{I}(\Xi)\Xi))$. Try to solve $\partial_t u = F(u)\xi$.
(Simplified) para-modelled ansatz: $u = \mathscr{R} u^\pi = \pi_<(u^\pi, \Pi) + u^\sharp$ with $u^\pi \in \mathscr{D}^{3\alpha}$. Equation for $u^\sharp$:

$$\partial_t u^\sharp = -\partial_t \pi_<(u^\pi, \Pi) + F(u)\xi = \pi_<(u^\pi, D\Pi) - \pi_<(F(u^\pi) \star \xi^\pi, \Pi) + \text{smooth}.$$

To have $u^\sharp \in C^{3\alpha}$: choose expansion u^π so that all coefficients for terms of homogeneity $< 3\alpha - 1$ cancel. Obtain a priori bounds on $\|u^\sharp\|_{3\alpha}$ and then on $\|u^\pi\|_{\mathscr{D}^{3\alpha}}$. Thus at least local existence of solutions.

Stochastic Quantization

Stochastic quantization of $(\Phi^4)_3 : \xi \in C^{-5/2-}$, $u \in C^{-1/2-}$,
$u = u_1 + u_2 + u_{\geqslant 3}$.

$$\mathcal{L}u = \xi + \lambda(u^3 - 3c_1 u - c_2 u)$$

$$\mathcal{L}u_1 + \mathcal{L}u_{\geqslant 2} = \xi + \lambda(u_1^3 - 3c_1 u_1) + 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_{\geqslant 2}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ $\mathcal{L}u_1 = \xi \Rightarrow u_1 \in C^{-1/2-}$, $\mathcal{L}u_2 = \lambda(u_1^3 - 3c_1 u_1) \Rightarrow u_2 \in C^{1/2-}$

$$\mathcal{L}u_{\geqslant 3} = 3\lambda(u_{\geqslant 2}(u_1^2 - c_1)) + 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3} u_2 u_1) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3 - \lambda c_2 u$$

▷ Ansatz: $u_{\geqslant 3} = 3\lambda u_{\geqslant 2} \prec X + u^\sharp$, with $\mathcal{L}X = (u_1^2 - c_1)$

$$\mathcal{L}u^\sharp = -3\lambda \mathcal{L}u_{\geqslant 2} \prec X + 3\lambda \mathrm{D}u_{\geqslant 2} \prec \mathrm{D}X + 3\lambda(u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u) + 3\lambda(u_{\geqslant 2} \succ (u_1^2 - c_1$$

$$+ 3\lambda(u_2^2 u_1) + 6\lambda(u_{\geqslant 3}(u_2 u_1)) + 3\lambda(u_{\geqslant 3}^2 u_1) + \lambda u_{\geqslant 2}^3$$

$$u_{\geqslant 2} \circ (u_1^2 - c_1) - c_2 u = (u_2 \circ (u_1^2 - c_1) - c_2 u_1) + (u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2})$$

$$(u_{\geqslant 3} \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) = (3\lambda(u_{\geqslant 2} \prec X) \circ (u_1^2 - c_1) - c_2 u_{\geqslant 2}) + u^\sharp \circ (u_1^2 - c_1)$$

$$= u_{\geqslant 2}(3\lambda(X \circ (u_1^2 - c_1)) - c_2) + 3\lambda C(u_{\geqslant 2}, X, (u_1^2 - c_1)) + u^\sharp \circ (u_1^2 - c_1)$$

▷ Basic objects:
$(u_1^2 - c_1), (u_1^3 - 3c_1 u_1), (3\lambda(X \circ (u_1^2 - c_1)) - c_2), (u_2 u_1), (u_2^2 u_1)$

Controlled paths/distributions

Controlled paths are paths which “looks like” a *given* path which often is random (but not necessarily).

A "good" quantification of this proximity allows a great deal of computations to be carried on explicitly on the base path and then extends them to all controlled paths.

A mix of functional analytic arguments and probabilistic ones.

Basic analogies

- Itô processes

$$\mathrm{d}X_t = f_t \mathrm{d}M_t + g_t \mathrm{d}t$$

- Amplitude modulation

$$f(t) = g(t)\sin(\omega t)$$

with $|\operatorname{supp}\hat{g}| \ll \omega$.

Small detour : Young integral

Take $f \in \mathscr{C}^\rho$, $g \in \mathscr{C}^\gamma$ with $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{f \prec Dg}_{\mathscr{C}^{\gamma-1}} + \underbrace{f \circ Dg + f \succ Dg}_{\mathscr{C}^{\gamma+\rho-1}}$$

then

$$\int fDg = \underbrace{\int f \prec Dg}_{\mathscr{C}^{\gamma}} + \underbrace{\int (f \circ Dg + f \succ Dg)}_{\mathscr{C}^{\gamma+\rho}}$$
$$= f \prec g + \mathscr{C}^{\gamma+\rho}.$$

Compare with standard estimate for the Young integral in Hölder spaces (valid when $\gamma + \rho > 1$):

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}).$$

Expansion in smalleness of increments vs. Expansion in regularity