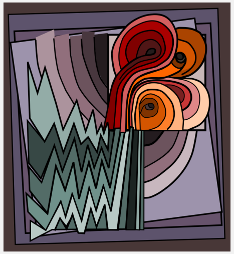
“Critical phenomena in statistical physics, continuum theories and SPDEs” · Warwick · December 14th, 2023

Wilson–Ito diffusions



Massimiliano Gubinelli · University of Oxford

[made with TeXmacs]

prologue

the origins

Itô's original paper

(Japanese version 1942, M.A.M.S. 1951)

Differential Equations Determining a Markoff Process*

KIYOSI ITÔ

More generally, for a simple Markoff process with its states being represented by the real numbers and having continuous parameter, the problem of determining quantities corresponding to $p_{ij}^{(k)}$ mentioned above and of constructing the corresponding Markoff process once these quantities are given has been investigated systematically by Kolmogoroff[3], who reduced the problem to the study of differential equations or integro-differential equations satisfied by the transition probability function.

W. Feller[4] has proved under fairly strong assumptions that these equations possess a unique solution and furthermore that the solution exhibits the properties of transition probability function.

However, if we adopt more strict point of view such as the one J. Doob[5] has applied toward his investigation of stochastic processes, it seems to us that the aforementioned work done by Feller is not quite adequate. For example, even though the differential equation determining the transition probabil-

ity function of a continuous stochastic process was solved in §3 of that paper, no proof was given of the fact that it is possible to introduce by means of this solution a probability measure on some "continuous" function space.

The objective of this article, then, is:

1) to formulate the problem precisely, and
2) to give a rigorous proof, à la Doob, for the existence of continuous parameter stochastic processes.

§1. Definition of Differentiation of a Markoff Process

Let $\{y_t\}$ be a (simple) Markoff process and denote by $F_{t_0 t}$ the conditional probability distribution[1] of $y_t - y_{t_0}$ given that "y_{t_0} is determined". $F_{t_0 t}$ is clearly a P_{y_t}-measurable (ρ) function[2] of y_t, where ρ denotes the Lévy distance among probability distributions.

Definition 1.1.[3]
If

$$F_{t_0 t}^{*[1/t-t_0]} \tag{1}$$

(here $[a]$ is the integer part of the number a, and "$*k$" denotes the k-fold convolution) converges in probability with respect to the Lévy distance ρ as $t \to t_0 + 0$, then we call the limit random variable (taking values in the space of probability distributions) the derivative of $\{y_t\}$ at t_0 and denote it by

$$D_{t_0}\{y_t\} \text{ or } Dy_{t_0}. \tag{2}$$

Corollary 1.1. *Dy_{t_0} is an infinitely divisible probability distribution.*[4]

bability distribution.

Dy_{t_0} obtained above is a function of t_0 as well as of y_{t_0}, and so, we denote it by $L(t_0, y_{t_0})$ corresponds precisely to the "basic transition probability" discussed in the Introduction.

The precise formulation of the problem of Kolmogoroff, then, is to solve the equation

$$Dy_t = L(t, y_t) \tag{4}$$

when the quantity $L(t, y)$ is given.

§2. A Comparison Theorem

Let us prove a comparison theorem for Dy_{t_0} which we shall make use of later.

Theorem 2.1. *Let $\{y_t\}$, $\{z_t\}$ be simple Markoff processes satisfying the following conditions:*

(1) $y_{t_0} = z_{t_0}$.
(2) $E(y_t - z_t \mid y_{t_0}) = o(t - t_0)$, *where o is the Landau symbol.*
(3) $\sigma(y_t - z_t \mid y_{t_0}) = o(\sqrt{t - t_0})$.

(Here $E(x \mid y)$ denotes the conditional expectation of x given y and $\sigma(x \mid y)$ denotes the conditional standard deviation of x given y. Also, the quantity o may depend on t_0 or y_{t_0}). Then, whenever Dz_{t_0} exists, Dy_{t_0} exists also, and $Dy_{t_0} = Dz_{t_0}$ holds.

H. Föllmer, “On Kiyosi Itô's Work and its Impact” (Gauss prize laudatio 2006)

In 1987 Kiyosi Itô received the Wolf Prize in Mathematics. The laudatio states that “he has given us a full understanding of the infinitesimal development of Markov sample paths. This may be viewed as Newton's law in the stochastic realm, providing a direct translation between the governing partial differential equation and the underlying probabilistic mechanism. Its main ingredient is the differential and integral calculus of functions of Brownian motion. The resulting theory is a cornerstone of modern probability, both pure and applied”.

yet...

