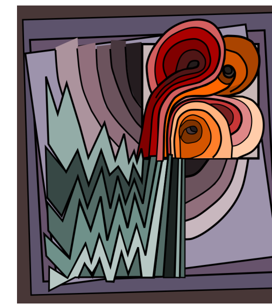
NEMSA workshop· Warwick · June 7th, 2023

Stochastic quantisation of the fractional Φ^4_3 Euclidean QFT in the full subcritical regime



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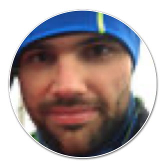
Part I · Euclidean QFTs & stochastic analysis

Part II · the flow equation method for the fractional Φ^4_3



joint work with

Paolo Rinaldi (Bonn)

Part I · Euclidean QFTs & stochastic analysis

EQFs – for mathematicians

an EQFT is a prob. measure μ on $\mathscr{S}'(\mathbb{R}^d)$ satisfying Osterwalder–Schrader axioms

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1. **Regularity**: $\|\varphi\|_*$ is some norm on $\mathscr{S}'(\mathbb{R}^d)$ and $\vartheta > 0$

$$\int_{\mathscr{S}'(\mathbb{R}^d)} e^{\vartheta\|\varphi\|_*}\mu(\mathrm{d}\varphi) < \infty$$

2. **Euclidean covariance**: the Euclidean group G (rotations R + translations h)

$$\int_{\mathscr{S}'(\mathbb{R}^d)} F(\varphi(R\cdot+h))\mu(\mathrm{d}\varphi) = \int_{\mathscr{S}'(\mathbb{R}^d)} F(\varphi)\mu(\mathrm{d}\varphi)$$

3. **Reflection positivity**: Let $\theta(x_1,\ldots x_d) = (-x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then

$$\int \overline{F(\theta\varphi)}F(\varphi)\mu(\mathrm{d}\varphi) \geqslant 0$$

Gaussian free field

GFF · simplest example of EQF · Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ s.t.

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean · $m>0$ is the *mass* · $G(0)=+\infty$ if $d\geqslant 2$: not a function · distribution of regularity

$$\alpha < (2-d)/2$$

▷ can be used to construct a QFT but the theory is free: no interaction

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variation · fractional Laplacian covariance $s\in(0,1)$

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} (a-\Delta)^{-1}(x-y)\rho(\mathrm{d}a) = (m^2+(-\Delta)^s)^{-1}(x-y)$$

interactions?

what about non-Gaussian EQFs? hard!

♠ idea: try to maintain the "Markovianity" of the GFF μ · heuristically

$$\nu(\mathrm{d}\varphi) = \frac{e^{\int_\Lambda V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi),$$

with $\Lambda = \Lambda_+ \cup \theta\Lambda_+$ and $V : \mathbb{R} \to \mathbb{R}$ so that

$$\int_\Lambda V(\varphi(x))\mathrm{d}x = \int_{\Lambda_+} V(\varphi(x))\mathrm{d}x + \int_{\Lambda_+} V((\theta\varphi)(x))\mathrm{d}x$$

☞ Reflection positivity holds:

$$\int \overline{F(\theta\varphi)}F(\varphi)\nu(\mathrm{d}\varphi) = \int \frac{\overline{F(\theta\varphi)e^{\int_{\Lambda_+} V(\theta\varphi(x))\mathrm{d}x}}F(\varphi)e^{\int_{\Lambda_+} V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi) \geqslant 0.$$

☞ unfortunately (even if we can make sense of it) will not be translation invariant · we need $\Lambda \to \mathbb{R}^d$

non-Gaussian Euclieand fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{Z}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side L

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi)e^{-\overbrace{\frac{1}{2}\sum_{x\in\mathbb{Z}^d_{\varepsilon,L}}|\nabla_\varepsilon\varphi(x)|^2+m^2\varphi(x)^2+V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}}\mathrm{d}\varphi$$

ε is an <u>UV regularisation</u> and L the <u>IR regularisation</u>

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$ · e.g. take V_ε polynomial bounded below ($d = 2, 3$)

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon\xi^2)$$

the limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$

❸ study the possible limit points [the $\boldsymbol{\Phi}^4_d$ **measure**] · uniqueness? non-uniqueness? correlations? description?

stochastic quantisation

Parisi–Wu ('81) introduced a stationary stochastic evolution associated with the EQF

$$\partial_t \Phi(t,x) = -\frac{\delta S(\Phi(t,x))}{\delta \Phi} + \eta(t,x), \qquad t \geqslant 0, x \in \mathbb{R}^d$$

with η space-time white noise

$$\langle \Phi(t,x_1) \cdots \Phi(t,x_n) \rangle = \frac{1}{Z} \int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(t,x_1) \cdots \varphi(t,x_n) e^{-S(\varphi)} \mathrm{d}\varphi, \qquad t \in \mathbb{R}$$

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transport interpretation: the map

$$\gamma \sim \eta \mapsto \Phi(t,\cdot) \sim \nu$$

sends the Gaussian measure of the space-time white noise γ to the EQF ν

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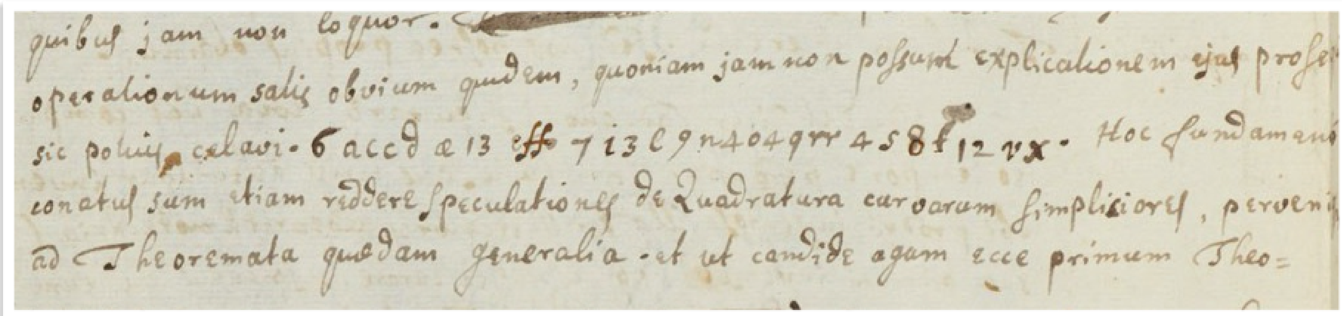
what is stochastic quantisation?


quibus jam non loquor.
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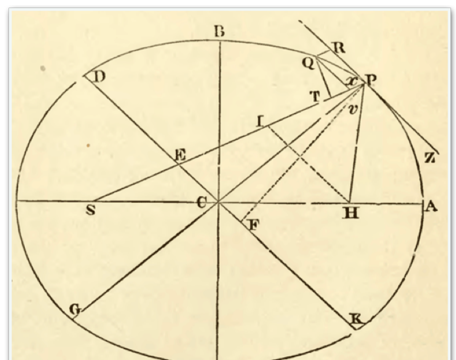
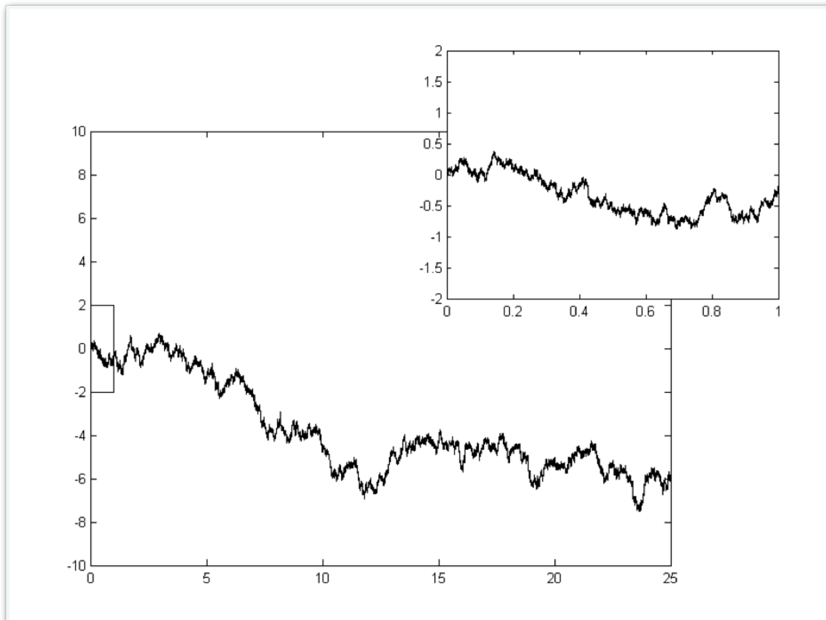
Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa (Newton)



