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Massimiliano Gubinelli

for Stochastic Processes

Claudia Klüppelberg

President
Bernoulli Society for Mathematical Statistics and Probability

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A variational method for Euclidean quantum fields

Massimiliano Gubinelli · University of Bonn

▹ Ito & Dœblin introduced a variety of analysis adapted to the sample paths of a stochastic process, following Levy's pionnering work on independent increment processes.

▹ Consider a family of kernels $(P_t)_{t\geqslant 0}$ on $\mathbb{R}^d$ satisfying Chapman–Kolmogorov equation

$$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$$

which defines a probability $\mathbb{P}$ on $C(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$: the law of a continuous Markov process.

▹ Sample paths have a *"tangent" process*. Ito identified it as a particular Lévy process: the Brownian motion $(W_t)_t$.

▹ Stochastic calculus: from the local picture to the global structure via *stochastic differential equation* (SDE)

$$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$$

▹ These are the basic building blocks of **stochastic analysis**

▹ The SDE describe rigorously the Gaussian small-time asymptotics of the diffusion:

$$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{1}{2\delta t}(y-x+b(x)\delta t)a(x)^{-1}(y-x+b(x)\delta t)} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}, \qquad \delta t \ll 1$$

▹ Like in analysis, the fact that we can consider infinitesimal changes simplify the analsysi and make appear universal objects:

- polynomials → calculus, Taylor expansion
- Brownian motion and its functionals → Ito calculus, stochastic Taylor expansion

▹ **Euclidean quantum fields** (EQFs) are particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$:

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

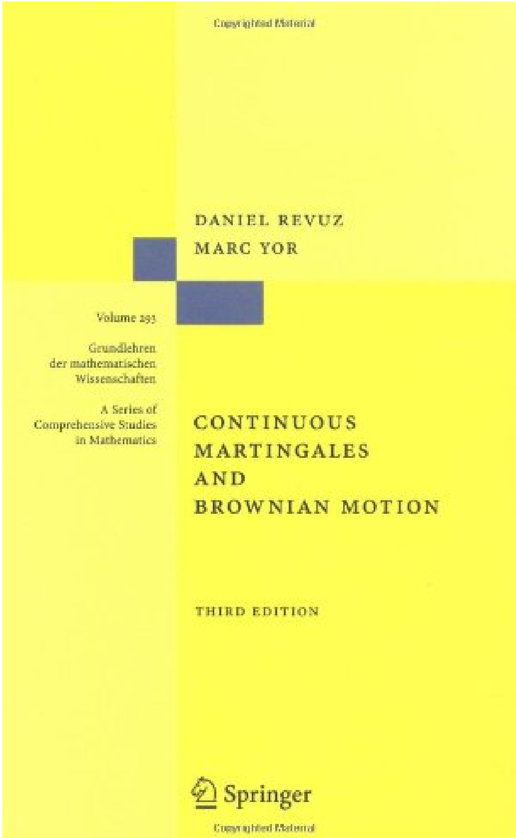
$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + p(\varphi(x))\mathrm{d}x$$

for some non-linear function $p: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, e.g. a polynomial bounded below, exponentials, trig funcs.

Introduced in the '70-'80 as a tool to constructs models of (bosonic) quantum field theories in the sense of Wightman via the reconstruction theorem of Osterwalder–Schrader.

A very ill-defined representation:

- Large scale problems: the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- Small scale problems: sample paths are not expected to be function, but only distributions, the quantity $p(\varphi(x))$ does not make sense.



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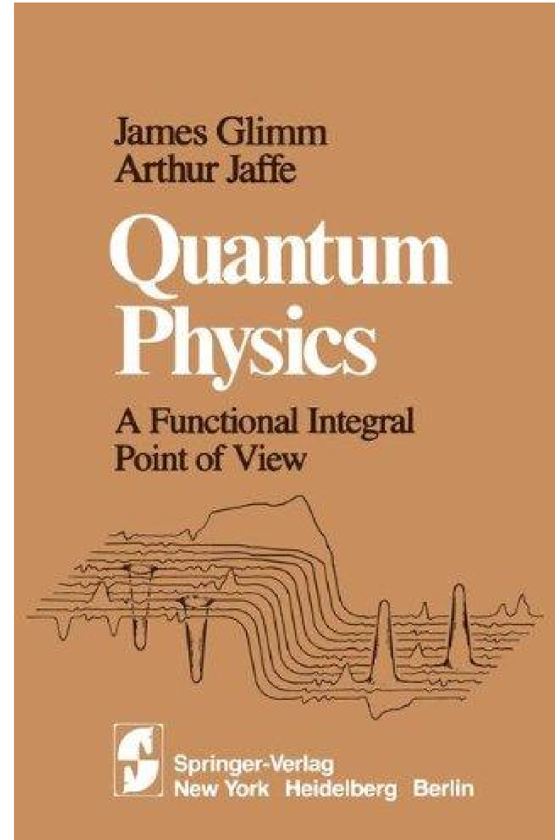

535 pages

Does a **stochastic analysis of EQFs** exists?