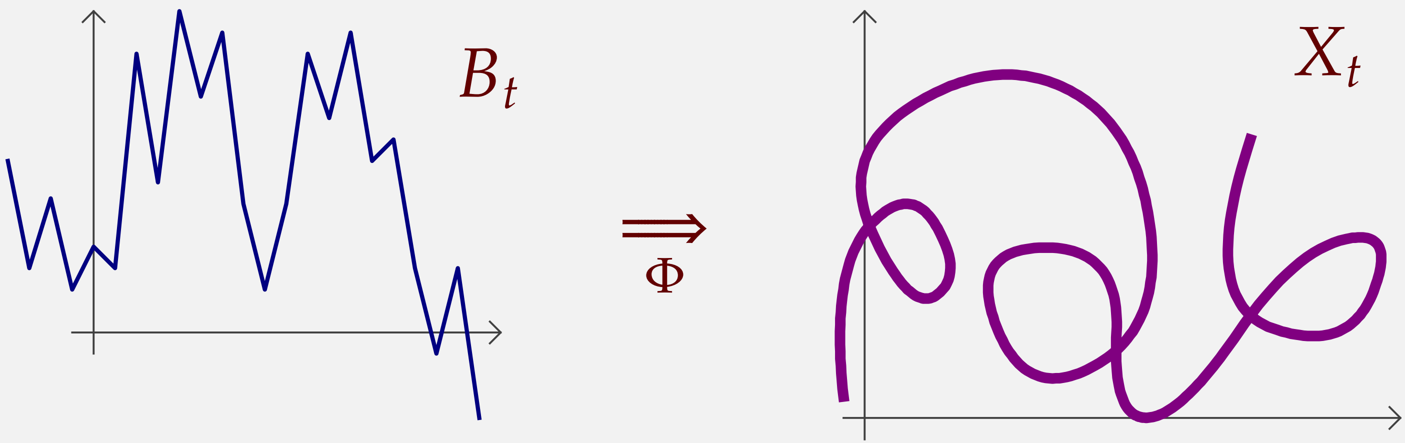
But when Kiyosi Itô came to Princeton in 1954, at that time a stronghold of probability theory with William Feller as the central figure, his new approach to diffusion theory did not attract much attention. Feller was mainly interested in the general structure of one-dimensional diffusions with local generator

$$F = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$$

motivated by his intuition that a "one-dimensional diffusion traveler makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m" [...]

Ito's brillant idea

Ito arrived to his calculus while trying to understand Feller's theory of diffusions an evolution in the space of probability measures and he introduced stochastic differential equations to define a map (**the Itô map**) which send Wiener measure to the law of a diffusion.



☞ useful byproduct: pathwise coupling between B and X

stochastic analysis today

> [...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognised as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.
>
> [D. Stroock, "Elements of stochastic calculus and analysis ", Springer, 2018]

Nowadays: Ito integral, Ito formula, stochastic differential equations, Girsanov's formula, Doob's transform, stochastic flows, Tanaka formula, local times, Malliavin calculus, Skorokhod integral, white noise analysis, martingale problems, rough path theory…

act I

the quest for equations

Euclidean (quantum) fields

conceptually: stationary Markovian d dimensional fields / Gibbsian continuous stochastic fields

probability measures ν on $\mathscr{S}'(\mathbb{R}^d)$ · (Feynman–Kac) path integral formalism

$$\nu(\mathrm{d}\varphi) \approx \frac{e^{-S(\varphi)}}{Z}\mathscr{D}\varphi \approx \frac{e^{-\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}}{Z'}\mu(\mathrm{d}\varphi), \qquad \mu(\mathrm{d}\varphi) \approx \frac{e^{-S_0(\varphi)}}{Z}\mathscr{D}\varphi$$

$$S(\varphi) = \underbrace{\int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 + m^2\varphi(x)^2}_{S_0(\varphi)} + \underbrace{\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x}_{\mathscr{V}(\varphi)}$$

natural probabilistic objects

heuristic description · large scale & small scale problems · need for renormalisation

How to set up stochastic analysis for Euclidean fields?

Gaussian free field

▷ GFF · simplest example of EQFT · Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ s.t.

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2+|k|^2}\frac{\mathrm{d}k}{(2\pi)^d} = (m^2-\Delta)^{-1}(x-y), \qquad x,y\in\mathbb{R}^d$$

and zero mean · $m>0$ is the *mass* · $G(0)=+\infty$ if $d\geqslant 2$: not a function · distribution of regularity

$$\alpha<(2-d)/2$$

▷ can be used to construct a QFT but the theory is free: no interaction

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variation · fractional Laplacian covariance $s\in(0,1)$

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} (a-\Delta)^{-1}(x-y)\rho(\mathrm{d}a) = (m^2+(-\Delta)^s)^{-1}(x-y)$$

the standard recipe for non-Gaussian Euclidean fields

❶ go on a periodic lattice: $\mathbb{R}^d \to \mathbb{Z}^d_{\varepsilon,L} = (\varepsilon\mathbb{Z}/2\pi L\mathbb{N})^d$ with spacing $\varepsilon > 0$ and side $2\pi L$

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}}\int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi)e^{-\overbrace{\frac{1}{2}\varepsilon^d\sum_{x\in\mathbb{Z}^d_{\varepsilon,L}}|\nabla_\varepsilon\varphi(x)|^2+m^2\varphi(x)^2+V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}}\mathrm{d}\varphi$$

ε is an UV regularisation and L the IR regularisation

❷ choose V_ε appropriately so that $\nu^{\varepsilon,L} \to \nu$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. E.g. take V_ε polynomial bounded below. $d = 2, 3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon\xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure.