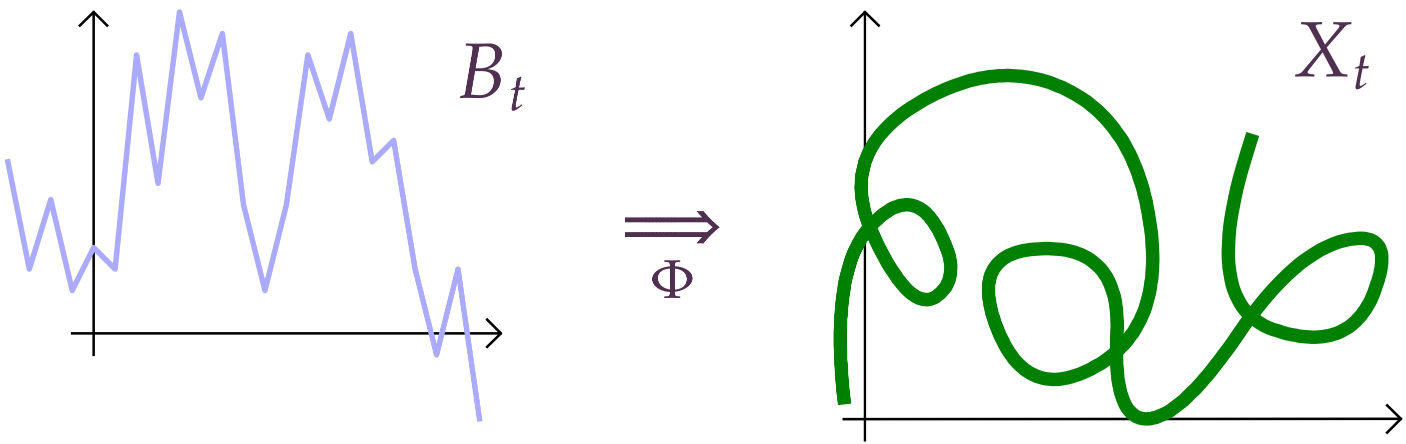
[Given an equation involving any number of fluent quantities to find the fluxions, and vice versa]

The word “*random*” comes from a French hunting term: “*randon*” designates the erratic course of the deer which zigzags trying to escape the dogs. The word also gave “*randonnée*” (hiking) in French.



Ito's idea

Ito arrived to his calculus while trying to understand Feller's theory of diffusions as an evolution in the space of probability measures and he introduced stochastic differential equations to define a map (**the Ito map**) which send Wiener measure to the law of a diffusion.



[...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.

(D. Stroock, “Elements of stochastic calculus and analysis ”, Springer, 2018)

nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…

analysis vs. stochastic analysis

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y) \in \mathcal{O} \subseteq \mathbb{R}^2$	global description	$P_t(x, \mathrm{d}y)$
$\alpha(x-x_0)^2 + \beta(y-y_0)^2 = \gamma$	·	$P_{t+s}(x,\mathrm{d}y) = \int P_s(x,\mathrm{d}z) P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t) \approx x(t) + a\delta t + o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$at + bt^2 + \cdots$	building block	$(W_t)_t$
$(\ddot{x}(t), \ddot{y}(t)) = F(x(t), y(t))$	local/global link	$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$

☞ other examples: rough paths, regularity structures, SLE, . . .

stochastic quantisation as a stochastic analysis

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x,\mathrm{d}y)$	global description	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi$
$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$		$\left\langle F(\varphi)\frac{\delta S(\varphi)}{\delta\varphi}+\frac{\delta F(\varphi)}{\delta\varphi}\right\rangle=0$
t	change parameter	t
$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$	local description	$\phi(t+\delta t)\approx\alpha\phi(t)+\beta\delta X(t)+\cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X=\frac{1}{2}[(\Delta_x-m^2)X]+\xi$
$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$	local/global link	$\partial_t\phi=\frac{1}{2}[(\Delta_x-m^2)\phi-V'(\phi)]+\xi$

Part II · the flow equation method for the fractional Φ^4_3

[P. Rinaldi, **MG** · Stochastic quantisation of the fractional Φ^4_3 Euclidean quantum field in the full subcritical regime · WIP]

fractional Φ^4_3 model

probability measure $\nu_{\varepsilon,M}$ on $\Omega_{\varepsilon,M} = \{\varphi \colon \mathbb{T}^d_{\varepsilon,M} \to \mathbb{R}\}$, $\mathbb{T}^d_{\varepsilon,M} = (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^d$, $\varepsilon > 0$, $M \in \mathbb{N}$

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) = \frac{\exp(-S_{\varepsilon,M}(\varphi))}{Z_{\varepsilon,M}} \prod_{x \in \mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x)$$

$$S_{\varepsilon,M}(\varphi) = \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon,M}} \left[\frac{1}{2}\varphi(x)(-\Delta_\varepsilon)^s \varphi(x) + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4}\varphi(x)^4 - \frac{r_\varepsilon}{2}\varphi(x)^2 \right]$$

* $m > 0$ *mass*, $\lambda > 0$ *coupling constant*, $r_\varepsilon > 0$ *mass renormalization*

Theorem. *Let $d = 3$ and $s \in (3/4, 1]$. There exists a choice of the sequence of mass renormalisations $(r_\varepsilon)_{\varepsilon > 0}$, with $r_\varepsilon = r_\varepsilon(\lambda) \to +\infty$ as $\varepsilon \to 0$, such that*

$\longrightarrow$ $(\nu_{\varepsilon,M})_{\varepsilon,M}$ *is a tight family of probability measure on $\mathscr{S}'(\mathbb{R}^3)$ as $M \to \infty$ and $\varepsilon \to 0$*

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stochastic quantisation

finite system of SDEs for $\phi^{(\varepsilon,M)}\colon \Lambda_{\varepsilon,M} \to \mathbb{R}$, $\Lambda_{\varepsilon,M} := \mathbb{R} \times \mathbb{T}^d_{\varepsilon,M}$

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]\phi^{(\varepsilon,M)} + \lambda(\phi^{(\varepsilon,M)})^3 - r_\varepsilon \phi^{(\varepsilon,M)} = 2^{1/2}\xi^{(\varepsilon,M)}$$

$\lambda > 0, r_\varepsilon > 0 \cdot \xi^{(\varepsilon,M)}$ *space–time white noise*

$$\mathbb{E}[\xi^{(\varepsilon,M)}(t,x)\xi^{(\varepsilon,M)}(s,y)] = \delta(t-s)\varepsilon^d \mathbb{1}_{x=y}, \qquad (t,x),(s,y) \in \Lambda_{\varepsilon,M}$$

$\exists$ stationary solution $\phi^{(\varepsilon,M)}$ s.t. $\mathrm{Law}(\phi^{(\varepsilon,M)}(t)) = \nu_{\varepsilon,M}$ for all $t \in \mathbb{R}$

▷ PDE estimates in weighted Sobolev spaces + positivity of the fractional Laplacian

$\Longrightarrow$ tightness of $(\phi^{(\varepsilon,M)})_M \Longrightarrow \exists$ of sub-sequential limits as $M \to \infty$

▷ any accum. point $\phi^{(\varepsilon)}$ is solution in $\Lambda_\varepsilon := \mathbb{R} \times (\varepsilon\mathbb{Z})^d$ of an ∞ system of SDEs