- Can we describe EQFs “locally” in term of “simpler” EQFs?
- … and then connect the “local” and “global” pictures?

Diffusion process	Euclidean quantum field
$P_t(x, \mathrm{d}y)$	$\nu(\mathrm{d}y)$
$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-S(\varphi)} \mathrm{d}\varphi$
$P_{\delta t}(x, \mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}},$	(local description)?
t	(change parameter)?
$(W_t)_t$	(basic block)?
$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$	(local-to-global link)?

The basic problem to solve is to identify:

- a **change parameter** along which consider "change" (*time* for diffusions)
- a suitable **building block** for the infinitesimal changes (*Brownian motion* for diffusion)

Many different (but roughly equivalent) ways to solve this identification problem:

- **Parabolic stochastic quantisation.** the parameter is an additional "fictious" coordinate $t \in \mathbb{R}$, playing the röle of a simulation time. The EQF is viewed as the invariant measure of a Markov process (SDE). Building block is a space-time white noise. [Parisi/Wu, Nelson, Jona-Lasinio/Mitter, Albeverio/Röckner, Da Prato/Debbusche, Hairer, Mourrat/Weber, G./Hofmanova, many others...]
- **Elliptic stochastic quantisation.** the parameter is a coordinate $z \in \mathbb{R}^2$. Building block is a white noise in $\mathbb{R}^{d+2}$. An elliptic stochastic partial differential equation describes the EQF as a function of the white noise. Link with supersymmetry. [Parisi/Sourlas, Klein/Landau/Perez, Albeverio/De Vecchi/G.]
- **the variational method.**

I've reported at SPA 2018 about recent progresses in understanding this "new" stochastic analysis. I will describe more in detail the variational method today.

The variational method

[Barashkov–Gubinelli, Duke J. 2020]

- change parameter: the scale $t \in \mathbb{R}_{\geqslant 0}$ of spatial variation of the sample paths.
- basic block: the scale-by-scale decomposition of the Gaussian free field, i.e. a centred Gaussian random field φ with covariance

$$\mathbb{E}[\varphi(x)\varphi(y)] = (m^2 - \Delta)^{-1}(x - y) \approx |x - y|^{-(d-2)/2}.$$

- local-to-global link: a stochastic optimal control problem or alternatively an ∞-dim Hamilton-Jacobi-Bellmann equation for the associated value function.

The HJB equation realises the continuous renormalization group à la Wilson and Wegner. [Wilson, Wegner, Polchinski, Salmhofer, Brydges/Kennedy, Mitter, Gawedzki/Kupiainen, Brydges/Bauerschmidt/Slade, Bauerschmidt/Bodineau, also many others...]

See also recent work of Bauerschmidt/Bodineau and Bauerschmidt/Hofstetter on HJB for sine-Gordon.

The basic building block: "Brownian motion" $(W_t)_{t\geqslant 0}$

$$W_t(x) \approx \int_{|k|\leqslant t} \frac{e^{ik\cdot x}\xi(\mathrm{d}k)}{(|k|^2+m^2)^{1/2}}, \qquad x\in\mathbb{R}^d,$$

with ξ a white noise in $\mathbb{R}^d$.