❸ study the possible limit points [the $\boldsymbol{\Phi^4_d}$ **measure**] · ask interesting questions: uniqueness? non-uniqueness? decay of correlations? intrinsic description?

some models

▷ $d=1$ · time-reversal symmetric, translation invariant, Markov diffusions. the generator is given by an implicit expression involving the ground state Ψ of the Hamiltonian H

$$\mathscr{L} = \nabla \log \Psi \cdot \nabla + \frac{1}{2}\Delta \qquad H = -\Delta + x^2 + V(x).$$

▷ $d=2$ · various choices $(a_\varepsilon \to +\infty)$

$$V_\varepsilon(\xi) = \lambda \xi^{2l} + \sum_{k=0}^{2l-1} a_{k,\varepsilon}\xi^k, \quad V_\varepsilon(\xi) = a_\varepsilon \cos(\beta\xi)$$

$$V_\varepsilon(\xi) = a_\varepsilon \cosh(\beta\xi), \qquad V_\varepsilon(\xi) = a_\varepsilon \exp(\beta\xi)$$

▷ $d=3$ · “only” 4th order (6th order is critical)

▷ $d=4$ · all the possible limits are Gaussian (see Aizenmann–Duminil Copin)

stochastic equations for the free Gaussian free field

Gaussian free field $\mu : \mathbb{E}[\varphi(x)\varphi(y)] = (m^2 - \Delta)^{-1}(x-y)$ · ξ white noise

❶ "Gaussian map":

$$\varphi(x) = (m^2 - \Delta)^{-1/2}\xi(x), \qquad (m^2 - \Delta)\varphi(x) = (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❷ Stochastic mechanics (Nelson):

$$\partial_{x_0}\varphi(x_0, \bar{x}) = -(m^2 - \Delta_{\bar{x}})^{1/2}\varphi(x_0, \bar{x}) + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu):

$$\varphi(x) \sim \phi(t, x) \qquad \partial_t \phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) + c\xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x) \sim \phi(z, x) \qquad (-\Delta_z)\phi(z, x) = -(m^2 - \Delta_x)\phi(z, x) + c\xi(z, x), \qquad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

stochastic equations for non-Gaussian EQFTs $(V \neq 0)$

❶ Shifted Gaussian map (Albeverio/Yoshida) **[does not have the right properties!]**

$$(m^2 - \Delta)\varphi(x) + V'(\varphi(x)) = (m^2 - \Delta)^{1/2}\xi(x), \qquad x \in \mathbb{R}^d$$

❷ Stochastic mechanics (Nelson): ground-state transformation **[implicit!]**

$$\partial_{x_0}\varphi(x_0, \bar{x}) = [\nabla_{\varphi(x_0,\bar{x})} \log \Psi(\varphi)] + \xi(x_0, \bar{x}), \qquad x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{d-1}$$

❸ Parabolic stochastic quantization (Parisi–Wu): Langevin diffusion

$$\varphi(x) \sim \phi(t, x) \qquad \partial_t \phi(t, x) = -(m^2 - \Delta_x)\phi(t, x) - V'(\phi(t, x)) + c\xi(t, x), \qquad t \in \mathbb{R}, x \in \mathbb{R}^d$$

❹ Elliptic stochastic quantization (Parisi–Sourlas):

$$\varphi(x) \sim \phi(z, x) \qquad (-\Delta_z)\phi(z, x) = -(m^2 - \Delta_x)\phi(z, x) - V'(\phi(z, x)) + c\xi(z, x), \quad z \in \mathbb{R}^2, x \in \mathbb{R}^d$$

Nelson's Markov field equations

11. Remarks on Markov field equations

E. NELSON

11.1. Introduction

WE have no new existence theorems in constructive quantum field theory to present here, but we wish to indicate a new direction which looks promising and which certainly poses many interesting questions.

Only the theory of a neutral scalar field with a quartic self-interaction will be considered. This theory has been much studied in dimension $d=2$ and Glimm and Jaffe have pioneered the study in dimension $d=3$. (For references, see [4]—in particular, see the first article by Glimm, Jaffe, and Spencer and the reference listed there.)

We wish to stress field equations, and so we will begin with a heuristic discussion from that point of view. The simplest non-linear relativistic field equation with good formal properties is

$$(\square + m^2)A = -gA^3 + \alpha A, \tag{11.1}$$

corresponding to the interaction Lagrangian density $-(g/4)A^4+(\alpha/2)A^2$. We could of course absorb the linear term αA in the term m^2A, but we prefer not to. Here m^2 and g are positive.