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]\phi^{(\varepsilon)} + \lambda(\phi^{(\varepsilon)})^3 - r_\varepsilon \phi^{(\varepsilon)} = \xi^{(\varepsilon)}$$

▷ **subcriticality** : at small scales the non-linear term is a perturbation of the linear part of the equation with the white-noise source · $\phi^{(\varepsilon)}$ behaves like the solution $X^{(\varepsilon)}$ of the linear equation

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s]X^{(\varepsilon)} = \xi^{(\varepsilon)}$$

☞ as $\varepsilon \to 0$ $X^{(\varepsilon)}$ has distributional limit in Besov spaces of regularity $< s - d/2$.

scale decomposition

$\mathscr{L}_\varepsilon := \partial_t + m^2 + (-\Delta_\varepsilon)^s \cdot F^\varepsilon(\psi) := -\lambda\psi^3 + r_\varepsilon \psi + \xi^{(\varepsilon)} \cdot (J_\sigma)$ scale decomposition $J_1 = \mathrm{Id}$

$$\mathscr{L}_\varepsilon \phi^\varepsilon = F^\varepsilon(\phi^{(\varepsilon)}) \quad \Longrightarrow \quad \phi^\varepsilon_\sigma := J_\sigma \phi^{(\varepsilon)} \quad \Longrightarrow \quad \mathscr{L}_\varepsilon \phi^\varepsilon_\sigma = J_\sigma(F^\varepsilon(\phi^{(\varepsilon)}))$$

$(F^\varepsilon_\sigma)_{\sigma\in(0,1)} : C(\Lambda_\varepsilon) \to \mathscr{S}'(\Lambda_\varepsilon)$ family of smooth functionals s.t. $\boldsymbol{F^\varepsilon_1 = F^\varepsilon}$,

$$\boldsymbol{F^\varepsilon(\phi^\varepsilon) = F^\varepsilon_\mu(\phi^\varepsilon_\mu) + R^\varepsilon_\mu} \qquad \boldsymbol{R^\varepsilon_\mu := \int_\mu^1 [\partial_\sigma F^\varepsilon_\sigma(\phi^\varepsilon_\sigma) + \mathrm{D}F^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\cdot(\partial_\sigma \phi^\varepsilon_\sigma)]\mathrm{d}\sigma}$$

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for any choice of $(F^\varepsilon_\sigma)_{\sigma\in[0,1]}$, $(\phi^\varepsilon_\mu, R^\varepsilon_\mu)_{\mu\in(0,1)}$ satisfies

$$\begin{cases} \mathscr{L}_\varepsilon \phi^\varepsilon_\mu = J_\mu(F^\varepsilon_\mu(\phi^\varepsilon_\mu) + R^\varepsilon_\mu) \\ R^\varepsilon_\mu = \int_\mu^1 H^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F^\varepsilon_\sigma(\phi^\varepsilon_\sigma)\cdot \dot{G}_\sigma R^\varepsilon_\sigma]\mathrm{d}\sigma \end{cases}$$

with

$$H^\varepsilon_\sigma(\psi) := \partial_\sigma F^\varepsilon_\sigma(\psi) + \mathrm{D}F^\varepsilon_\sigma(\psi) \cdot \dot{G}_\sigma F^\varepsilon_\sigma(\psi)$$

⊡ (fractional) PDE estimates

Theorem. *let*

$$f_\sigma := \partial_t \phi_\sigma + (-\Delta)^s \phi_\sigma + \lambda \phi_\sigma^3,$$

then for $\bar\mu$ *large enough, there exists an universal constant* $C > 0$,

$$\boldsymbol{|||\phi|||_{\bar\mu} \lesssim 1 + C\lambda^{-1/3} |||f|||_{\#,\bar\mu}^{1/3}}$$

where

$$|||\phi|||_{\bar\mu} \approx \sup_{\sigma\in(\bar\mu,1)} [\![\sigma]\!]^{\gamma} \|\phi_\sigma\|, \qquad |||f|||_{\#,\bar\mu} \approx \sup_{\sigma\in(\bar\mu,1)} [\![\sigma]\!]^{3\gamma} \|f_\sigma\|$$

Theorem. *Let* (ϕ, R) *be solution of*

$$\begin{cases} \mathscr{L}\phi_\mu = J_\mu(F_\mu(\phi_\mu) + R_\mu) \\ R_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F_\sigma(\phi_\sigma) \cdot \dot{G}_\sigma R_\sigma]\mathrm{d}\sigma \end{cases}$$

Let $(F_\sigma)_\sigma$ *be s.t.,*

$$|||\sigma \mapsto J_\sigma F_\sigma(\psi_\sigma) - (-\lambda\psi_\sigma^3)|||_\# \leqslant [\![\bar{\mu}]\!]^\zeta [C_F(1 + |||\psi|||)^M + (1 + |||\psi|||)^2 |||\mathscr{L}\psi|||_\#],$$

for some (non-random) $M, \zeta > 0$ *and random* C_F *almost surely finite together with some suitable estimates for* $H_\sigma(\phi_\sigma)$ *and* $\mathrm{D}F_\sigma(\phi_\sigma)$.

Then there exists a (random) $\bar{\mu} \in (0,1)$ *such that*

$$\boldsymbol{|||\phi|||_{\bar{\mu}} \lesssim 1, \qquad |||\mu \mapsto J_\mu R_\mu^\varepsilon|||_{\#,\bar{\mu}} \lesssim 1.}$$

⚂ appoximate effective force

Theorem. *There exists a random functional* $(F^{\varepsilon}_{\mu})_{\mu\in(0,1)}$ *satisfying the boundary condition*

$$F_1^{\varepsilon}(\psi)=-\lambda\psi^3-r_{\varepsilon}\psi+\xi^{(\varepsilon)}$$

and the bound

$$|||\sigma\mapsto J_{\sigma}F_{\sigma}(\psi_{\sigma})-(-\lambda\psi_{\sigma}^3)|||_{\#}\ \leqslant\ [\![\bar{\mu}]\!]^{\zeta}[C_F(1+|||\psi|||)^M+(1+|||\psi|||)^2|||\mathscr{L}\psi|||_{\#}]$$

where $C_F=\|F^{\varepsilon,\mathfrak{A}}\|$ *is a finite random constant together with some suitable estimates for*

$$H_{\sigma}(\phi_{\sigma}):=\partial_{\sigma}F_{\sigma}^{\varepsilon}+\mathrm{D}F_{\sigma}^{\varepsilon}\cdot\dot{G}_{\sigma}F_{\sigma}^{\varepsilon}$$

and $\mathrm{D}F_{\sigma}(\phi_{\sigma})$ *such that, for any large* p*,*

$$\sup_{\varepsilon>0}\mathbb{E}[\|F^{\varepsilon,\mathfrak{A}}\|^p]<\infty$$

⚃ random flow equation

▷ look for approximate solutions to

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma \cdot (\dot{G}_\sigma F_\sigma) = 0, \qquad F_1 = F$$

in the space of polynomial in the fields

▷ write $F_\sigma = \sum_k F_\sigma^{(k)}$, where we denote with $F_\sigma^{(k)}$ the component of degree k

$$F^{(k)}(\Psi)(z) = \int_{\Lambda^k} F^{(k)}(z; z_1, \ldots, z_k)\Psi(z_1)\cdots\Psi(z_k)\mathrm{d}z_1\cdots\mathrm{d}z_k, \qquad z \in \Lambda.$$

▷ also derivatives of the field: $\Psi^{A_i}(z_i) := \partial^{A_i}\Psi(z_1)$ (jet bundle)

$$F^{(k)}(\Psi)(z) = \sum_{A_1,\ldots,A_k} \int_{\Lambda^k} F^{\langle A_1,\ldots,A_k\rangle}(z; z_1, \ldots, z_k)\Psi^{A_1}(z_1)\cdots\Psi^{A_k}(z_k)\mathrm{d}z_1\cdots\mathrm{d}z_k.$$