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-\int_{\mathbb{R}^d} p(W_T(x))\mathrm{d}x} \overbrace{\frac{e^{-S_0(\varphi)}\mathrm{d}\varphi}{Z}}^{\text{Gaussian Free Field}}$$

$$S_0(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 \mathrm{d}x.$$

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \lim_{\Lambda\uparrow\mathbb{R}^d} \lim_{T\to\infty} \frac{\mathbb{E}\left[e^{-\int_\Lambda p(W_T(x))\mathrm{d}x} O(W_\infty)\right]}{\mathbb{E}\left[e^{-\int_\Lambda p(W_T(x))\mathrm{d}x}\right]}$$

$\Lambda\to\infty$ infinite volume limit, $T\to\infty$ small scales limit

Stochastic control representation for general functional of a Brownian motion

Boué-Dupuis (BD) formula

$$-\log \mathbb{E}[e^{-V_\Lambda(W_T)+F(W_\cdot)}] = \inf_u \mathbb{E}\left[V_\Lambda(W_T+Z_T) - F(W_\cdot + Z_\cdot) + \frac{1}{2}\int_0^\infty \| u_s \|^2_{L^2(\mathbb{R}^d)} \mathrm{d}s \right]$$

$$\frac{\partial}{\partial t} Z_t(x) \approx \int_{|k|\approx t} \frac{e^{ik\cdot x}\hat{u}_t(k)}{(|k|^2+m^2)^{1/2}} \mathrm{d}k = (J_t u_t)(x), \qquad (u_t)_{t\geq 0} \text{ adapted to } (W_t)_{t\geq 0}$$

$$V_\Lambda(W) = \int_\Lambda p(W(x))\mathrm{d}x$$

In the r.h.s. we have a stochastic process $(W_t + Z_t)_{t\geq 0}$ controlled by u with a quadratic running cost.

Note that

$$\sup_t \|Z_t\|_{H^1}^2 \lesssim \int_0^\infty \|u_s\|_{L^2(\mathbb{R}^d)}^2 ds.$$

Summing up:

$$-\log \int_{\mathscr{S}'(\mathbb{R}^d)} e^{f(\varphi)} \nu(d\varphi) \approx \lim_{\Lambda \uparrow \mathbb{R}^d, T \to \infty} [\mathscr{W}_{\Lambda,T}(f) - \mathscr{W}_{\Lambda,T}(0)]$$

$$\mathscr{W}_{\Lambda,T}(f) = \inf_u \mathbb{E}\left[V_\Lambda(W_T + Z_T) - f(W_\infty + Z_\infty) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2(\mathbb{R}^d)}^2 ds \right]$$

▹ Existence and uniqueness of the limits $T \to \infty$ and $\Lambda \uparrow \mathbb{R}^d$ is reduced to a good understanding of a stochastic variational problem.

Let us discuss first the $T \to \infty$ limit for fixed Λ (which we take to be a torus $\mathbb{T}^d$).

The Φ^4_2 EQF

Take $d=2$ and $p(\xi)=p_T(\xi)=\xi^4+a_T\xi^2+b_T$. By choosing a_T, b_T appropriately we have

$$V_{\Lambda,T}(W_T+Z_T)=\int_{\mathbb{T}^d}\left\{[\![W_T^4]\!]+4[\![W_T^3]\!]Z_T+6[\![W_T^2]\!]Z_T^2+3W_TZ_T^3+Z_T^4\right\}$$

where $[\![W_T^k]\!]$ are Wick products and martingales in T.

By direct estimation one has

$$\left|\int_{\mathbb{T}^d}4[\![W_T^3]\!]Z_T\right|+\left|\int_{\mathbb{T}^d}6[\![W_T^2]\!]Z_T^2\right|+\left|\int_{\mathbb{T}^d}3W_TZ_T^3\right|\leqslant Q([\![W_T^k]\!])+\delta\left[\int_{\mathbb{T}^d}Z_T^4+\frac{1}{2}\sup_t\|Z_t\|^2_{H^1}\right]$$

From which

$$V_\Lambda(W_T+Z_T)+\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2(\mathbb{R}^d)}ds\geqslant -Q([\![W_T^k]\!])+(1-\delta)\left[\int_{\mathbb{T}^d}Z_T^4+\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2(\mathbb{R}^d)}ds\right]\geqslant -Q([\![W_T^k]\!])$$

▹ Now for all $k \geqslant 0$ and $p \geqslant 1$ and $\kappa < 0$ we have

$$\sup_T \mathbb{E} \| [\![W_T^k]\!] \|_{B^{-\kappa}_{\infty,\infty}(\mathbb{T}^2)}^p < \infty$$

where $B^{-\kappa}_{\infty,\infty}(\mathbb{T}^2)$ are the Hölder-Besov spaces of the torus $\mathbb{T}^2$.

All the Wick products $[\![W_T^k]\!]$ converge in $\mathscr{S}'$ as distributions of small negative regularity.