The Euclidean approach to the problem of quantized solutions to (11.1) is, in rough outline, as follows. The Wightman distributions (vacuum expe-

11.3. The Markov field equation

Let us compute:

$$\begin{aligned}
\hat{E}\phi(x)-\mu &= \frac{1}{N}\int_{-\infty}^{\infty}(\xi-\mu)\exp\left[\left(-\frac{g}{4}\xi^4+\frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]\mathrm{d}\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty}\exp\left[\left(-\frac{g}{4}\xi^2+\frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\left(-\sigma^2\frac{\mathrm{d}}{\mathrm{d}\xi}\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]\right)\mathrm{d}\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty}\left(\sigma^2\frac{\mathrm{d}}{\mathrm{d}\xi}\exp\left[\left(-\frac{g}{4}\xi^4+\frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\right)\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]\mathrm{d}\xi \\
&= \frac{1}{N}\int_{-\infty}^{\infty}\sigma^2(-g\xi^3+\alpha\xi)\varepsilon^d\exp\left[\left(-\frac{g}{4}\xi^4+\frac{\alpha}{2}\xi^2\right)\varepsilon^d\right]\times \\
&\qquad\qquad\times\exp\left[-\frac{(\xi-\mu)^2}{2\sigma^2}\right]\mathrm{d}\xi \\
&= \sigma^2\hat{E}[-g\phi(x)^3+\alpha\phi(x)]\varepsilon^d.
\end{aligned}$$

We may write this as

$$\phi(x)-\mu = \sigma^2[-g\phi(x)^3+\alpha\phi(x)]\varepsilon^d+\sigma^2\omega(x)\varepsilon^d, \tag{11.8}$$

where $\omega(x)$ is a function of $\phi(x)$ and $\phi(y)$ for the nearest neighbours y of x (because (11.8) is the definition of $\omega(x)$) and

$$\hat{E}\omega(x)=0. \tag{11.9}$$

Nelson, E. `Remarks on Markov Field Equations'. *Functional Integration and Its Applications (Proc. Internat. Conf., London, 1974)*, 1975, 136–43.

an (pre)history of (Langevin) stochastic quantisation (personal & partial)

- 1981 · Parisi/Wu – stochastic quantisation for gauge theories (SQ)
- 1985 · Jona-Lasinio/Mitter · "On the stochastic quantization of field theory" (rigorous SQ for Φ^4_2 on bounded domain)
- 1988 · Damgaard/Hüffel · review book on SQ (theoretical physics)
- 1990 · Funaki · Control of correlations via SQ (smooth reversible dynamics)
- 1990–1994 · Kirillov · "Infinite-dimensional analysis and quantum theory as semimartingale calculus", "On the reconstruction of measures from their logarithmic derivatives", "Two mathematical problems of canonical quantization."
- 1993 · Ignatyuk/Malyshev/Sidoravicius · "Convergence of the Stochastic Quantization Method I,II" [Grassmann variables + cluster expansion]
- 2000 · Albeverio/Kondratiev/Röckner/Tsikalenko · "A Priori Estimates for Symmetrizing Measures…" [Gibbs measures via IbP formulas]
- 2003 · Da Prato/Debussche · "Strong solutions to the stochastic quantization equations"
- 2014 · Hairer – Regularity structures, local dynamics of Φ^4_3
- 2017 · Mourrat/Weber · global solutions for Φ^4_2, coming down from infinity for Φ^4_3
- 2018 · Albeverio/Kusuoka · "The invariant measure and the flow associated to Φ^4_3…"
- 2021 · Hofmanova/G. – Global space-time solutions for Φ^4_3 and verification of axioms (CMP)
- 2022 · Hairer/Steele – "optimal" tail estimates (JSP)
- 2020–2021 · Chandra/Chevyrev/Hairer/Shen · SQ for Yang–Mills 2d/3d (local theory) (arXiv)

act II

a new class of equations

a new class of equations for Euclidean fields

Goal · identify a rigorous framework to analyse Euclidean fields

Let φ_∞ be a random field on $\mathbb{R}^d$, possibly distributional.

❶ We endow it with a decomposition over scales $(\varphi_a)_{a\geqslant 0}$ where φ_a is a description of φ_∞ including fluctuations at scales larger than $1/a$. $\varphi_a \to \varphi_\infty$ as $a\to\infty$ and $a\mapsto\varphi_a$ is continuous in some topology over smooth fields.

❷ Let $(\mathscr{F}_a)_a$ the filtration generated by φ_a. An **observable** is a martingale wrt. this filtration. The observable $(\mathscr{O}_a)_a$ is supported on a set $U\subseteq\mathbb{R}^d$ if

$$\mathscr{O}_a - \hat{\mathscr{O}}_a(\varphi_a, \nabla\varphi_a, \cdots) \to 0, \qquad \text{as } a\to\infty$$

where $\hat{\mathscr{O}}_a$ is a functional of φ_a which depends on the fields only on a $1/a$-enlargement of the set U. A field of observables $x\in\mathbb{R}^d\mapsto(\mathscr{O}_a(x))_a$ is **local** if $\mathscr{O}_a(x)$ is supported on $\{x\}$ for all x.