▷ example:

$$\Psi^3(z) = \int_{\Lambda^3} \delta^{(3)}(z; z_1, z_2, z_3)\Psi(z_1)\Psi(z_2)\Psi(z_3)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$$

norms

approximate the equation letting

$$F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F_\sigma^{[\ell]} \qquad \partial_\sigma F_\sigma^{[\ell+1]} + \sum_{\ell'=0}^{\ell} \mathrm{D}F_\sigma^{[\ell-\ell']} \cdot \dot{G}_\sigma F_\sigma^{(\ell')} = 0$$

$F^{\mathfrak{a}} \equiv F^{[\ell],\langle A_1,\ldots,A_k\rangle}$ for $\mathfrak{a} = (\ell, A_1, \ldots, A_k)$, $F^{\mathfrak{A}} := (F_\sigma^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A},\sigma\in(0,1)}$

$$\|F^{\mathfrak{a}}\| = \sup_{z\in\Lambda} \int_{\Lambda^k} |F^{[\ell],\langle A_1,\ldots,A_k\rangle}(z;z_1,\ldots,z_k)| \mathrm{d}z_1 \cdots \mathrm{d}z_k$$

$$\|F^{\mathfrak{A}}\| := \sup_{\mathfrak{a}\in\mathfrak{A}} \left[\sup_{\sigma\geqslant\mu\geqslant 0} [\![\sigma]\!]^{-[\mathfrak{a}]} \|F_\sigma^{\mathfrak{a}}\|_{\mu,\sigma} \right] \qquad [\mathfrak{a}] := -\alpha + \delta\ell(\mathfrak{a}) + \beta k(\mathfrak{a}) + |A(\mathfrak{a})|$$

(for suitable parameters α, β and δ to be fixed) · this means bounds of the form

$$\|F_\sigma^{\mathfrak{a}}\|_{\mu,\sigma} \lesssim \|F^{\mathfrak{A}}\| \, [\![\sigma]\!]^{[\mathfrak{a}]}$$

⚄ cumulants

cumulants of $(F^{\mathfrak{a}})_{\mathfrak{a}} \rightsquigarrow$ deterministic kernels $(\mathscr{F}^{\boldsymbol{a}})_{\boldsymbol{a}}$ defined by

$$\mathscr{F}^{\boldsymbol{a}} := \kappa_n(F^{\mathfrak{a}_1}, \cdots, F^{\mathfrak{a}_n}), \qquad \boldsymbol{a} \in \boldsymbol{A} = \{(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : \mathfrak{a}_k \in \mathfrak{A}, n \leqslant N\}$$

$$\|\mathscr{F}^A\| := \sup_{\boldsymbol{a} \in A} \left[\sup_{\sigma \geqslant \mu \geqslant 0} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma} \right]^{1/n(\boldsymbol{a})}$$

$$[\boldsymbol{a}] = -\rho + \theta n(\boldsymbol{a}) + \delta L(\boldsymbol{a}) + \beta K(\boldsymbol{a}) + |A(\boldsymbol{a})|$$

Theorem. ***Kolmogorov-type argument:*** *from estimates on the norm $\|\mathscr{F}^A\|$ to those on $\|F^{\mathfrak{A}}\|$ as*

$$\{\mathbb{E}[\|F^{\mathfrak{A}}\|^n]\}^{1/n} \lesssim \|\mathscr{F}^A\|$$

Duch's flow equation

Lemma. *Cumulants satisfy a flow equation*

$$\partial_\sigma \mathscr{F}_\sigma^{a} = \sum_{b} \mathscr{A}_{b}^{a}(\dot{G}_\sigma, \mathscr{F}_\sigma^{b}) + \sum_{b,c} \mathscr{B}_{b,c}^{a}(\dot{G}_\sigma, \mathscr{F}_\sigma^{b}, \mathscr{F}_\sigma^{c}),$$

$$[\![\sigma]\!]^{-[a]} \| \mathscr{A}_{b}^{a}(\dot{G}_\sigma, \mathscr{F}_\sigma^{b}) \|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[b]-1} \| \mathscr{F}_\sigma^{b} \|_{\mu,\sigma}.$$

$$[\![\sigma]\!]^{-[a]} \| \mathscr{B}_{b,c}^{a}(\dot{G}_\sigma, \mathscr{F}_\sigma^{b}, \mathscr{F}_\sigma^{c}) \|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[b]-[c]-1} \| \mathscr{F}_\sigma^{b} \|_{\mu,\sigma} \| \mathscr{F}_\sigma^{c} \|_{\mu,\sigma}.$$

[P. Duch · Renormalization of singular elliptic stochastic PDEs using flow equation · arXiv:2201.05031]

classification of cumulants

the structure of the flow equation for cumulants propagates estimates of the form

$$\sup_{\sigma \geqslant \mu} [\![\sigma]\!]^{-[a]} \|\mathscr{F}_\sigma^a\|_{\mu,\sigma} < \infty \qquad \Longrightarrow \qquad \|\mathscr{F}_\sigma^a\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{[a]}$$

* **irrelevant cumulants**$\rightsquigarrow [a] > \mathbf{0}$: the flow equation can be solved *backwards* starting from the final condition at $\sigma = 1$.
* **relevant cumulants**$\rightsquigarrow [a] < \mathbf{0}$: the flow equation cannot be integrated close to $\sigma = 1$;

  $\Longrightarrow$use the only freedom we have: fine tuning of the final condition

$$F(\psi) = -\lambda\psi^3 + r_\varepsilon \psi + \xi$$

* **marginal cumulants**$\rightsquigarrow [a =]\mathbf{0}$: we shall handle them as the irrelevant.

relevant cumulants

♟ relevant cumulants $\cdot\, \mathscr{F}^{\,a}$ s.t.

$$[\boldsymbol{a}] = -\rho + \theta\, n(\boldsymbol{a}) + \delta\, L(\boldsymbol{a}) + \beta\, K(\boldsymbol{a}) + |A(\boldsymbol{a})| < 0$$

$\Longrightarrow$ the only relevant cumulant is $\boldsymbol{\kappa_1(F_\sigma^{[\ell](1)})}$

♞ $\kappa_1(F_\sigma^{[\ell](1)})$ is a non-local object **!** but the only freedom we have is the fine tuning a constant (r_ε)!

♝ Taylor expansion

$$\begin{aligned} F_\sigma^{[\ell](1)}(\psi)(z) &= \int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)\psi(z_1)\mathrm{d}z_1 \\ &= \sum_{A:0\leqslant|A|\leqslant 1} \partial^A\psi(z)\int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)(z_1-z)^A\mathrm{d}z_1 + \\ &\quad +\sum_{A:1<|A|\leqslant 2} c_A\int_0^1 \mathrm{d}t\int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z;z_1)(z_1-z)^A\partial^A\psi(z+t(z_1-z))\mathrm{d}z_1 \end{aligned}$$

$\Longrightarrow$ we have **local terms** + some non-local remainder terms

localisation

♘ modify the flow equation by introducing the operators

$$(\boldsymbol{L}F_\sigma)(\Psi) = \sum_{\mathfrak{b}:k(\mathfrak{b})=1} \sum_{A:0\leqslant|A|\leqslant 1} \Psi^A(z)\int_\Lambda (z_1-z)^A F_\sigma^{[\ell(\mathfrak{b})]\langle 0\rangle}(z;z_1)\mathrm{d}z_1 + \sum_{\mathfrak{b}:k(\mathfrak{b})\neq 1} F_\sigma^{\mathfrak{b}}(\Psi)$$

$$(\boldsymbol{R}F_\sigma)(\Psi) = \sum_{\mathfrak{b}:k(\mathfrak{b})=1} \sum_{A:1<|A|\leqslant 2} c_A\int_0^1 \mathrm{d}t\int_\Lambda F_\sigma^{[\ell(\mathfrak{b})]\langle 0\rangle}(z;z_1)(z_1-z)^A\Psi^A(z+t(z_1-z))\mathrm{d}z_1$$

$$\partial_\sigma F_\sigma + (\boldsymbol{L}+\boldsymbol{R})[\mathrm{D}F_\sigma\cdot(\dot{G}_\sigma F_\sigma)] = 0 \Rightarrow \partial_\sigma \mathscr{F}_\sigma^{a} = \textstyle\sum_b \tilde{\mathscr{A}}_b^a(\dot{G}_\sigma, \mathscr{F}_\sigma^b) + \sum_{b,c} \tilde{\mathscr{B}}_{b,c}^a(\dot{G}_\sigma, \mathscr{F}_\sigma^b, \mathscr{F}_\sigma^c)$$