▹ Lower bound.

$$\inf_u \mathbb{E}\left[V_\Lambda(W_T + Z_T) + \frac{1}{2}\int_0^\infty \| u_s \|_{L^2(\mathbb{R}^d)}^2 \mathrm{d}s \right] \geqslant -\mathbb{E}\, Q([\![W_T^k]\!]) > -\infty.$$

▹ Upper bound. On the other hand

$$\inf_u \mathbb{E}\left[V_\Lambda(W_T + Z_T) + \frac{1}{2}\int_0^\infty \| u_s \|_{L^2(\mathbb{R}^d)}^2 \mathrm{d}s \right] \leqslant \mathbb{E}[V_\Lambda(W_T)] \leqslant \mathbb{E} Q'([\![W_T^k]\!]) < +\infty.$$

▹ We have bounds uniform in T. Convergence is then (non-trivial) matter of choosing the right spaces.

The Φ_3^4 EQF

▹ Take now $d=3$ always $p(\xi)=\xi^4+\cdots$.

Things get trickier. We have now

$$\sup_T \left[\mathbb{E}\|W_T\|^p_{B^{-1/2-\kappa}_{\infty,\infty}(\mathbb{T}^d)} + \mathbb{E}\|[\![W_T^2]\!]\|^p_{B^{-1-\kappa}_{\infty,\infty}(\mathbb{T}^d)} + \frac{1}{\log T}\mathbb{E}\|[\![W_T^3]\!]\|^p_{B^{-3/2-\kappa}_{\infty,\infty}(\mathbb{T}^d)} \right] < \infty.$$

Wick's monomials tends to be more irregular distributions. Analytic estimates get worser, in particular no hope to control directly

$$\left|\int_{\mathbb{T}^d} [\![W_T^3]\!] Z_T\right| + \left|\int_{\mathbb{T}^d} [\![W_T^2]\!] Z_T^2\right|$$

in terms of the L^4 and H^1 norms of Z_T.

▹ **Idea:** implement approximate minimizers to guess appropriate cancellations.

▷ Variational functional

$$\int_{\mathbb{T}^d}\Big\{[\![W_T^4]\!]+4[\![W_T^3]\!]Z_T+6[\![W_T^2]\!]Z_T^2+3W_TZ_T^3+Z_T^4+\overbrace{c_T([\![W_T^2]\!]+W_TZ_T+Z_T^2)}^{\text{additional renorm.}}\Big\}+\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2(\mathbb{R}^d)}ds$$

▷ Ito formula

$$\int_{\mathbb{T}^d}[4[\![W_T^3]\!]Z_T+6[\![W_T^2]\!]Z_T^2]=\int_0^T\int_{\mathbb{T}^d}[4[\![W_s^3]\!]+12[\![W_s^2]\!]Z_s]\dot{Z}_s ds+\text{martingale}$$

$$=\int_0^T\int_{\mathbb{T}^d}J_s[4[\![W_s^3]\!]+12[\![W_s^2]\!]Z_s]u_s ds+\text{martingale}$$

▷ **(Stochastic) Euler-Lagrange equation** (take care to obtain an adapted solution!)

$$J_s[4[\![W_s^3]\!]+12[\![W_s^2]\!]Z_s]+u_s+\cdots=0$$

▹ Approximate minimizer allow to introduce a new variational parameter l:

$$u_s = -J_s[4[\![W_s^3]\!] + 12[\![W_s^2]\!]Z_s] + \cdots = -J_s[4[\![W_s^3]\!] + 12[\![W_s^2]\!] \succ Z_s] + l_s$$

where we use the paraproduct decomposition

$$[\![W_s^2]\!]Z_s = [\![W_s^2]\!] \succ Z_s + [\![W_s^2]\!] \preccurlyeq Z_s$$

in order to isolate the most irregular part in this product.

In $[\![W_s^2]\!] \succ Z_s$ the factor Z_s behaves in estimates like a constant.