E.g. if φ_∞ is a function:

$$\mathscr{O}_a(x) = \mathbb{E}[\varphi_\infty(x)|\mathscr{F}_a]$$

❸ We assume that the **scale dynamics** of $(\varphi_a)_a$ is given by an Itô diffusion:

$$\mathrm{d}\varphi_a = B_a \mathrm{d}a + \mathrm{d}M_a, \qquad \mathrm{d}\langle M \otimes M\rangle_a = D_a^2 \mathrm{d}a$$

with adapted drift B_a and diffusivity "matrix" D_a^2. We want that the dynamics is specified only in terms of features of φ_∞ "brought back" to the scale a. So we postulate:

a) the existence of local observables for the "microscopic" drift $(f_a)_{a\geqslant 0}$ and for the "microscopic" diffusivity $(\Sigma_a^2)_{a\geqslant 0}$

b) that the characteristics B_a, D_a^2 of the diffusion at scale a are given by some spatial averaging of the microscopic characteristics:

$$B_a = \dot{C}_a f_a, \qquad D_a^2 = \dot{C}_a^{1/2} \Sigma_a^2 \dot{C}_a^{1/2}$$

where $(C_a)_a$ are spatial averaging operators at scale a and $\dot{C}_a = \partial_a C_a$. E.g.

$$(C_a f)(x) = \int_{\mathbb{R}^d} a^d \chi(a(x-y)) f(y) \mathrm{d}y$$

where $\chi: \mathbb{R}^d \to \mathbb{R}$ is a mass one, positive and positive definite function with support on the unit ball. Note that we could allow also random averaging: $C_a = C_a(\varphi_a)$.

Wilson–Ito diffusions

[I. Bailleul, I. Chevyrev, MG. Wilson–Ito diffusions. arXiv (July 2023)]

Definition. *A **Wilson–Ito diffusion** $(\varphi_a)_a$ is the solution of the SDE*

$$\mathrm{d}\varphi_a = \dot{C}_a f_a \mathrm{d}a + \dot{C}_a^{1/2} \Sigma_a \mathrm{d}W_a, \qquad a \geqslant 0$$

where W is a cylindrical Brownian motion, f_a, Σ_a^2 are local observables for the microscopic drift and diffusivity and C_a is a local averaging operator at scale a.

It describes the random field φ_∞.

Covariant under change of scales · $A = A(a)$, $\tilde{\varphi}_a := \varphi_{A(a)}$ then $\tilde{C}_a = C_{A(a)}$ and

$$\mathrm{d}\tilde{\varphi}_a = \dot{\tilde{C}}_a f_{A(a)} \mathrm{d}a + \dot{\tilde{C}}_a^{1/2} \Sigma_{A(a)} \mathrm{d}\tilde{W}_a,$$

where $\tilde{W}$ is a cylindrical Brownian motion.

Trivial example · $f_a = 0$, $\Sigma_a = 1$. Then $\varphi_\infty = \int_0^\infty \dot{C}_a^{1/2} \mathrm{d}W_a$ is a white noise, so in general the solutions are distributions.

are there non-trivial examples?

approximation strategy · fix some $A>0$ and functional F_A and solve the forward-backward SDE:

$$\mathrm{d}\phi_a = \dot{C}_a \mathbb{E}_a[F_A(\phi_A)]\mathrm{d}a + \dot{C}_a^{1/2}\mathrm{d}W_a, \qquad a \in (0, A).$$

where $\mathbb{E}_a = \mathbb{E}[*|\mathscr{F}_a]$. Try to send $A \to \infty$ and at the same time make F_A more and more local.

coherent germs · another approach is to “guess” the drift $f_a \approx F_a(\varphi_a)$ where F_a is the “germ”

$$f_a = F_a(\varphi_a) + R_a, \qquad R_a \to 0 \text{ as } a \to \infty,$$

then we have a forward–backwards system for (φ_a, R_a):

$$\begin{cases} \mathrm{d}\phi_a = \dot{C}_a(F_a(\phi_a) + R_a)\mathrm{d}a + \dot{C}_a^{1/2}\mathrm{d}W_a, \\ R_a = \mathbb{E}_a[\int_a^\infty \mathring{\mathscr{L}}_b F_b(\phi_b)\mathrm{d}b + \int_a^\infty \mathrm{D}F_b(\phi_b)\dot{C}_b R_b \mathrm{d}b] \end{cases}$$

$$\mathring{\mathscr{L}}_b = \partial_b + \frac{1}{2}\Delta_{\dot{C}_b} + F_b(\phi_b)\dot{C}_b \mathrm{D}$$