☞ same estimates + relevant cumulant $\kappa_1(F_\sigma^{[\ell]\langle 0\rangle})$ now **local** by construction

⤳∃ constants $(r_\sigma^\ell)_{\ell\geqslant 0}$ such that

$$\kappa_1(F^{[\ell]\langle 0\rangle}(z,z_1)) = \mathbb{E}(F^{[\ell]\langle 0\rangle}(z,z_1)) = r_\sigma^\ell\delta(z-z_1).$$

⚅ inductive procedure

solve and estimate the flow equation by induction over ℓ

$$M_\ell := \sup_{\boldsymbol{a}: L(\boldsymbol{a}) \leqslant \ell} \left[\sup_{\sigma \geqslant \mu} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma} \right]^{1/n(\boldsymbol{a})} < \infty$$

start with $M_0 < \infty$ · assume $M_\ell < \infty$ and take $\mathscr{F}_\sigma^{\boldsymbol{a}}$ with $L(\boldsymbol{a}) = \ell + 1$

☞ for $\mathscr{F}_\sigma^{\boldsymbol{a}}$ **irrelevant** ($[\boldsymbol{a}] > 0$) ⇒ solve the equation **backwards** from the final condition $\mathscr{F}_1^{\boldsymbol{a}} = 0$:

$$\mathscr{F}_\sigma^{\boldsymbol{a}} = \int_\sigma^1 \left[\sum_{\boldsymbol{b}} \tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a}} (\dot{G}_\eta, \mathscr{F}_\eta^{\boldsymbol{b}}) + \sum_{\boldsymbol{b},\boldsymbol{c}} \tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}} (\dot{G}_\eta, \mathscr{F}_\eta^{\boldsymbol{b}}, \mathscr{F}_\eta^{\boldsymbol{c}}) \right] \mathrm{d}\eta$$

$$\Rightarrow \qquad \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma} \leqslant C M_\ell^{n(\boldsymbol{a})+1} \int_\sigma^1 [\![\eta]\!]^{[\boldsymbol{a}]-1} \mathrm{d}\eta \lesssim C [\![\sigma]\!]^{[\boldsymbol{a}]} M_\ell^{n(\boldsymbol{a})+1}$$

this shows that we cannot do like that for $[\boldsymbol{a}] < 0$...

☞ for $\mathscr{F}_\sigma^a$ **relevant** ($[a]<0$) ⇒ solve the equation **forwards** by fixing $r_{\mu_0}^{\ell+1}$ to be some arbitrary value at some reference scale $\mu_0<1$

$$
\begin{aligned}
[\![\sigma]\!]^{-[a]}\|\mathscr{F}_\sigma^a\|_{\mu,\sigma} &\leqslant [\![\sigma]\!]^{-[a]}|r_{\mu_0}^{\ell+1}| + [\![\sigma]\!]^{-[a]}\int_{\mu_0}^{\sigma}\|\partial_{\sigma'}\mathscr{F}_{\sigma'}^a\|_{\mu,\sigma'}\mathrm{d}\sigma' \\
&\lesssim |r_{\mu_0}^{\ell+1}| + [\![\sigma]\!]^{-[a]} C M_\ell^{n(a)+1}\int_{\mu_0}^{\sigma}[\![\sigma']\!]^{[a]-1}\mathrm{d}\sigma' \\
&\lesssim |r_{\mu_0}^{\ell+1}| + C M_\ell^{n(a)+1}
\end{aligned}
$$

Lemma.

$$
\|\mathscr{F}^A\| = \sup_{a:L(a)\leqslant\bar{\ell},N(a)\leqslant N}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[a]}\|\mathscr{F}_\sigma^a\|_{\mu,\sigma}\right]^{1/n(a)} = M_{\bar{\ell}} \leqslant C(1+M_0)^{2^{\bar{\ell}}} < \infty
$$

☞ this complete the proofs of uniform apriori estimates on the SPDE → tightness

summary

interplay of techniques

PDEs ⊕ Renormalization Group ⊕ Stochastic Analysis

* **PDEs**
  * ▷ coercive a priori estimates to deal with the so-called large field problem
* **Renormalization Group**
  * ▷ **flow equation** for the effective force + handling of renormalization;
* **Stochastic Analysis**
  * ▷ flow Equation for cumulants + Kolmogorov argument

☞ insensitive of how close we are to the critical case

☞ renormalisation conditions as good boundary conditions when solving the flow equation

☞ no need of requiring small coupling constant

---

the end

(no human has been harmed with $\TeX$ to produce this presentation)

Boué–Dupuis formula

Theorem. *Let* $(B_t)_{t\geqslant 0}$ *be a Brownian motion on* $\mathbb{R}^n$*, then for any bounded* $F: C(\mathbb{R}_+;\mathbb{R}^n) \to \mathbb{R}$ *we have*

$$\log \mathbb{E}[e^{F(B_\bullet)}] = \sup_{u\in\mathbb{H}_a} \mathbb{E}\left[F(B_\bullet + I(u)_\bullet) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

with $u:\Omega\times\mathbb{R}_+\to\mathbb{R}^n$ *adapted to* B *and with*

$$I(u)_t := \int_0^t u_s \mathrm{d}s.$$

$$\frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s \approx H(\mathrm{Law}(B_\bullet + I(u)_\bullet)|\mathrm{Law}(B_\bullet)).$$

[M. Boué and P. Dupuis, A Variational Representation for Certain Functionals of Brownian Motion, *Ann. Prob.* 26(4), 1641–59]

Boué–Dupuis for the $d=2$ GFF

$$\mathbb{E}[W_t(x)W_s(y)] = (t\wedge s)(m^2-\Delta)^{-1}(x-y), \qquad t,s\in[0,1].$$

The BD formula gives

$$-\log\int e^{-F(\phi)}\mu(\mathrm{d}\phi) = -\log\mathbb{E}[e^{-F(W_1)}] = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[F(W_1+Z_1)+\frac{1}{2}\int_0^1\|u_s\|_{L^2}^2\mathrm{d}s\right],$$

where

$$Z_t = (m^2-\Delta)^{-1/2}\int_0^t u_s\mathrm{d}s, \qquad u_t = (m^2-\Delta)^{1/2}\dot{Z}_t$$

$$-\log\mathbb{E}[e^{-F(W_1)}] = \inf_{Z\in H^a}\mathbb{E}[F(W_1+Z_1)+\mathscr{E}(Z_\bullet)],$$

with

$$\mathscr{E}(Z_\bullet) := \frac{1}{2}\int_0^1\|(m^2-\Delta)^{1/2}\dot{Z}_s\|_{L^2}^2\mathrm{d}s = \frac{1}{2}\int_0^1(\|\nabla\dot{Z}_s\|_{L^2}^2+m^2\|\dot{Z}_s\|_{L^2}^2)\mathrm{d}s$$

Φ^4_2 in a bounded domain Λ

Fix a compact region $\Lambda \Subset \mathbb{R}^2$ and consider the Φ^4_2 measure θ_Λ on $\mathscr{S}'(\mathbb{R}^2)$ with interaction in Λ and given by

$$\theta_\Lambda(\mathrm{d}\phi) := \frac{e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)}{\int e^{-\lambda V_\Lambda(\phi)}\mu(\mathrm{d}\phi)} \qquad \phi \in \mathscr{S}'(\mathbb{R}^2) \tag{1}$$

with interaction potential $V_\Lambda(\phi) := \int_\Lambda \phi^4 - c\int_\Lambda \phi^2$. For any $f: \mathscr{S}'(\mathbb{R}^d) \to \mathbb{R}$ (non necessarily linear) let

$$e^{-\mathscr{W}_\Lambda(f)} := \int e^{-f(\phi)}\theta_\Lambda(\mathrm{d}\phi).$$