▹ Then

$$Z_T = \int_0^T J_s u_s \mathrm{d}s = -\int_0^T J_s^2[4[\![W_s^3]\!] + 12[\![W_s^2]\!] \succ Z_s]\mathrm{d}s + \overbrace{\int_0^T J_s l_s \mathrm{d}s}^{K_T}$$

▹ Renormalized form of the variational problem

Substituting the reparametrized control into the functional we get:

$$\int_{\mathbb{T}^d}\left\{[\![W_T^4]\!]+4[\![W_T^3]\!]Z_T+6[\![W_T^2]\!]Z_T^2+3W_TZ_T^3+Z_T^4\right\}+\frac{1}{2}\int_0^\infty \|u_s\|_{L^2(\mathbb{R}^d)}^2 ds$$

$$=\Psi(Z_T,K_T)+\int_{\mathbb{T}^d}Z_T^4+\frac{1}{2}\int_0^\infty \|l_s\|_{L^2(\mathbb{R}^d)}^2 ds$$

where now (provided c_T is appropriately chosen)

$$|\Psi(Z_T,K_T)|\leqslant\delta\int_{\mathbb{T}^d}Z_T^4+\frac{1}{2}\int_0^\infty \|l_s\|_{L^2(\mathbb{R}^d)}^2 ds$$

allowing to get uniform bounds as in the $d=2$ case.

▹ The $T\to\infty$ limit is then obtained via Γ-convergence of this variational representation. *(after 30+ pages fighting for some compactness...)*

Wrap up (so far)

The variational approach introduces a nice *stochastic analysis* of some EQFs:

- change parameter : a scale parameter $t \in \mathbb{R}_{\geqslant 0}$;
- basic building block: the scale-by-scale decomposition $(W_t)_{t \geqslant 0}$ of the Gaussian free field:

$$W_T \approx \sum_k \frac{\mathbb{1}_{|k| \leqslant T}}{(|k|^2 + m^2)^{1/2}} g_k$$

- local description: approximate minimizer via stochastic Euler-Lagrange equation:

$$u_s = -J_s[4\llbracket W_s^3 \rrbracket + 12\llbracket W_s^2 \rrbracket Z_s] + \cdots$$

- local-to-global link: renormalized variational problem.

Small scales behaviour

The local description is precise enough to obtain interesting results like the singularity of the Φ_3^4 measure ν on the torus $\mathbb{T}^3$ with respect to the Gaussian free field (GFF) μ.

The reason is that under ν the scale-by-scale canonical field $X_t = W_t + Z_t$ behaves like (recall the EL equations)

$$X_t = W_t + Z_t = W_t - \int_0^T J_s^2 [4 [\![W_s^3]\!] + 12 [\![W_s^2]\!] \succ Z_s] \mathrm{d}s + \cdots$$

and this allows to show that a quantity like

$$\int_{\mathbb{T}^d} [\![\varphi^4]\!] \approx \int_{\mathbb{T}^d} [\![X_T^4]\!]$$

has different almost sure behaviour as $T \to \infty$ under ν and under the GFF measure μ.

[Barashkov/G. See also Hairer.]

The semiclassical limit is about letting $\hbar \to 0$ in

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu^{\hbar}(\mathrm{d}\varphi) = \frac{1}{Z^{\hbar}} \int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi) e^{-\frac{S(\varphi)}{\hbar}} \mathrm{d}\varphi.$$

- Physically it corresponds to the limit where quantum fluctuations become negligible.
- Probabilistically it leads to a Laplace principle where $W \to \hbar^{1/2} W$ (small noise limit).

The variational formulation for $\Phi^4_{2,3}$ on $\mathbb{T}^d$ gives readily that the family $(\nu^{\hbar})_{\hbar>0}$ satisfies a large deviation principle with the classical action as rate function

$$-\hbar \log \nu^{\hbar}(A) \approx \inf_{\varphi \in A} \int_{\mathbb{T}^d} \frac{1}{2} |\nabla\varphi(x)|^2 + \frac{1}{2} m^2 |\varphi(x)|^2 + |\varphi(x)|^4 \mathrm{d}x.$$

[Barashkov/G. See also previous work by Jona-Lasinio/Mitter]

Infinite volume limit

is far less understood. Let's discuss Φ^4_2. [Barashkov/G. in progress]

- Small scale limit can be taken right away if we want by letting $t \in [0,1]$ and

$$\mathbb{E}[W_t(x)W_s(x)] = (m^2 - \Delta)^{-1}(x-y)(t \wedge s), \qquad t,s \in [0,1].$$

- The functional (ignoring the source term with f) is now

$$\int_\Lambda \{ [\![W_1^4]\!] + 4[\![W_1^3]\!]Z_1 + 6[\![W_1^2]\!]Z_1^2 + 3W_1 Z_1^3 + Z_1^4 \} + \overbrace{\frac{1}{2}\int_0^1 \| u_s \|^2_{L^2(\mathbb{R}^2)} ds}^{\approx \| Z_1 \|^2_{H^1}}, \qquad Z_t = \int_0^t (m^2 - \Delta)^{-1/2} u_s ds.$$