Fully open problem in generality · I know very little about it · (some examples later)

a linear force

assume

$$f_a = \mathbb{E}_a[-A\phi_\infty] + \mathbb{E}_a[h(\phi_\infty)]$$

where A is a positive linear operator, e.g. $A = m^2 - \Delta$. Then, with $C_{\infty,a} := C_\infty - C_a$

$$\phi_\infty = \phi_a + \int_a^\infty \dot{C}_a(\mathbb{E}_a[-A\phi_\infty] + \mathbb{E}_a[h(\phi_\infty)])\mathrm{d}a + \int_a^\infty \dot{C}_a^{1/2}\mathrm{d}W_a$$

$$\mathbb{E}_a[\phi_\infty] = \phi_a - C_{\infty,a}A\mathbb{E}_a[\phi_\infty] + C_{\infty,a}\mathbb{E}_a[h(\phi_\infty)]$$

Let $\psi_a := (1 + C_{\infty,a}A)^{-1}\phi_a$ then $\psi_\infty = \phi_\infty$ and

$$\mathrm{d}\psi_a = \dot{Q}_a\mathbb{E}_a[h(\psi_\infty)]\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a, \qquad \dot{Q}_a := \partial_a(A^{-1}(1 + C_{\infty,a}A)^{-1})$$

the Gaussian field

$$X_a^Q := \int_0^a \dot{Q}_c^{1/2}\mathrm{d}W_c$$

has covariance $Q_\infty - Q_0 = (1 + A)^{-1}$, i.e. is a GFF when $A = m^2 - \Delta$.

gradient Wilson–Ito diffusions

Assume now also that $h(\psi_\infty) = -\mathrm{D}V_\infty(\psi_\infty)$, and let $(V_a)_{a\geqslant 0}$ be the solution to the *Polchinski equation* (HJB)

$$\partial_a V_a - \frac{1}{2}\mathrm{D}V_a \dot{Q}_a \mathrm{D}V_a + \frac{1}{2}\dot{Q}_a \mathrm{D}^2 V_a = 0$$

then one can prove that

$$\mathbb{E}_a[h(\psi_\infty)] = -\mathbb{E}_a[\mathrm{D}V_\infty(\psi_\infty)] = -\mathrm{D}V_a(\psi_a)$$

and by performing Doob's h-transform with $\mathrm{d}\mathbb{Q} = e^{-V_0(0) + V_a(\psi_a)}\mathrm{d}\mathbb{P}$ also that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[G(\psi_a)] &= \mathbb{E}_{\mathbb{Q}}[G(\psi_a) e^{V_0(0) - V_a(\psi_a)}] = \mathbb{E}_{\mathbb{P}}\left[G(X_a^Q) e^{V_0(0) - V_a(X_a^Q)}\right] \\ &= \frac{\mathbb{E}_{\mathbb{P}}\left[G(X_a^Q) e^{-V_\infty(X_\infty^Q)}\right]}{\mathbb{E}_{\mathbb{P}}\left[e^{-V_\infty(X_\infty^Q)}\right]} \end{aligned}$$

for any nice function G. In particular the law of ψ_∞ is given by the Gibbs measure

$$\nu_\infty(\mathrm{d}\psi) = \frac{e^{-V_\infty(\psi)} \mu^{Q_\infty}(\mathrm{d}\psi)}{\int e^{-V_\infty(\psi)} \mu^{Q_\infty}(\mathrm{d}\psi)}.$$

Euclidean fields as Wilson–Ito fields

The class of Wilson–Itô fields comprises as a particular case the Euclidean quantum fields constructed as perturbations of a Gaussian field. They are obtained by solving **Polchinski FBSDEs** of the form

$$\mathrm{d}\psi_a = -\dot{Q}_a \mathbb{E}_a[\mathrm{D}\, V_\infty(\psi_\infty)]\, \mathrm{d}a + \dot{Q}_a^{1/2} \mathrm{d}W_a.$$

Optimal control formulation · Let $u_a := -\dot{Q}_a^{1/2}\mathbb{E}_a[\mathrm{D}\, V_\infty(\psi_\infty)]$, test it with adapted $(v_a)_a$ and integrate:

$$\mathbb{E}\left[\int_0^\infty \langle v_a, u_a\rangle \mathrm{d}a + \left\langle \int_0^\infty \dot{Q}_a^{1/2} v_a \mathrm{d}a, \mathrm{D}V_\infty(\psi_\infty)\right\rangle\right] = 0.$$

It is the first-order condition for the minimisation of the functional

$$\Psi(u) := \mathbb{E}\left[V_\infty(\psi_\infty^u) + \frac{1}{2}\int_0^\infty \langle u_a, u_a\rangle \mathrm{d}a\right]$$

over all adapted controls $(u_a)_{a\geqslant 0}$, where

$$\psi_a^u := \int_0^a \dot{Q}_b^{1/2} u_b \mathrm{d}b + \int_0^a \dot{Q}_b^{1/2} \mathrm{d}W_b,$$

is the controlled process.

rigorous results

While Wilson–Ito fields are very young (few months), we have already established some results in the same flavour by looking at FBSDE or at the stochastic control formulation of Euclidean fields.