We have the variational representation, $Z = Z_1$, $Z_\bullet = (Z_t)_{t\in[0,1]}$:

$$\mathscr{W}_\Lambda(f) = \inf_{Z\in H^a} F^{f,\Lambda}(Z_\bullet) - \inf_{Z\in H^a} F^{0,\Lambda}(Z_\bullet)$$

where

$$F^{f,\Lambda}(Z_\bullet) := \mathbb{E}[f(W+Z) + \lambda V_\Lambda(W+Z) + \mathscr{E}(Z_\bullet)].$$

---

renormalized potential

$$V_\Lambda(W+Z) = \int_\Lambda \Big\{ \underbrace{W^4 - cW^2}_{\mathbb{W}^4} + 4\underbrace{\Big[W^3 - \frac{c}{4}W\Big]}_{\mathbb{W}^3} Z + 6\underbrace{\Big[W^2 - \frac{c}{6}\Big]}_{\mathbb{W}^2} Z^2 + 4WZ^3 + Z^4 \Big\}$$

take $c = 12\mathbb{E}[W^2(x)] = +\infty$

$$V_\Lambda(W+Z) = \int_\Lambda \{4\mathbb{W}^3 Z + 6\mathbb{W}^2 Z^2 + 4WZ^3 + Z^4\} + \cdots$$

$$\mathbb{W}^n \in \mathscr{C}^{-n\kappa}(\Lambda) = B^{-n\kappa}_{\infty,\infty}(\Lambda)$$

Here $B^{-\kappa}_{\infty,\infty}(\Lambda)$ is an Hölder–Besov space. A distribution $f \in \mathscr{S}'(\mathbb{T}^d)$ belongs to $B^{\alpha}_{\infty,\infty}(\Lambda)$ iff for any $n \geqslant 0$

$$\|\Delta_n f\|_{L^\infty} \leqslant (2^n)^{-\alpha} \|f\|_{B^\alpha_{\infty,\infty}(\Lambda)}$$

where $\Delta_n f = \mathscr{F}^{-1}(\varphi_n(\cdot)\mathscr{F}f)$ and φ_n is a function supported on an annulus of size $\approx 2^n$. We have $f = \sum_{n\geqslant 0} \Delta_n f$. If $\alpha > 0$ $B^\alpha_{\infty,\infty}(\mathbb{T}^d)$ is a space of functions otherwise they are only distributions.

Euler–Lagrange equation for minimizers

Lemma. *There exists a minimizer $Z = Z^{f,\Lambda}$ of $F^{f,\Lambda}$. Any minimizer satisfies the Euler–Lagrange equations*

$$\begin{aligned}&\mathbb{E}\Big(4\lambda\int_\Lambda Z^3 K + \int_0^1 \int_\Lambda (\dot{Z}_s(m^2-\Delta)\,\dot{K}_s)\mathrm{d}s\Big)\\ = \;&\mathbb{E}\Big(\int_\Lambda f'(W+Z)\,K + \lambda\int_\Lambda (\mathbb{W}^3 + \mathbb{W}^2 Z + 12\,W Z^2)\,K\Big)\end{aligned}$$

for any K adapted to the Brownian filtration and such that $K \in L^2(\mu, H)$.

▷ technically one really needs a relaxation to discuss minimizers, we ignore this all along this talk. the actualy object of study is the law of the pair $(\mathbb{W}, Z)$ and not the process Z. (similar as what happens in the Φ^4_3 paper)

apriori estimates

we use polynomial weights $\rho(x) = (1 + \ell|x|)^{-n}$ for large $n > 0$ and small $\ell > 0$.

Theorem. *There exists a constant C independent of $|\Lambda|$ such that, for any minimizer Z of $F^{f,\Lambda}(\mu)$ and any spatial weight $\rho: \Lambda \to [0,1]$ with $|\nabla\rho| \leqslant \varepsilon\,\rho$ for some $\varepsilon > 0$ small enough, we have*

$$\mathbb{E}\left[4\lambda\int_\Lambda \rho\, Z_1^4 + \int_0^1 \int_{\mathbb{R}^2} ((m^2 - \Delta)^{1/2}\rho^{1/2}\dot{Z}_s)^2 \mathrm{d}s\right] \leqslant C.$$

Proof. test the Euler–Lagrange equations with $K = \rho Z$ and then estimate the bad terms with the good terms and objects only depending on $\mathbb{W}$, e.g.

$$\left|\int_\Lambda \rho\, \mathbb{W}^3 Z\right| \leqslant C_\delta \|\mathbb{W}^3\|_{H^{-1}(\rho^{1/2})}^2 + \delta\|Z\|_{H^1(\rho^{1/2})}^2,$$

$$\left|\int_\Lambda \rho\, \mathbb{W}^2 Z^2\right| \leqslant C_\delta \|\rho^{1/8}\, \mathbb{W}^2\|_{C^{-\varepsilon}}^4 + \delta(\|\rho^{1/4}\,\bar{Z}\|_{L^4}^4 + \|\rho^{1/2}\,\bar{Z}\|_{H^{2\varepsilon}}^2), \cdots$$

tightness and bounds

$$\mathscr{W}_\Lambda(f) = \inf_Z F^{f,\Lambda}(Z) - \inf_Z F^{0,\Lambda}(Z) = F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

Therefore

$$F^{f,\Lambda}(Z^{f,\Lambda}) - F^{0,\Lambda}(Z^{f,\Lambda}) \leqslant \mathscr{W}_\Lambda(f) \leqslant F^{f,\Lambda}(Z^{0,\Lambda}) - F^{0,\Lambda}(Z^{0,\Lambda})$$

and since, for any g,

$$F^{f,\Lambda}(Z^{g,\Lambda}) - F^{0,\Lambda}(Z^{g,\Lambda}) = \mathbb{E}[f(W + Z^{g,\Lambda}) + \lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})]$$

$$-\mathbb{E}[\lambda V_\Lambda(W + Z^{g,\Lambda}) + \mathscr{E}(Z^{g,\Lambda})] = \mathbb{E}[f(W + Z^{g,\Lambda})]$$

$$\mathbb{E}[f(W + Z^{f,\Lambda})] \leqslant \mathscr{W}_\Lambda(f) \leqslant \mathbb{E}[f(W + Z^{0,\Lambda})]$$

Consequence: tightness of $(\theta_\Lambda)_\Lambda$ in $\mathscr{S}'(\mathbb{R}^2)$ and optimal exponential bounds (cfr. Hairer/Steele)

$$\sup_\Lambda \int \exp(\delta \|\phi\|^4_{W^{-\kappa,4}(\rho)}) \theta_\Lambda(\mathrm{d}\phi) < \infty.$$

Euler–Lagrange equation in infinite volume

moreover

$$\int f(\phi)\,\theta_\Lambda(\mathrm{d}\phi) = \mathbb{E}[f(X + Z^{0,\Lambda})]$$

the family $(Z^{f,\Lambda})_\Lambda$ is converging (provided we look at the relaxed problem) and any limit point $Z = Z^f$ satisfies a EL equation:

$$\mathbb{E}\left\{\int_{\mathbb{R}^2} f'(W+Z)\,K + 4\lambda\int_{\mathbb{R}^2} [\![(W+Z)^3]\!]\,K + \int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\,\dot{K}_s \mathrm{d}s\right\} = 0$$

for any test process K (adapted to $\mathbb{W}$ and to Z).

<u>**a kind of stochastic "elliptic" problem**</u>

the stochastic equation

rewrite the EL equation as

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \left(f'(W_1+Z_1) + 4\lambda [\![(W_1+Z_1)^3]\!] + \dot{Z}_s(m^2-\Delta)\right) \dot{K}_s \mathrm{d}s\right\} = 0$$

then

$$\mathbb{E}\left\{\int_0^1 \int_{\mathbb{R}^2} \mathbb{E}\left[f'(W_1+Z_1) + 4\lambda [\![(W_1+Z_1)^3]\!] + (m^2-\Delta)\dot{Z}_s \middle| \mathscr{F}_s\right] \dot{K}_s \mathrm{d}s\right\} = 0$$

which implies that

$$(m^2-\Delta)\dot{Z}_s = -\mathbb{E}\left[f'(W_1+Z_1) + 4\lambda [\![(W_1+Z_1)^3]\!] \middle| \mathscr{F}_s\right]$$

Open questions

- Uniqueness??
- Γ-convergence of the variational description of $\mathscr{W}_\Lambda(f)$?

not clear. We lack sufficient knowledge of the dependence on f of the solutions to the EL equations above.