- Note that $W_1 \in B^{-\kappa}_{\infty,\infty}([-L,L]^d)$ but

$$\| W_1 \|_{B^{-\kappa}_{\infty,\infty}([-L,L]^d)} \approx \log^{1/2}(L), \quad L \to \infty.$$

Euler-Lagrange equations

$$\mathbb{E}_\mu\left(\int_{\mathbb{R}^2} f'(W_1+Z_1)\,K_1 + 4\int_\Lambda [\![(W_1+Z_1)^3]\!]\,K_1 + \int_0^1\int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\,\dot{K}_s \mathrm{d}s \right) = 0, \qquad \forall K. \tag{1}$$

A strange (new) kind of stochastic elliptic equation (in weak form).

- ▹ One can still get apriori estimates. Take weight $\rho(x) = (1+|x|)^{-\ell}$ and $K_s = \rho Z_s$

$$\mathbb{E}\left[\int_{\mathbb{R}^2} f'(W_1+Z_1)\,\rho Z_1 + 4\int_\Lambda \rho([\![(W_1+Z_1)^3]\!] - Z_1^3) Z_1 + \underbrace{4\int_\Lambda \rho Z_1^4 + \int_0^1\int_{\mathbb{R}^2} \dot{Z}_s(m^2-\Delta)\,\rho\dot{Z}_s \mathrm{d}s}_{\text{coercive terms}} \right] = 0$$

- ▹ Some analysis then gives

$$\mathbb{E}_\mu\left(4\int_\Lambda \rho Z_1^4 + \int_0^1\int_{\mathbb{R}^2} |(m^2-\Delta)^{1/2}(\rho^{1/2}\dot{Z}_s)|^2 \mathrm{d}s \right) \lesssim \mathbb{E}_\mu Q([\![W_1^k]\!]) < +\infty.$$

uniformly in Λ. This allows to take the limit in (1).

Description of the infinite volume measure?

$$-\log \int_{\mathscr{S}'(\mathbb{R}^d)} e^{f(\varphi)} \nu(\mathrm{d}\varphi) \approx \lim_{\Lambda \uparrow \mathbb{R}^d} [\mathscr{W}_\Lambda(f) - \mathscr{W}_\Lambda(0)]$$

$$\mathscr{W}_\Lambda(f) = \inf_Z \mathbb{E}\left[V_\Lambda(W_1 + Z_1) - f(W_1 + Z_1) + \frac{1}{2}\int_0^1 \|(m^2 - \Delta)^{1/2} \dot{Z}_s\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}s \right]$$

$$= \mathbb{E}\left[V_\Lambda(W_1 + Z_1^f) - f(W_1 + Z_1^f) + \frac{1}{2}\int_0^1 \|(m^2 - \Delta)^{1/2} \dot{Z}_s^f\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}s \right]$$

▹ Still open.

Main obstacle : show that $Z^f - Z$ decays at large distance for local observables f ≈ stability of the Euler-Lagrange equations.

This is a main open problem in all approaches to stochastic analysis of $\Phi^4_{2,3}$

Outlook

Goal: understand the stochastic analysis of EQFs
(at least for superrenormalizable models)

- Identify "building blocks" and describe EQFs (non-perturbatively) in terms of these simpler objects.
- Small scales behaviour/renormalization: well understood in most models in some of the approaches (see e.g. recent results of Hairer et al. on Yang-Mills fields).
- Coercivity (large fields problem) plays a key role for global control and infinite volume limit. So far, not understood at all for YM.
- Variational description of the infinite volume limit? A possible replacement for DLR approach or Dyson–Schwinger equations.
- A complete example of the program carried out to a large extent: Barashkov in his thesis verifies OS axioms for Sine-Gordon within the variational approach: existence and uniqueness of infinite volume limit, control of the Laplace transform, decay of correlations.
- Possible to use it for proving weak-universality (e.g. for Φ^4_3)

Open problems

- Infinite volume limit, stability of the EL equations.
- How to apply these ideas to gauge theories/geometric models? Higgs model, Yang-Mills? Coercivity not well understood.
- Grassmann fields? Is there a replacement for the variational structure?
- Small coupling regime?
- Decay of correlations at high temperature?
- Use the approach for lattice unbounded spin systems?
- What about mass-less models on the lattice: $\nabla\varphi$ models?
- ...

Many of these problems are still open also for the other varieties of stochastic analysis thanks