- N. Barashkov and MG · A Variational Method for Φ_3^4 · *Duke Mathematical Journal* (2020)
- N. Barashkov and MG · The Φ_3^4 Measure via Girsanov's Theorem · EJP (2021)
- N. Barashkov's PhD thesis · University of Bonn (2021)
- N. Barashkov and MG · On the Variational Method for Euclidean Quantum Fields in Infinite Volume, Prob. Math. Phys. (2023+)
- N. Barashkov · A Stochastic Control Approach to Sine Gordon EQFT · arXiv (2022)
- R. Bauerschmidt, M. Hofstetter · Maximum and Coupling of the Sine-Gordon Field · Ann. Prob. (2022)
- F. C. De Vecchi, L. Fresta, and MG · A stochastic analysis of subcritical Euclidean fermionic field theories · arXiv (2022)
- N. Barashkov, T. S. Gunaratnam, M. Hofstetter · Multiscale Coupling and the Maximum of φ_2^4 Models on the Torus · arXiv (2023)
- R. Bauerschmidt, T. Bodineau, B. Dagallier · Stochastic Dynamics and the Polchinski Equation: An Introduction · arXiv (2023)
- MG and S. J. Meyer · An FBSDE for Sine–Gordon up to 6π. In preparation.

more on observables

☞ Observables should be parametrized by approximatively local functions.

Recall that $\mathring{\mathscr{L}}_a = \partial_a + \frac{1}{2}\Delta_{\dot{C}_a} + F_a(\varphi_a)\dot{C}_a \mathrm{D}$ is the approximate generator of the WI diffusion and $\mathscr{L}_a = \partial_a + \frac{1}{2}\Delta_{\dot{C}_a} + f_a \dot{C}_a \mathrm{D}$ is the "true" generator.

The observable $(O_a)_a$ can be recovered from their germs $(\mathring{O}_a)_a$ via

$$O_a = \mathring{O}_a(\phi_a) + R_a^O,$$

$$R_a^O = \mathbb{E}_a \int_a^\infty \mathscr{L}_b \mathring{O}_b(\phi_b)\mathrm{d}b = \mathbb{E}_a \int_a^\infty \mathring{\mathscr{L}}_b \mathring{O}_b(\phi_b)\mathrm{d}b + \mathbb{E}_a \int_a^\infty R_b^f \dot{C}_b \mathrm{D}\mathring{O}_b(\phi_b)\mathrm{d}b.$$

the prefactorization algebra of observables

☞ Observables are martingales so they do not form an algebra.

If O^1, O^2 have some coherent germs $\mathring{O}^1, \mathring{O}^2$ with disjoint supports, say they are at distance strictly larger than $1/a_0$, we can set for $a \geqslant a_0$

$$\mathring{O}_a^{(12)} := \mathring{O}_a^{(1)}\, \mathring{O}_a^{(2)}.$$

Since $\mathrm{Tr}\, \dot{C}_a^{1/2} \sigma_a^2 \dot{C}_a^{1/2} \mathrm{D}\mathring{O}_a^{(1)} \mathrm{D}\mathring{O}_a^{(2)} = 0$, for $a \geqslant a_0$, because of the support condition, one has

$$\mathring{\mathscr{L}}_a \mathring{O}_a^{(12)} = (\mathring{\mathscr{L}}_a \mathring{O}_a^{(1)}) \mathring{O}_a^{(2)} + \mathring{O}_a^{(1)} (\mathring{\mathscr{L}}_a \mathring{O}_a^{(2)})$$

and the germ $\mathring{O}_a^{(12)}$ is under control provided $\mathring{\mathscr{L}}_a \mathring{O}_a^{(i)}$ decay fast enough in a and defines a new observable $O^{(12)}$.

This defines a structure of *pre-factorization algebra* on the observables.

gauge theories

Take the field φ to be a connection on a principal bundle over $\mathbb{R}^d$, with finite dimensional compact structure group G and Lie algebra $\mathfrak{g}$. Recall that the space of connections is affine with $\mathfrak{g}$-valued 1-forms as underlying vector space.

The gauge group consists of G-valued functions and acts on connections by

$$g\cdot\varphi = \mathrm{Ad}_g\varphi - (\mathrm{d}g)\, g^{-1}$$

and forms by

$$g\cdot f = \mathrm{Ad}_g f\,.$$

We assume moreover that the force field is a gauge covariant function of the underlying field $f_a = f_a(\varphi_a)$ (taking values in the space of 1-forms) :

$$g\cdot f_a(\psi) = f_a\,(g\cdot\psi),$$

for every $g\in\mathfrak{G}$ and connection ψ.