---

exponential interaction

we can study similarly the model with

$$V^{\xi}(\varphi)=\int_{\mathbb{R}^2}\xi(x)[\![\exp(\beta\varphi(x))]\!]\mathrm{d}x$$

for $\beta^2<8\pi$ and $\xi:\mathbb{R}^2\to[0,1]$ a smooth spatial cutoff function

$$V^{\xi}(W+Z)=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))\underbrace{[\![\exp(\beta W(x))]\!]\mathrm{d}x}_{M^{\beta}(\mathrm{d}x)}$$

$$=\int_{\mathbb{R}^2}\xi(x)\exp(\beta Z(x))M^{\beta}(\mathrm{d}x),$$ [Gaussian multiplicative chaos]

BD formula

$$\begin{aligned}\mathcal{W}^{\xi,\exp}(f) &= -\log\int\exp(-f(\phi))\mathrm{d}\nu^{\xi}\\ &= \inf_{Z\in\mathfrak{H}_a}\mathbb{E}\left[f(W+Z)+\int\xi\exp(\beta Z)\mathrm{d}M^{\beta}+\frac{1}{2}\int_0^1\int((m^2-\Delta)^{1/2}\dot{Z}_t)^2\mathrm{d}t\right]\end{aligned}$$

▷ the function $Z\mapsto V^{\xi}(W+Z)$ is convex!

variational description of the infinite volume limit

▷ thanks to convexity the EL equations have a unique limit Z in the ∞ volume limit

▷ moreover we have the Γ-convergence of the variational description:

$$\mathcal{W}_{\mathbb{R}^2}(f) = \lim_{n\to\infty}\left[-\log\int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp}\right]$$

$$= \lim_{n\to\infty}\left[\mathcal{W}_{\xi_n}(f) - \mathcal{W}_{\xi_n}(0)\right] = \inf_K G^{f,\infty,\exp}(K)$$

with functional

$$G^{f,\infty,\exp}(K) = \mathbb{E}\Big[f(W+Z+K) + \underbrace{\int \exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^\beta + \mathcal{E}(K)}_{\geqslant 0}\Big]$$

which depends via Z on the infinite volume measure for the exp interaction.

end

Part III · the FBSDE for Grassmann measures

Euclidean Fermions

Fermions: quantum particles satisfying Fermi–Dirac statistics

EQFT: Wick rotation of QFT. $t \to \tau = it$, $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ Euclidean space. Wightman functions → Schwinger functions.

$$\Psi, \Psi^* \to \psi, \bar{\psi}.$$

☞ K. Osterwalder and R. Schrader. Euclidean Fermi fields and a Feynman-Kac formula for Boson-Fermions models. *Helvetica Physica Acta*, 46:277–302, 1973.

Euclidean fermion fields $\psi, \bar{\psi}$ form a Grassmann algebra

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \quad (\psi_\alpha^2 = 0).$$

Schwinger functions

- Schwinger functions are given by a Berezin integral on $\Lambda = \mathrm{GA}(\psi, \bar{\psi})$

$$\langle O(\psi,\bar{\psi})\rangle = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-S_E(\psi,\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi} e^{-S_E(\psi,\bar{\psi})}} = \frac{\langle O(\psi,\bar{\psi}) e^{-V(\psi,\bar{\psi})}\rangle_C}{\langle e^{-V(\psi,\bar{\psi})}\rangle_C}$$

$$S_E(\psi,\bar{\psi}) = \frac{1}{2}(\psi, C\,\bar{\psi}) + V(\psi,\bar{\psi}) \qquad \langle O(\psi,\bar{\psi})\rangle_C = \frac{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, O(\psi,\bar{\psi}) e^{-\frac{1}{2}(\psi, C\bar{\psi})}}{\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{-\frac{1}{2}(\psi, C\bar{\psi})}}$$

- Under $\langle\cdot\rangle_C$ the variables $\psi, \bar{\psi}$ are “Gaussian” (Wicks' rule):

$$\langle \psi(x_1)\cdots\psi(x_{2n})\rangle_C = \sum_\sigma (-1)^\sigma \langle \psi(x_{\sigma(1)})\psi(x_{\sigma(2)})\rangle_C \cdots \langle \psi(x_{\sigma(2n-1)})\psi(x_{\sigma(2n-1)})\rangle_C$$

algebraic probability

▷ a non-commutative probability space $(\mathscr{A},\omega)$ is given by a C^*-algebra $\mathscr{A}$ and a *state* ω, a linear normalized positive functional on $\mathscr{A}$ (i.e. $\omega(aa^*)\geqslant 0$).

▷ a random variable is an algebra homomorphism into $\mathscr{A}$

☞ L. Accardi, A. Frigerio, and J. T. Lewis. Quantum stochastic processes. *Kyoto University. Research Institute for Mathematical Sciences. Publications*, 18(1):97–133, 1982. 10.2977/prims/1195184017

example. (classical) random variable X with values on a manifold $\mathscr{M}$?

$$\Omega \xrightarrow{X} \mathscr{M} \xrightarrow{f} \mathbb{R}$$

$$f\in L^\infty(\mathscr{M};\mathbb{C})\to X(f)\in\mathscr{A}=L^\infty(\Omega;\mathbb{C}),\qquad X(fg)=X(f)X(g),\quad X(f^*)=X(f)^*.$$

algebraic data: $\mathscr{A}=L^\infty(\Omega;\mathbb{C})$, $\omega(a)=\int_\Omega a(\omega)\mathbb{P}(\mathrm{d}\omega)$, $X\in\mathrm{Hom}_*(L^\infty(\mathscr{M}),\mathscr{A})$.

Grassmann probability

▷ random variables with values in a Grassmann algebra Λ are algebra homomorphisms

$$\mathscr{G}(V) = \mathrm{Hom}(\Lambda V, \mathscr{A})$$

The embedding of ΛV into $\mathscr{A}$ allows to use the topology of $\mathscr{A}$ to do analysis on Grassmann algebras.

$$d_{\mathscr{G}(V)}(X, Y) := \|X - Y\|_{\mathscr{G}(V)} = \sup_{v \in V, |v|_V = 1} \|X(v) - Y(v)\|_{\mathscr{A}},$$

analogy. Gaussian processes in Hilbert space. Abstract Wiener space. "a convenient place where to hang our (analytic) hat on".

back to QFT: IR & UV problems

QFT requires to consider the formula (Fermionic path integral)

$$\langle O(\psi,\bar{\psi})\rangle_{C,V} = \frac{\langle O(\psi,\bar{\psi})e^{-V(\psi,\bar{\psi})}\rangle_C}{\langle e^{-V(\psi,\bar{\psi})}\rangle_C}$$

with local interaction

$$V(\psi,\bar{\psi}) = \int_{\mathbb{R}^d} P(\psi(x),\bar{\psi}(x))\mathrm{d}x$$

and singular covariance kernel (due to reflection positivity)

$$\langle \bar{\psi}(x)\psi(y)\rangle \propto |x-y|^{-\alpha}$$

this gives an ill-defined representation

- **large scale (IR) problems**
- **small scale (UV) problems**

well understood in the constructive QFT literature (Gawedzki, Kupiainen, Lesniewski, Rivasseau, Seneor, Magnen, Feldman, Salmhofer, Mastropietro, Giuliani, . . .)

what about stochastic quantisation for Grassmann measures?