Given a connection φ, denote by $h^{xy}(\varphi)$ the φ-holonomy along the geodesic from x to y. Recall that

$$h^{xy}(g\cdot\varphi) = g(x)\,h^{xy}(\varphi)\,g(y)^{-1}. \tag{1}$$

Let $\chi_a(x,y)$ be a symmetric function of (x,y). We define a map $\dot{C}_a^{1/2}(\varphi)$ acting on 1-forms by

$$(\dot{C}_a^{1/2}(\varphi)\,\omega)(x) := \frac{1}{a^{1/2}}\int \chi_a(x,y)\mathsf{Ad}_{h^{xy}(\varphi)}\omega(y)\mathrm{d}y.$$

The operator $\dot{C}_a^{1/2}$ is symmetric and $\dot{C}_a(\varphi) := (\dot{C}_a^{1/2}(\varphi))^2$ is symmetric and non-negative. By (1), it is also gauge covariant:

$$g\cdot(\dot{C}_a(\varphi)\,\omega) = \dot{C}_a(g\cdot\varphi)\,(g\cdot\omega).$$

The tangent space of the gauge orbit at a given connection φ is spanned by the elements of the form $\mathrm{d}_\varphi h$, where d_φ is the φ-covariant derivative and h an arbitrary (smooth enough) $\mathfrak{g}$-valued 0-form,

$$\mathrm{d}_\varphi h = \sum_i (\partial_i h + [\varphi_i, h])\mathrm{d}x_i.$$

Two scale-dependent families of connections $(\varphi_a)_{a\geqslant 0}$ and $(\varphi'_a)_{a\geqslant 0}$ are said to be *gauge equivalent* if there exists a scale-dependent family of gauge transforms $(g_a)_{a\geqslant 0}$ such that

$$\varphi'_a = g_a \cdot \varphi_a$$

for all $a \geqslant 0$.

We note that if $(\varphi_a)_{a\geqslant 0}$ is a solution of the Wilson–Itô equation with $\sigma_a \equiv 1$, and $(g_a)_{a\geqslant 0}$ is an adapted process that is differentiable in a and takes values in $C^1(M, \mathfrak{G})$ then

$$\varphi_a^g := g_a \cdot \varphi_a$$

satisfies the equation

$$\begin{aligned} \mathrm{d}\varphi_a^g &= g_a \cdot (\mathrm{d}\varphi_a) - \mathrm{d}_{\varphi_t^a}(\dot{g}_t g_t^{-1} \mathrm{d}a) \\ &= (\dot{C}_a f_a(\varphi_a^g) - \mathrm{d}_{\varphi_a^g}(\dot{g}_a g_a^{-1}))\,\mathrm{d}a + \dot{C}_a^{1/2}(\varphi_a^g)\mathrm{d}W_a. \end{aligned}$$

Proposition. *For any adapted process* $(h_a)_{a\geqslant 0}$ *with values in* $C^1(M,\mathfrak{g})$ *the solution* $(\varphi_a^{(h)})_{a\geqslant 0}$ *to the equation*

$$\mathrm{d}\varphi_a^{(h)} = \big((\dot{C}_a f_a)(\varphi_a^{(h)}) + \mathrm{d}_{\varphi_a^{(h)}} h_a\big)\,\mathrm{d}a + \dot{C}_a^{1/2}(\varphi_a^{(h)})\,\mathrm{d}W_a$$

is gauge equivalent to the solution of a Wilson–Itô equation

$$\mathrm{d}\varphi_a^{[h]} = (\dot{C}_a f_a)(\varphi_a^{[h]})\,\mathrm{d}a + \dot{C}_a^{1/2}(\varphi_a^{[h]})\mathrm{d}W_a^h \tag{2}$$

driven by another Brownian motion W^h. *So the law of the gauge orbit of* $(\varphi_a^{(0)})_{a\geqslant 0}$ *is well-defined.*

The proof proceeds by solving the ordinary differential equation $\dot{g}_a g_a^{-1} = h_a$ and remarking that $\varphi_a^{[h]} := g_a \cdot \varphi_a^{(h)}$ solves (2) with $\mathrm{d}W_a^h := g_a \cdot \mathrm{d}W_a$, which is a Brownian motion by Itô isometry.

As Parisi–Wu stochastic quantization scheme this approach allows to quantize gauge theories without path integrals and their associated ghosts or BRS symmetries. Adding terms of the form $\mathrm{d}_{\varphi_a} h_a$ in the drift allows us to perform gauge-fixing in analogy to the Zwanziger–DeTurck–Sadun trick.

some remarks on Wilson–Ito diffusions

- our the working hypothesis is that Wilson–Ito diffusions are natural mechanism to generate and analyse random fields
- they emerge from simple and natural assumptions and covers in principle much more than those theories that can be reached perturbatively from a Gaussian functional integral, e.g. from the path-integral picture
- they can be used for gauge theories and fields on manifolds and for Grassmann fields
- they allow for rigorous non-perturbative results in the whole space
- (hopefully) they provide a new framework for the stochastic analysis of Euclidean fields
- numerical simulations?
- still lot to understand: FBSDEs are non-trivial to analyse but PDE methods seems applicable similarly to Parisi–Wu style stochastic quantisation.

thanks