☞ Ignatyuk/Malyshev/Sidoravicius | "Convergence of the Stochastic Quantization Method I,II", 1993. [Grassmann variables + cluster expansion]

weak topology + solution of equations in law + infinite volume limit but no removal of the UV cutoff

*

☞ "Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions" | joint work with Sergio Albeverio, Luigi Borasi, Francesco C. De Vecchi. arXiv:2004.09637 (PTRF)

algebraic probability viewpoint + strong solutions via Picard interation + infinite volume limit but no removal of the UV cutoff

☞ "A stochastic analysis of subcritical Euclidean fermionic field theories" | joint work with Francesco C. De Vecchi and Luca Fresta. arXiv:2210.15047

alg. prob. + forward-backward SDE + infinite volume limit & removal of IR cutoff in the whole subcritical regime

Grassmann stochastic analysis

▷ filtration $(\mathscr{A}_t)_{t\geqslant 0}$, conditional expectation $\omega_t\colon \mathscr{A}\to\mathscr{A}_t$,

$$\omega_t(ABC) = A\omega_t(B)C, \qquad A, C\in\mathscr{A}_t.$$

▷ Brownian motion $(B_t)_{t\geqslant 0}$ with $B_t\in\mathscr{G}(V)$

$$\omega(B_t(v)B_s(w)) = \langle v, Cw\rangle(t\wedge s), \qquad t, s\geqslant 0, v, w\in V.$$

$$\|B_t - B_s\| \lesssim |t-s|^{1/2}.$$

▷ Ito formula

$$\Psi_t = \Psi_0 + \int_0^t B_u(\Psi_u)\mathrm{d}u + X_t, \qquad \omega(X_t\otimes X_s) = C_{t\wedge s}$$

$$\omega_s(F_t(\Psi_t)) = \omega_s(F_s(\Psi_s)) + \int_s^t \omega_s[\partial_u F_u(\Psi_u) + \mathscr{L}F_u(\Psi_u)]\mathrm{d}u,$$

$$\mathscr{L}_u F_u = \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle B_u, \mathrm{DF}_u\rangle$$

the forward-backward SDE

[joint work with Francesco C. De Vecchi and Luca Fresta]

let Ψ be a solution of

$$\mathrm{d}\Psi_s = \dot{C}_s\, \omega_s(\mathrm{D}V(\Psi_T))\mathrm{d}s + \mathrm{d}X_s, \qquad s \in [0,T], \quad \Psi_0 = 0.$$

where $(X_t)_t$ is Gaussian martingale with covariance $\omega(X_t \otimes X_s) = C_{t\wedge s}$. Then

$$\omega(\mathrm{e}^{V(X_T)})\omega(\mathrm{e}^{-V(\Psi_T)}) = 1$$

and

$$\omega(O(\Psi_T)) = \frac{\omega(O(X_T)\mathrm{e}^{V(X_T)})}{\omega(\mathrm{e}^{V(X_T)})} = \frac{\langle O(\psi)e^{V(\psi)}\rangle_{C_T}}{\langle e^{V(\psi)}\rangle_{C_T}}$$

for any O.

▷ this FBSDE provides a stochastic quantisation of the Grassmann Gibbs measure along the interpolation $(X_t)_t$ of its Gaussian component

the backwards step

let F_t be such that $F_T = \mathrm{D}V$. By Ito formula

$$B_s := \omega_s(\mathrm{D}V(\Psi_T)) = \omega_s(F_T(\Psi_T))$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

$$= F_s(\Psi_s) + \int_s^T \omega_s\left[\left(\partial_u F_u(\Psi_u) + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u(\Psi_u) + \langle B_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle\right)\right]\mathrm{d}u$$

letting $R_t = B_t - F_s(\Psi_s)$ we have now the forwards-backwards system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

with

$$Q_u := \partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{DF}_u\rangle$$

solution theory

▷ standard interpolation for $C_\infty = (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}$, $\gamma \leqslant d/2$. $\chi \in C^\infty(\mathbb{R}_+)$, compactly supported around 0:

$$C_t := (1+\Delta_{\mathbb{R}^d})^{\gamma-d/2}\chi(2^{-2t}(-\Delta_{\mathbb{R}^d})), \qquad \|\dot{C}\|_{\mathscr{L}(L^\infty,L^\infty)} \lesssim 2^{2\gamma-d}, \|\dot{C}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim 2^{2\gamma}$$

▷ the system

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s)+R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[Q_u(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

can be solved by standard fixpoint methods for small interaction, uniformly in the volume since X stays bounded as long as $T < \infty$:

$$\|X_t\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{\gamma t}.$$

▷ decay of correlations can be proved by coupling different solutions (Funaki '96).

▷ limit $T \to \infty$ requires renormalization when $\gamma \in [0, d/2]$.

relation with the continuous RG

if we take F such that $Q = 0$ we have $R = 0$ and then

$$\Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s))\mathrm{d}s + X_t,$$

with

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u + \langle F_u, \dot{C}_u \mathrm{D}\mathrm{F}_u \rangle = 0, \qquad F_T = \mathrm{D}V.$$

define the effective potential V_t by the solution of the HJB equation

$$\partial_u V_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} V_u + \langle \mathrm{D}V_u, \dot{C}_u \mathrm{D}\mathrm{V}_u \rangle = 0, \qquad V_T = V.$$

then $F_t = \mathrm{D}V_t$ and the FBSDE computes the solution of the RG flow equation along the interacting field.

▷ so far a full control of the Fermionic HJB equation has not been achieved (work by Brydges, Disertori, Rivasseau, Salmhofer, . . .). Fermionic RG methods rely on a discrete version of the RG iteration.

approximate flow equation

thanks for the FBSDE we are not bound to solve exactly the flow equation and we can proceed to approximate it.

▷ **linear approximation.** take

$$\partial_u F_u + \frac{1}{2}\mathrm{D}^2_{\dot{C}_u} F_u = 0, \qquad F_T = \mathrm{D}V.$$

this corresponds to Wick renormalization of the potential V:

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u)\rangle]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{D}F_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

the key difficulty is to show uniform estimates for

$$\int_t^T \omega_t[\langle F_u(\Psi_u), \dot{C}_u F_u(\Psi_u)\rangle]\mathrm{d}u$$

as $T \to \infty$. we cannot expect better than $\|\Psi_t\| \approx \|X_t\| \approx 2^{\gamma t}$.

polynomial truncation

a better approximation is to truncate the equation to a (large) finite polynomial degree

$$\partial_u F_u + \frac{1}{2} \mathrm{D}^2_{\dot{C}_u} F_u + \Pi_{\leqslant K} \langle F_u, \dot{C}_u \mathrm{D} F_u \rangle = 0$$

where $\Pi_{\leqslant K}$ denotes projection on Grassmann polynomials of degree $\leqslant K$ and take

$$F_t(\psi) = \sum_{k \leqslant K} F_t^{(k)} \psi^{\otimes k}.$$

With this approximation one can solve the flow equation and get estimates

$$\|F_t^{(k)}\| \leqslant \frac{2^{(\alpha - \beta k)t}}{(k+1)^2}, \qquad t \geqslant 0,$$

with $\alpha = 3\beta$, $\beta = d/2 - \gamma$, provided the initial condition $F_T = \mathrm{D}V$ is appropriately renormalized.

FBSDE in the full subcritical regime

with the truncation Π_K we have

$$\begin{cases} \Psi_t = \int_0^t \dot{C}_s\,(F_s(\Psi_s) + R_s)\mathrm{d}s + X_t, \\ R_t = \int_t^T \omega_t[\Pi_{>K}\langle F_u, \dot{C}_u \mathrm{DF}_u\rangle(\Psi_u)]\mathrm{d}u + \int_t^T \omega_t[\langle R_u, \dot{C}_u \mathrm{DF}_u(\Psi_u)\rangle]\mathrm{d}u \end{cases}$$

but now observe that

$$\|\Psi_t\| \approx \|X_t\| \lesssim 2^{\gamma t} \qquad \|F_t^{(k)}\Psi_t^{\otimes k}\| \lesssim 2^{(\gamma k - \beta(k-3))t}$$

which is exponentially small for k large as long as $\gamma \leqslant d/4$ (full subcrititcal regime).

now the term

$$\int_t^T \omega_t[\Pi_{>K}\langle F_u, \dot{C}_u \mathrm{DF}_u\rangle(\Psi_u)]\mathrm{d}u$$

can be controlled uniformly as $T \to \infty$ and also the full FBSDE system. (!)

thanks